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EXISTENCE OF NONTRIVIAL SOLUTIONS FOR A CLASS OF ELLIPTIC SYSTEMS

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Abstract. Using a version of the generalized mountain pass theorem, we obtain the existence of nontrivial solutions for a class of superquadratic elliptic systems.

1. Introduction and statement of results

Consider the elliptic system

$$\begin{gathered}
-\Delta u = H_v(u,v,x), \quad \text{in } \Omega,\\
-\Delta v = H_u(u,v,x), \quad \text{in } \Omega,\\
u=0, \quad v=0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, and H_u denotes the partial derivative of H with respect to u.

The system (1.1) has been already studied in the recent works [1, 2, 4, 5, 6, 7, 8, 10, 11] and the reference therein. Using the generalized mountain pass theorem in its infinite dimensional setting, Benci and Rabinowitz [1] studied a special case of the system

$$\begin{gathered}
-\Delta w = H_w(w,z,x),\\
\Delta z = H_z(w,z,x),
\end{gathered} \tag{1.2}$$

which is equivalent to system (1.1).

In Clément, De Figueiredo and Mitidieri [4] discussed the existence of a positive solution for the system below subjected to Dirichlet boundary conditions:

$$-\Delta u = f(v), \quad -\Delta v = g(u), \quad \text{in } \Omega. \tag{1.3}$$

In this case, the Hamiltonian is $H(u,v) = F(v) + G(u)$, where $F(t) = \int_0^t f(s)ds$, and similarly G is a primitive of g. The approach in [4] for system (1.3) was via a Topological argument, using a theorem of Krasnoselski on Fixed Point Index for compact mappings in cones in Banach spaces.

Using a variational approach through a version of the generalized mountain pass theorem, De Figueiredo and Felmer [8] obtained the existence of nontrivial solutions for system (1.1), which extends the results in [1] and [4]. Felmer and Wang [10] proved the existence of infinitely many strong solutions for the elliptic system (1.1).

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De Figueiredo and Ding [7] studied the existence and multiplicity of solutions of the elliptic system (1.2). For more details on semilinear elliptic systems of the Hamiltonian types, we refer the reader to [6] and the references therein.

We say that (u, v) is a strong solution of (1.1) if

$$u \in W^{2,p/(p-1)}(\Omega) \cap W_0^{1,p/(p-1)}(\Omega), \quad v \in W^{2,q/(q-1)}(\Omega) \cap W_0^{1,q/(q-1)}(\Omega)$$

and (u, v) satisfies $-\Delta u = H_v(u, v, x)$ and $-\Delta v = H_u(u, v, x)$ a.e. in Ω.

In this article, motivated by [8], we study the existence of strong solutions for the elliptic system (1.1). This kind of Hamiltonian was studied recently by Chen and Tang [3] in the context of Hamiltonian systems.

Here and in the sequel, we assume that $p \geq \alpha > p-1 > 0$ and $q \geq \beta > q-1 > 0$ such that

(i) $\frac{1}{\alpha} + \frac{1}{\beta} < 1$,

(ii) $\{2 - (\frac{1}{p} + \frac{1}{q})\} \max\{\frac{p}{\alpha}, \frac{q}{\beta}\} < 1 + \frac{2}{N}$,

(iii)

$$\frac{p-1}{p}\frac{q}{\beta} < 1, \quad \frac{q-1}{q}\frac{p}{\alpha} < 1.$$

We will always assume $N \geq 3$. If $N = 2$ or $N = 1$, we need less restrictive assumptions. Furthermore, in the case $N \geq 5$, we also impose

(iv)

$$(1 - \frac{1}{p}) \max\{\frac{p}{\alpha}, \frac{q}{\beta}\} < \frac{N+4}{2N}, \quad (1 - \frac{1}{q}) \max\{\frac{p}{\alpha}, \frac{q}{\beta}\} < \frac{N+4}{2N}.$$

Our main results are the following theorems.

Theorem 1.1. *Suppose that H satisfies:*

(H0) *$H : \mathbb{R}^2 \times \overline{\Omega} \to \mathbb{R}$ is of class C^1;*

(H1) *$H(u, v, x) \geq 0$ for all $(u, v, x) \in \mathbb{R}^2 \times \overline{\Omega}$;*

(H2) *There exists $c_0 > 0$ such that*

$$\frac{1}{\alpha} H_u(u, v, x) \cdot u + \frac{1}{\beta} H_v(u, v, x) \cdot v \geq H(u, v, x) > 0$$

for all $(u, v) \in \mathbb{R}^2$, $|(u, v)| \geq c_0$ and $x \in \overline{\Omega}$;

(H3)

$$\lim_{|(u,v)| \to 0} \frac{H(u, v, x)}{|u|^{1+\alpha/\beta} + |v|^{1+\beta/\alpha}} = 0$$

uniformly for $x \in \Omega$;

(H4) *There exists $c_1 > 0$ such that*

$$|H_u(u, v, x)| \leq c_1(|u|^{p-1} + |v|^{(p-1)q/p} + 1),$$
$$|H_v(u, v, x)| \leq c_1(|v|^{q-1} + |u|^{(q-1)p/q} + 1)$$

for all $(u, v) \in \mathbb{R}^2$ and $x \in \overline{\Omega}$.

Then problem (1.1) *possesses at least one nontrivial strong solution.*

Remark 1.2. For Hamiltonian systems, the corresponding superquadratic condition (H2) is due to Felmer [9]. The hypothesis (H3) was introduced in [3].

Theorem 1.3. *Suppose that H satisfies* (H1)–(H4) *and*

(H0') *$H : \mathbb{R}^2 \times \overline{\Omega} \to \mathbb{R}$ is of class $C^{1,\varepsilon}$;*

(H5) *$H_u(u,v,x) \geq 0$, $H_v(u,v,x) \geq 0$ for all $(u,v) \in \mathbb{R}^2$, $u \geq 0$, $v \geq 0$, $x \in \overline{\Omega}$;*
(H6) *$H_u(u,v,x) = 0$ when $u = 0$, $H_v(u,v,x) = 0$ when $v = 0$.*

Then (1.1) *possesses at least one positive solution (u,v) with $u(x) > 0$, $v(x) > 0$ if $x \in \Omega$.*

Remark 1.4. It is easy to show that our Theorems 1.1 and 1.3 generalize Theorems 0.1 and 0.3 in [8]. There are functions H satisfying our Theorems and not satisfying the corresponding results in [8]. In fact, for $\alpha > 1, \beta > 1$ satisfying $1/\alpha + 1/\beta < 1$, let

$$H(u,v,x) = a_1(|u|^{1+\alpha/\beta} + |v|^{1+\beta/\alpha})^{\gamma_1} + a_2(|u|^{1+\alpha/\beta} + |v|^{1+\beta/\alpha})^{\gamma_2},$$

where $a_1 > 0$, $a_2 > 0$, $1 < \gamma_1 < \alpha\beta/(\alpha+\beta) < \gamma_2$. Choose $\gamma_2 = (\alpha\beta+1)/(\alpha+\beta)$, $p = \alpha + 1/\beta$, $q = \beta + 1/\alpha$, then H satisfies our Theorems and does not satisfy the corresponding results in [8].

Remark 1.5. If $H(u,v) = |u|^p/p + |v|^q/q$ then one could use a fourth-order approach and then assumption (iv) would not be necessary (see [2, 5]). We do not know if (iv) can be avoided for general Hamiltonians.

2. Proof of main results

To set up our problem variationally, we shall have to utilize fractional Sobolev spaces. For more details and references we cite [8]. Consider the spaces E^s, which are obtained as the domains of fractional powers of the operator

$$-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega),$$

where Δ denotes the Laplacian and $H^2(\Omega)$, $H_0^1(\Omega)$ are the usual Sobolev spaces. Namely $E^s = D((-\Delta)^{s/2})$ for $0 \leq s \leq 2$, and the corresponding operator is denoted by

$$A^s : E^s \to L^2(\Omega).$$

The spaces E^s are Hilbert spaces with inner product

$$(u,v)_{E^s} = \int_\Omega A^s u A^s v \, dx.$$

Its associated norm is denoted by $\|u\|_{E^s}$. In E^s, we find the Poincaré's inequality for the operator A^s

$$\|A^s u\|_{L^2(\Omega)} \geq \lambda_1^{s/2} \|u\|_{L^2(\Omega)} \quad \text{for all } u \in E^s,$$

where λ_1 is the first eigenvalue of $-\Delta$.

Next, we define the spaces on which we set up the problem. For numbers $s > 0$ and $t > 0$ with $s + t = 2$, we define the Hilbert space $E = E^s \times E^t$ and the bilinear form $B : E \times E \to \mathbb{R}$ by the formula

$$B((u,v),(\phi,\psi)) = \int_\Omega (A^s u A^t \psi + A^s \phi A^t v) dx.$$

The bilinear form B is continuous and symmetric. There exists a selfadjoint bounded linear operator $L : E \to E$ such that

$$B(z,\eta) = (Lz,\eta)_E$$

for all z, $\eta \in E$. Here $(\cdot,\cdot)_E$ denotes the natural inner product in E induced by E^s and E^t. We can also define the quadratic form $\mathcal{Q}: E \to \mathbb{R}$ associated to B and L as

$$\mathcal{Q}(z) = \frac{1}{2}(Lz, z)_E = \int_\Omega A^s u A^t v\, dx \tag{2.1}$$

for all $z = (u, v) \in E$. The operator L defined above can be written as [8, Proposition 1.1]

$$L(u, v) = ((A^s)^{-1}A^t v, (A^t)^{-1}A^s u). \tag{2.2}$$

We define the subspaces

$$E^+ = \{(u, A^{-t}A^s u)|u \in E^s\}, \quad E^- = \{(u, -A^{-t}A^s u)|u \in E^s\}, \tag{2.3}$$

which give a natural splitting $E = E^+ \oplus E^-$. The spaces E^+ and E^- are the positive and negative eigenspaces of L, they are consequently orthogonal with respect to the bilinear form B; that is,

$$B(z^+, z^-) = 0, \quad \forall z^+ \in E^+,\ \forall z^- \in E^-.$$

We also find that

$$\frac{1}{2}\|z\|_E^2 = \mathcal{Q}(z^+) - \mathcal{Q}(z^-), \tag{2.4}$$

where $z = z^+ + z^-$, $z^\pm \in E^\pm$.

Now we will choose the numbers s and t defining the orders of the Sobolev spaces involved. From inequality (ii), we see the existence of $s, t \in \mathbb{R}$, $s + t = 2$ such that

$$(1 - \frac{1}{p})\max\{\frac{p}{\alpha}, \frac{q}{\beta}\} < \frac{1}{2} + \frac{s}{N} \tag{2.5}$$

and

$$(1 - \frac{1}{q})\max\{\frac{p}{\alpha}, \frac{q}{\beta}\} < \frac{1}{2} + \frac{t}{N}. \tag{2.6}$$

By (iii) and (iv), if $N \geq 5$, we can choose $s > 0$ and $t > 0$. Since $p/\alpha \geq 1$ and $q/\beta \geq 1$, we obtain from (2.5) and (2.6) that

$$\frac{1}{p} > \frac{1}{2} - \frac{s}{N}, \quad \frac{1}{q} > \frac{1}{2} - \frac{t}{N}. \tag{2.7}$$

These last inequalities and Sobolev Embedding Theorem give the compact inclusions (see [8, Theorem 1.1])

$$E^s \hookrightarrow L^p(\Omega), \quad E^t \hookrightarrow L^q(\Omega).$$

Now we can define a functional $\Phi : E \to \mathbb{R}$ as

$$\Phi(z) = \mathcal{Q}(z) - \mathcal{H}(z) = \int_\Omega A^s u A^t v\, dx - \int_\Omega H(u, v, x)dx \tag{2.8}$$

for $z = (u, v) \in E$. The functional Φ is of class C^1. The functional

$$\mathcal{H}(u, v) = \int_\Omega H(u(x), v(x), x)dx$$

is of class C^1 and its derivative is given by

$$\mathcal{H}'(u, v)(\phi, \psi) = \int_\Omega H_u(u, v, x)\phi + H_v(u, v, x)\psi dx$$

for all $(u, v), (\phi, \psi) \in E$. Moreover $\mathcal{H}' : E \to E$ is a compact operator (see [8]).

For details and proof of the aspects discussed so far, we refer the reader to [8]. In particular, see in [8] that critical points of Φ correspond to the strong solutions of (1.1).

For our proofs, we introduce the following abstract critical point theorem due to Felmer [9]. We consider a Hilbert space E with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We assume that E has a splitting $E = X \oplus Y$, where the subspaces X and Y are not necessarily orthogonal and both of them can be infinite dimensional. Let $\Phi : E \to \mathbb{R}$ be a functional having the structure

$$\Phi(z) = \frac{1}{2}\langle Lz, z\rangle + \mathcal{H}(z).$$

- (I1) $L : E \to E$ is a linear, bounded, selfadjoint operator.
- (I2) $\mathcal{H}'$ is compact.
- (I3) There are two linear bounded, invertible operators $B_1, B_2 : E \to E$ satisfying: If $\omega \in \mathbb{R}_0^+$, the linear operator
$$\widehat{B}(\omega) = P_X B_1^{-1} \exp(\omega L) B_2 : X \to X$$
is invertible.

Here P_X denotes the projection of E onto X induced by the splitting $E = X \oplus Y$, and $\mathbb{R}_0^+$ is a set of nonnegative real numbers.

Let $\rho > 0$ and define

$$S = \{B_1 z : \|z\| = \rho, z \in Y\}. \tag{2.9}$$

For $z_+ \in Y$, $z_+ \neq 0$, $\sigma > \rho/\|B_1^{-1}B_2 z_+\|$ and $M > \rho$, we define

$$Q = \{B_2(\tau z_+ + z) : 0 \leq \tau \leq \sigma, \|z\| \leq M, z \in X\}. \tag{2.10}$$

We define ∂Q as the boundary of Q relative to the subspace

$$\{B_2(\tau z_+ + z) | \tau \in \mathbb{R}, z \in X\}.$$

Let us consider the class of functions

$$\Gamma = \{h \in C(E \times [0,1], E) : h \text{ satisfies the following three conditions}\}$$

1. $h(z,t) = \exp(\omega(z,t)L)z + K(z,t)$, where $\omega : E \times [0,1] \to \mathbb{R}_0^+$ is continuous and transforms bounded sets into bounded sets, and $K : E \times [0,1] \to E$ is compact.
2. $h(z,t) = z$ for all $z \in \partial Q$ and all $t \in [0,1]$.
3. $h(z,0) = z$ for all $z \in Q$.

Theorem 2.1 ([9]). *Let $\Phi : E \to \mathbb{R}$ be a C^1 functional satisfying the Palais-Smale condition and* (I1)–(I3)*. Furthermore assume that there is a constant $\delta > 0$ such that*

- (IS) $\Phi(z) \geq \delta$ *for all* $z \in S$,
- (IQ) $\Phi(z) \leq 0$ *for all* $z \in \partial Q$.

Then Φ possesses a critical point with critical value $d \geq \delta$ characterized by

$$d = \inf_{h\in\Gamma} \sup_{z\in Q} \Phi(h(z,1)).$$

Here, we define the operators B_1, B_2 and the splitting $E = E^s \times E^t = E^- \oplus E^+$. Let $X = E^-$ and $Y = E^+$. We define $B_1 : E \to E$ by

$$B_1(u,v) = (\rho^{\beta-1}u, \rho^{\alpha-1}v) \tag{2.11}$$

and $B_2 : E \to E$ by

$$B_2(u,v) = (\sigma^{\beta-1}u, \sigma^{\alpha-1}v). \tag{2.12}$$

Certainly B_1 and B_2 are bounded linear operators and both of them are invertible. From (2.9) and (2.11), we obtain

$$S = \{(\rho^{\beta-1}u, \rho^{\alpha-1}v) : \|(u,v)\| = \rho, (u,v) \in E^+\}. \tag{2.13}$$

By (2.10) and (2.12), we have

$$\begin{aligned} Q = \big\{&\tau(\sigma^{\beta-1}u_+, \sigma^{\alpha-1}v_+) + (\sigma^{\beta-1}u, \sigma^{\alpha-1}v) : 0 \le \tau \le \sigma, \\ &0 \le \|(u,v)\| \le M,\ (u,v) \in E^-\big\}, \end{aligned} \tag{2.14}$$

where $z_+ = (u_+, v_+) \in E^+$ with u_+ some fixed eigenvector of $-\Delta$. In what follows, we note that z_+ is an eigenvector of L associated to a positive eigenvalue (i.e. to 1). We assume $\|z_+\|_E = 1$. We denote by ∂Q the boundary of Q relative to the subspace

$$\{\tau(\sigma^{\beta-1}u_+, \sigma^{\alpha-1}v_+) + (\sigma^{\beta-1}u, \sigma^{\alpha-1}v) : \tau \in \mathbb{R},\ (u,v) \in E^-\}.$$

Now, we can give the proof of our Theorems.

Proof of Theorem 1.1. The proof is divided into several steps.

Step 1: Φ satisfies the Palais-Smale condition. See [8, Proposition 2.1].

Step 2: We claim that Φ satisfies (I1)–(I3). From (2.1) and (2.8), we have

$$\begin{aligned} \Phi(z) &= \mathcal{Q}(z) - \int_\Omega H(u,v,x)dx \\ &= \frac{1}{2}(Lz,z)_E - \int_\Omega H(u,v,x)dx. \end{aligned}$$

Taking $\mathcal{H}(z) = \int_\Omega H(z,x)dx$, we obtain

$$(\Phi'(z), \eta) = \langle Lz, \eta\rangle - \langle \mathcal{H}'(z), \eta\rangle,$$

where $z = (u,v)$ and $\eta = (\phi, \psi)$. So, $\Phi' = L - \mathcal{H}'$, where L is a linear bounded self-adjoint operator. And, from the growth hypothesis (H4), $\mathcal{H}'$ is a compact operator. Thus, Φ satisfies (I1) and (I2). From (2.2), one has

$$L(u,v) = ((A^s)^{-1}A^t v, (A^t)^{-1}A^s u).$$

It is well known that

$$\begin{gathered} \exp(\omega L) = 1 + \omega L + \frac{1}{2!}\omega^2 L^2 + \frac{1}{3!}\omega^3 L^3 + \frac{1}{4!}\omega^4 L^4 + \dots, \\ \cosh(\omega L) = 1 + \frac{1}{2!}\omega^2 L^2 + \frac{1}{4!}\omega^4 L^4 + \dots, \\ \sinh(\omega L) = \omega L + \frac{1}{3!}\omega^3 L^3 + \frac{1}{5!}\omega^5 L^5 + \dots. \end{gathered}$$

Hence, for $\omega \in \mathbb{R}$, the operator $\exp(\omega L) : E \to E$ is given by

$$\exp(\omega L)(u,v) = \cosh(\omega)(u,v) + \sinh(\omega)(A^{-s}A^t v, A^{-t}A^s u). \tag{2.15}$$

We can give an explicit formula for $\widehat{B}(u,v)$. For $z \in E^-$, one has $z = (u, -A^{-t}A^s u)$ with $u \in E^s$. From (2.11), (2.12) and (2.15), one sees

$$B_1^{-1} exp(\omega L) B_2 z = (\xi u, \eta A^{-t}A^s u),$$

where
$$\xi = \frac{\cosh(\omega)\sigma^{\beta-1} - \sinh(\omega)\sigma^{\alpha-1}}{\rho^{\beta-1}}, \quad \eta = \frac{-\cosh(\omega)\sigma^{\alpha-1} + \sinh(\omega)\sigma^{\beta-1}}{\rho^{\alpha-1}}.$$
Since the orthogonal projections $P^{\pm} : E \to E^{\pm}$ are given by the formula (see [8])
$$P^{\pm}(u,v) = \frac{1}{2}(u \pm A^{-s}A^t v, v \pm A^{-t}A^s u).$$
Using the formula for the projection into E^-, we obtain
$$\begin{aligned}
\widehat{B}(\omega)z &= P^-(\xi u, \eta A^{-t}A^s u)\\
&= \frac{1}{2}(\xi u - \eta u, \eta A^{-t}A^s u - \xi A^{-t}A^s u)\\
&= \frac{1}{2}((\xi - \eta)u, -(\xi-\eta)A^{-t}A^s u)\\
&= \frac{\theta}{2}(u, -A^{-t}A^s u),
\end{aligned}$$
where
$$\theta = \big\{\big(\frac{\sigma^{\beta-1}}{\rho^{\beta-1}} + \frac{\sigma^{\alpha-1}}{\rho^{\alpha-1}}\big)\cosh(\omega) - \big(\frac{\sigma^{\alpha-1}}{\rho^{\beta-1}} + \frac{\sigma^{\beta-1}}{\rho^{\alpha-1}}\big)\sinh(\omega)\big\}.$$
If we assume $\sigma > 1$ and $\rho < 1$, it is easy to see that θ is positive. In fact
$$\big(\frac{\sigma^{\beta-1}}{\rho^{\beta-1}} + \frac{\sigma^{\alpha-1}}{\rho^{\alpha-1}}\big) - \big(\frac{\sigma^{\alpha-1}}{\rho^{\beta-1}} + \frac{\sigma^{\beta-1}}{\rho^{\alpha-1}}\big) = \frac{(\rho^{\beta-1} - \rho^{\alpha-1})(\sigma^{\alpha-1} - \sigma^{\beta-1})}{\rho^{\alpha+\beta-2}}$$
is positive so that $\theta > 0$ independently of the value of $\omega \in \mathbb{R}$. It implies that $\widehat{B}(\omega)$ is invertible.

Step 3: We claim that (IS) is satisfied, that is, there exist $\rho > 0$ and $\delta > 0$ such that $\Phi(z) \geq \delta$, $\forall z \in S$, where S is defined by (2.13).

From hypothesis (H3) and (H4), for each $\varepsilon > 0$, we have
$$H(u,v,x) \leq \varepsilon\big(|u|^{1+\alpha/\beta} + |v|^{1+\beta/\alpha}\big) + c_2\big(|u|^p + |v|^q\big), \tag{2.16}$$
where $c_2 = c_2(\varepsilon) > 0$. Let $\tilde{z} = (u,v) \in E^+$ and take $z = (\rho^{\beta-1}u, \rho^{\alpha-1}v)$ for some $\rho > 0$. Then, by (2.16), one has
$$\begin{aligned}
&\int_\Omega H(u,v,x)dx\\
&\leq \varepsilon\Big(\rho^{(\beta-1)(1+\alpha/\beta)}\int_\Omega |u|^{1+\alpha/\beta}dx + \rho^{(\alpha-1)(1+\beta/\alpha)}\int_\Omega |v|^{1+\beta/\alpha}dx\Big)\\
&\quad + c_2\Big(\rho^{(\beta-1)p}\int_\Omega |u|^p dx + \rho^{(\alpha-1)q}\int_\Omega |v|^q dx\Big).
\end{aligned} \tag{2.17}$$
Since $\alpha \leq p$, $\beta \leq q$, by (i) and (2.7), one sees that
$$\frac{1}{1+\alpha/\beta} = \frac{\beta}{\alpha+\beta} > \frac{1}{p} > \frac{1}{2} - \frac{s}{N}$$
and
$$\frac{1}{1+\beta/\alpha} = \frac{\alpha}{\alpha+\beta} > \frac{1}{q} > \frac{1}{2} - \frac{t}{N}.$$
Hence, Sobolev Embedding Theorem gives the compact inclusions (see [8, Theorem 1.1])
$$E^s \hookrightarrow L^{1+\alpha/\beta}(\Omega), \quad E^t \hookrightarrow L^{1+\beta/\alpha}(\Omega).$$

By (2.17), there exist two positive constants c_3 and c_4 such that

$$\begin{aligned}\int_\Omega H(u,v,x)dx \le \varepsilon c_3\Big(&\rho^{(\beta-1)(1+\alpha/\beta)}\|\tilde z\|_E^{1+\alpha/\beta} + \rho^{(\alpha-1)(1+\beta/\alpha)}\|\tilde z\|_E^{1+\beta/\alpha}\Big)\\ &+ c_4\Big(\rho^{(\beta-1)p}\|\tilde z\|_E^p + \rho^{(\alpha-1)q}\|\tilde z\|_E^q\Big).\end{aligned} \tag{2.18}$$

As $(u,v)\in E^+$, then $v = A^{-t}A^s u$ and $u = A^{-s}A^t v$. We obtain

$$\mathcal{Q}(z) = \int_\Omega \rho^{\beta-1}A^s u \rho^{\alpha-1}A^t v\,dx = \rho^{\alpha+\beta-2}\int_\Omega A^s u A^t v\,dx. \tag{2.19}$$

It follows from (2.4) and (2.19) that

$$\mathcal{Q}(z) = \frac{1}{2}\rho^{\alpha+\beta-2}\|\tilde z\|_E^2. \tag{2.20}$$

If we consider $\rho = \|\tilde z\|_E$, from (2.18) and (2.20), we obtain

$$\begin{aligned}\Phi(z) &\ge \frac{1}{2}\rho^{\alpha+\beta} - \varepsilon c_3(\rho^{\beta+\alpha}+\rho^{\alpha+\beta}) - c_4(\rho^{\beta p}+\rho^{\alpha q})\\ &= \big(\frac{1}{2} - 2\varepsilon c_3\big)\rho^{\beta+\alpha} - c_4(\rho^{\beta p}+\rho^{\alpha q}).\end{aligned} \tag{2.21}$$

Since $1/\alpha + 1/\beta < 1$, $\alpha \le p$ and $\beta \le q$, one has $\beta+\alpha < \alpha q$ and $\beta + \alpha < \beta p$. Taking $\varepsilon = 1/(8c_3)$, if ρ is small enough, by (2.21), there exists $\delta > 0$ such that

$$\Phi(z) \ge \delta > 0, \quad \text{if } \|\tilde z\|_E = \rho$$

and this inequality holds for $z \in S$, according to the definition of S.

Step 4. We claim that (IQ) is satisfied, that is, there are constants $\sigma > 0$ and $M > 0$ such that $\Phi(z) \le 0$ for all $z \in \partial Q$, where Q is defined by (2.14). For $\tau \in \mathbb{R}^+$, $(u,v)\in E^-$, we take

$$z = \tau(\sigma^{\beta-1}u_+, \sigma^{\alpha-1}v_+) + (\sigma^{\beta-1}u, \sigma^{\alpha-1}v). \tag{2.22}$$

From (2.3), by the definitions of E^+ and E^-, one has

$$v_+ = A^{-t}A^s u_+, \quad v = -A^{-t}A^s u. \tag{2.23}$$

Then, from (2.22) and (2.23) we obtain

$$\begin{aligned}\mathcal{Q}(z) &= \int_\Omega (\tau\sigma^{\beta-1}A^s u_+ + \sigma^{\beta-1}A^s u)(\tau\sigma^{\alpha-1}A^s u_+ - \sigma^{\alpha-1}A^s u)dx\\ &= \sigma^{\alpha+\beta-2}\int_\Omega (\tau A^s u_+ + A^s u)(\tau A^s u_+ - A^s u)dx\\ &= \frac{1}{2}\sigma^{\alpha+\beta-2}(\tau^2 - \|(u,v)\|_E^2).\end{aligned} \tag{2.24}$$

By hypothesis (H1), we see that for $\tau = 0$,

$$\Phi(z) \le 0. \tag{2.25}$$

It follows from (H2) that there are constants $c_5 > 0$ and $c_6 > 0$ such that

$$H(u,v,x) \ge c_5(|u|^\alpha + |v|^\beta) - c_6.$$

So, we have

$$\int_\Omega H(z,x)dx \ge c_5\int_\Omega (\sigma^{\alpha(\beta-1)}|\tau u_+ + u|^\alpha + \sigma^{\beta(\alpha-1)}|\tau v_+ + v|^\beta)dx - c_6|\Omega|. \tag{2.26}$$

Now, every u can be decomposed as $u = \gamma u_+ + \hat{u}$, where $\hat{u}$ is orthogonal to u^+ in $L^2(\Omega)$, and $\gamma \in \mathbb{R}$. We obtain from Hölder's inequality that

$$(\tau + \gamma) \int_\Omega |u^+|^2 dx = \int_\Omega (\tau u^+ + u) u^+ dx \le \|\tau u^+ + u\|_{L^\alpha(\Omega)} \|u^+\|_{L^{\alpha'}(\Omega)}.$$

Hence, for some constant $c_7 > 0$, we get

$$\tau + \gamma \le c_7 \|\tau u^+ + u\|_{L^\alpha(\Omega)}. \tag{2.27}$$

Similarly, we obtain

$$\tau - \gamma \le c_7 \|\tau v^+ + v\|_{L^\beta(\Omega)}. \tag{2.28}$$

If $\gamma \ge 0$, we get from (2.24), (2.26) and (2.27) that

$$\Phi(z) \le \frac{1}{2} \sigma^{\alpha+\beta-2} \tau^2 - c_8 \tau^\alpha \sigma^{\alpha(\beta-1)} + c_6 |\Omega| \tag{2.29}$$

for some positive constant c_8. And, if $\gamma \le 0$, we conclude from (2.24), (2.26) and (2.28) that

$$\Phi(z) \le \frac{1}{2} \sigma^{\alpha+\beta-2} \tau^2 - c_8 \tau^\beta \sigma^{\beta(\alpha-1)} + c_6 |\Omega|. \tag{2.30}$$

Choosing $\tau = \sigma$, and taking σ large enough it follows from $1/\alpha + 1/\beta < 1$, (2.29) and (2.30) that

$$\Phi(z) \le 0. \tag{2.31}$$

Finally, we choose M. Given $\tau \in (0, \sigma)$, we deduce from (2.24) and (2.26) that

$$\Phi(z) \le \frac{1}{2} \sigma^{\alpha+\beta} - \frac{1}{2} \sigma^{\alpha+\beta-2} \|(u,v)\|_E^2 + c_6 |\Omega|.$$

So that if M is enough large and $\|(u,v)\|_E^2 = M$, one has

$$\Phi(z) \le 0. \tag{2.32}$$

Thus, from (2.25), (2.31) and (2.32), we have

$$\Phi(z) \le 0, \quad \forall z \in \partial Q.$$

Hence, the hypothesis of Theorem 2.1 is satisfied. Thus, there exists $z \in E$ such that $\Phi'(z) = 0$; i.e., z is an (s,t)-weak solution of (1.1). Next, [8, Theorem 1.2] gives that $z = (u,v)$ is such that $u \in W^{2,p/(p-1)}(\Omega) \bigcap W_0^{1,p/(p-1)}(\Omega)$ and $v \in W^{2,q/(q-1)}(\Omega) \bigcap W_0^{1,q/(q-1)}(\Omega)$. That is, (u,v) is a strong solution of (1.1).

Moreover, $(0,0)$ is a solution of (1.1). Since $\Phi(z) \ge \delta > 0$ and $\Phi(0,0) = 0$, it implies that (u,v) is not trivial. □

Proof of Theorem 1.3. Here, we define the functional $\widehat{\Phi} : E \to \mathbb{R}$ as

$$\widehat{\Phi}(z) = \mathcal{Q}(z) - \int_\Omega \widehat{H}(z,x) dx,$$

where

$$\widehat{H}(u,v,x) = \begin{cases} H(u,v,x), & \text{if } u \ge 0, v \ge 0, \\ H(0,v,x), & \text{if } u \le 0, v \ge 0, \\ H(u,0,x), & \text{if } u \ge 0, v \le 0, \\ 0, & \text{if } u \le 0, v \le 0. \end{cases}$$

From (H6), $\widehat{H}$ is of class $C^{1,\varepsilon}$. And, $\widehat{H}$ satisfies (H1), (H3) and (H4). Moreover, (H2) is satisfied in a restricted form. Obviously, the critical points of $\widehat{\Phi}$ correspond to the strong solutions of

$$\begin{gathered} -\Delta u = \widehat{H}_v(u,v,x), \quad \text{in } \Omega, \\ -\Delta v = \widehat{H}_u(u,v,x), \quad \text{in } \Omega, \\ u = 0, \quad v = 0, \quad \text{on } \partial\Omega. \end{gathered}$$

Since $\widehat{H}_u(u,v,x) \geq 0$ and $\widehat{H}_v(u,v,x) \geq 0$, by the maximum principle, we obtain that $u > 0$ and $v > 0$ in Ω. As the proof of Theorem 1.1, we can get that hypotheses of Theorem 2.1 still hold. Hence, (1.1) possesses at least one positive solution (u,v) with $u(x) > 0, v(x) > 0$ if $x \in \Omega$. □

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References

[1] V. Benci, P. Rabinowitz; *Critical point theorems for indefinite functionals*, Invent. Math. 52 (1979), 241-273.

[2] D. Bonheure, E. Moreira dos Santos, M. Ramos; *Ground state and non-ground state solutions of some strongly coupled elliptic systems*, Trans. Amer. Math. Soc. 364 (2012), no. 1, 447-491.

[3] S.-J. Chen, C.-L. Tang; *Periodic and subharmonic solutions of a class of superquadratic Hamiltonian systems*, J. Math. Anal. Appl. 297 (2004) 267-284.

[4] Ph. Clément, D. G. De Figueiredo, E. Mitidieri; *Positive solutions of semilinear elliptic systems*, Comm. Partial Differential Equations 17 (1992), 923-940.

[5] Ph. Clément, P. Felmer, E. Mitidieri; *Homoclinic orbits for a class of infinite-dimensional Hamiltonian systems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 24 (1997), no. 2, 367-393.

[6] D. G. De Figueiredo; *Semilinear elliptic systems: existence, multiplicity, symmetry of solutions*, Handbook of differential equations: stationary partial differential equations. Vol. V, 1-48, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008.

[7] D. G. De Figueiredo, Y. H. Ding; *Strongly indefinite functionals and multiple solutions of elliptic systems*, Trans. Amer. Math. Soc. 355 (2003), 2973-2989.

[8] D. G. De Figueiredo, P. L. Felmer; *On Superquadratic elliptic systems*, Trans. Amer. Math. Soc. 343 (1994), 99-116.

[9] P. L. Felmer; *Periodic solutions of superquadratic Hamiltonian systems*, J. Differential Equations 102 (1993) 188-207.

[10] P. L. Felmer, Z. Q. Wang; *Multiplicity for symmetric indefinite functionals: Application to Hamiltonian and elliptic systems*, Topol. Methods Nonlinear Anal. 12 (2) (1998), 207-226.

[11] J. Hulshof, R. C. A. M. van der Vorst; *Differential systems with strongly indefinite variational structure*, J. Funct. Anal. 114 (1993), 32-58.

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