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SOLUTIONS IN SEVERAL TYPES OF PERIODICITY FOR PARTIAL NEUTRAL INTEGRO-DIFFERENTIAL EQUATION

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Abstract. In this article we study the existence of mild solutions in several types of periodicity for partial neutral integro-differential equations with unbounded delays.

1. Introduction

In this article we study the existence of several types of mild solutions for the partial neutral integro-differential equation

$$\frac{d}{dt}(x(t)+f(t,x_t)) = Ax(t) + \int_0^t B(t-s)x(s)ds + g(t,x_t), \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

where $A : D(A) \subset X \to X$ and $B(t) : D(B(t)) \subset X \to X$, $t \geq 0$, are closed linear operators; $(X, \|\cdot\|)$ is a Banach space; the history $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to an abstract phase space $\mathcal{B}$ defined axiomatically, and $f, g : I \times \mathcal{B} \to X$ are appropriated functions.

The literature relative to ordinary neutral differential equations is very extensive, thus we suggest the Hale and Lunel book [20] concerning this matter. Referring to partial neutral functional differential equations, we cite the pioneer articles Hale [19] and Wu [37, 38, 39] for finite delay equations, Hernández and Henriquez [28, 29], Hernández [25] for the unbounded delay, Hernández and dos Santos [27] and Henríquez et al. [21, 24] and Dos Santos et al. [14, 16, 15] for partial neutral integro-differential equations with unbounded delay.

The existence of almost automorphic, asymptotically almost automorphic, almost periodic, asymptotically almost periodic, S-asymptotically ω-periodic and asymptotically ω-periodic solutions to differential equations is among the most attractive topics in mathematical analysis due to their possible applications in areas such as physics, economics, mathematical biology, engineering, etc. (cf. [1, 2, 3, 4, 5, 8, 10, 11, 12, 13, 16, 17, 23, 26, 33, 34, 41, 42, 43]). The concept of asymptotically almost automorphic, was introduced in the literature in the early

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eighties by N'Guérékata [32]. However, the literature concerning S-asymptotically ω-periodic functions with values in Banach spaces is recent (cf [4, 6, 7, 22, 23]). The existence of asymptotically almost automorpic, S-asymptotically ω-periodic functions and asymptotically ω-periodic for the partial neutral system (1.1)-(1.2) is an untreated topic in the literature and this fact is the main motivation of the present work.

This paper is organized in four sections. In Section 2 we mention a few results and notations related with resolvent of operators and of several types of periodicity. In Section 3 we study the existence of several types of periodicity mild solutions to the partial neutral system (1.1)-(1.2). In Section 4, we discuss the existence and uniqueness of several types of periodicity solution to a concrete partial neutral integro-differential equation with delay, as an illustration to our abstract results.

2. Preliminaries

Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. We denote by $\mathcal{L}(Z, W)$ the space of bounded linear operators from Z into W endowed with norm of operators, and we write simply $\mathcal{L}(Z)$ when $Z = W$. By $\mathbf{R}(Q)$ we denote the range of a map Q and for a closed linear operator $P : D(P) \subseteq Z \to W$, the notation $[D(P)]$ represents the domain of P endowed with the graph norm, $\|z\|_1 = \|z\|_Z + \|Pz\|_W$, $z \in D(P)$. In the case $Z = W$, the notation $\rho(P)$ stands for the resolvent set of P, and $R(\lambda, P) = (\lambda I - P)^{-1}$ is the resolvent operator of P. Furthermore, for appropriate functions $K : [0, \infty) \to Z$ and $S : [0, \infty) \to \mathcal{L}(Z, W)$, the notation $\widehat{K}$ denotes the Laplace transform of K, and $S * K$ the convolution between S and K, which is defined by $S * K(t) = \int_0^t S(t-s)K(s)ds$. The notation, $B_r(x, Z)$ stands for the closed ball with center at x and radius $r > 0$ in Z. As usual, $C_0([0, \infty), Z)$ represents the sub-space of $C_b([0, \infty), Z)$ formed by the functions which vanish at infinity and $C_\omega([0, \infty), X)$ denote the spaces $C_\omega([0, \infty), X) = \{x \in C_b([0, \infty), X) : x \text{ is } \omega\text{-periodic }\}$. If $k : \mathbb{R} \to W$, we denote $\|k\|_{W,\infty} = \sup_{s \in \mathbb{R}} \|k(s)\|_W$ or if $k : [0, \infty) \to W$, we denote $\|k\|_{W,\infty} = \sup_{s \in [0,\infty)} \|k(s)\|_W$.

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ similar at those in [30]. More precisely, $\mathcal{B}$ will denote a vector space of functions defined from $(-\infty, 0]$ into X endowed with a semi-norm denoted by $\|\cdot\|_{\mathcal{B}}$ and such that the following axioms hold:

- (A1) If $x : (-\infty, \sigma + b) \to X$ with $b > 0$ is continuous on $[\sigma, \sigma + b)$ and $x_\sigma \in \mathcal{B}$, then for each $t \in [\sigma, \sigma + b)$ the following conditions hold:
  - (i) x_t is in $\mathcal{B}$,
  - (ii) $\|x(t)\| \le H \|x_t\|_{\mathcal{B}}$,
  - (iii) $\|x_t\|_{\mathcal{B}} \le K(t - \sigma) \sup\{\|x(s)\| : \sigma \le s \le t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$,

  where $H > 0$ is a constant, and $K, M : [0, \infty) \mapsto [1, \infty)$ are functions such that $K(\cdot)$ and $M(\cdot)$ are respectively continuous and locally bounded, and H, K, M are independent of $x(\cdot)$.
- (A2) If $x(\cdot)$ is a function as in (A1), then x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + b)$.
- (B1) The space $\mathcal{B}$ is complete.
- (C1) If $(\varphi^n)_{n \in \mathbb{N}}$ is a sequence in $C_b((-\infty, 0], X)$ formed by functions with compact support such that $\varphi^n \to \varphi$ uniformly on compact, then $\varphi \in \mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$ as $n \to \infty$.

Definition 2.1. Let $S(t) : \mathcal{B} \to \mathcal{B}$ be the C_0-semigroup defined by $S(t)\varphi(\theta) = \varphi(0)$ on $[-t, 0]$ and $S(t)\varphi(\theta) = \varphi(t+\theta)$ on $(-\infty, -t]$. The phase space $\mathcal{B}$ is called a fading memory if $\|S(t)\varphi\|_{\mathcal{B}} \to 0$ as $t \to \infty$ for each $\varphi \in \mathcal{B}$ with $\varphi(0) = 0$.

Remark 2.2. In this work we assume there exists positive $\mathfrak{K}$ such that

$$\max\{K(t), M(t)\} \le \mathfrak{K}$$

for each $t \ge 0$. Observe that this condition is verified, for example, if $\mathcal{B}$ is a fading memory, see [30, Proposition 7.1.5].

Example 2.3. The phase space $C_r \times L^p(\rho, X)$. Let $r \ge 0$, $1 \le p < \infty$ and let $\rho : (-\infty, -r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [30]. Briefly, this means that ρ is locally integrable and there exists a non-negative, locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \le \gamma(\xi)\rho(\theta)$, for all $\xi \le 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(\rho, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $\rho\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(\rho, X)$ is defined by

$$\|\varphi\|_{\mathcal{B}} := \sup\{\|\varphi(\theta)\| : -r \le \theta \le 0\} + \Big(\int_{-\infty}^{-r} \rho(\theta)\|\varphi(\theta)\|^p d\theta\Big)^{1/p}.$$

The space $\mathcal{B} = C_r \times L^p(\rho; X)$ satisfies axioms (A1), (A2), (B1). Moreover, when $r = 0$ and $p = 2$, we can take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + (\int_{-t}^{0} \rho(\theta)\, d\theta)^{1/2}$, for $t \ge 0$ and

$$\mathfrak{K} = \Big(\sup_{s \le 0} |\gamma(s)^{1/2}| + \Big(1 + (\int_{-\infty}^{0} \rho(\theta) d\theta)^{1/2}\Big)\Big).$$

See [30, Theorem 1.3.8] for details.

For better comprehension of the subject we shall introduce the following definitions, hypothesis and results. Throughout the rest of the paper we always assume that the abstract integro-differential problem

$$\frac{dx(t)}{dt} = Ax(t) + \int_0^t B(t-s)x(s)\, ds, \tag{2.1}$$

$$x(0) = x \in X. \tag{2.2}$$

Definition 2.4. A one-parameter family of bounded linear operators $(\mathcal{R}(t))_{t \ge 0}$ on X is called a resolvent operator of (2.1)-(2.2) if the following conditions are satisifed.

(a) Function $\mathcal{R}(\cdot) : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous and $\mathcal{R}(0)x = x$ for all $x \in X$.
(b) For $x \in D(A)$, $\mathcal{R}(\cdot)x \in C([0, \infty), [D(A)]) \cap C^1([0, \infty), X)$, and

$$\frac{d\mathcal{R}(t)x}{dt} = A\mathcal{R}(t)x + \int_0^t B(t-s)\mathcal{R}(s)x ds, \tag{2.3}$$

$$\frac{d\mathcal{R}(t)x}{dt} = \mathcal{R}(t)Ax + \int_0^t \mathcal{R}(t-s)B(s)x ds, \tag{2.4}$$

for every $t \ge 0$,
(c) There exists constants $M > 0, \delta$ such that $\|\mathcal{R}(t)\| \le Me^{\delta t}$ for every $t \ge 0$.

Definition 2.5. A resolvent operator $(\mathcal{R}(t))_{t\geq 0}$ of (2.1)-(2.2) is called exponentially stable if there exists positive constants M, β such that $\|\mathcal{R}(t)\| \leq Me^{-\beta t}$.

In this work we assume that the following conditions are satisfied:

- (H1) Operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X, and there are constants $M_0 > 0, \omega \in \mathbb{R}$ and $\vartheta \in (\pi/2, \pi)$ such that $\rho(A) \supseteq \Lambda_{\omega,\vartheta} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, \, |\arg(\lambda - \omega)| < \vartheta\}$ and $\|R(\lambda, A)\| \leq \dfrac{M_0}{|\lambda - \omega|}$ for all $\lambda \in \Lambda_{\omega,\vartheta}$.
- (H2) For all $t \geq 0$, $B(t) : D(B(t)) \subseteq X \to X$ is a closed linear operator, $D(A) \subseteq D(B(t))$ and $B(\cdot)x$ is strongly measurable on $(0, \infty)$ for each $x \in D(A)$. There exists $b(\cdot) \in L^1([0, \infty))$ such that $\widehat{b}(\lambda)$ exists for $\mathrm{Re}(\lambda) > 0$ and $\|B(t)x\| \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\widehat{B} : \Lambda_{\omega,\pi/2} \to \mathcal{L}([D(A)], X)$ has an analytical extension (still denoted by $\widehat{B}$) to $\Lambda_{\omega,\vartheta}$ such that $\|\widehat{B}(\lambda)x\| \leq \|\widehat{B}(\lambda)\| \, \|x\|_1$ for all $x \in D(A)$, and $\|\widehat{B}(\lambda)\| = O(\frac{1}{|\lambda|})$ as $|\lambda| \to \infty$.
- (H3) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and positive constants C_i, $i = 1, 2$, such that $A(D) \subseteq D(A)$, $\widehat{B}(\lambda)(D) \subseteq D(A)$, $\|A\widehat{B}(\lambda)x\| \leq C_1\|x\|$ for every $x \in D$ and all $\lambda \in \Lambda_{\omega,\vartheta}$.

For $r > 0$, $\theta \in (\frac{\pi}{2}, \vartheta)$ and $w \in \mathbb{R}$, set

$$\Lambda_{r,\omega,\theta} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\lambda| > r, \, |\arg(\lambda - \omega)| < \theta\},$$

and $\omega + \Gamma^i_{r,\theta}$, $i = 1, 2, 3$, the paths

$$\begin{gathered}
\omega + \Gamma^1_{r,\theta} = \{\omega + te^{i\theta} : t \geq r\}, \\
\omega + \Gamma^2_{r,\theta} = \{\omega + re^{i\xi} : -\theta \leq \xi \leq \theta\}, \\
\omega + \Gamma^3_{r,\theta} = \{\omega + te^{-i\theta} : t \geq r\},
\end{gathered}$$

with $\omega + \Gamma_{r,\theta} = \bigcup_{i=1}^3 \omega + \Gamma^i_{r,\theta}$ oriented counterclockwise. In addition, $\Psi(G)$ is the set

$$\Psi(G) = \{\lambda \in \mathbb{C} : G(\lambda) := (\lambda I - A - \widehat{B}(\lambda))^{-1} \in \mathcal{L}(X)\}.$$

The next results establish that the operator family $(\mathcal{R}(t))_{t\geq 0}$ defined by

$$\mathcal{R}(t) = \begin{cases} \frac{1}{2\pi i} \int_{\omega+\Gamma_{r,\theta}} e^{\lambda t} G(\lambda) d\lambda, & t > 0, \\ I, & t = 0. \end{cases} \tag{2.5}$$

is an exponentially stable resolvent operator for (2.1)-(2.2).

Theorem 2.6 ([16, Corollary 3.1])**.** *Suppose that conditions* (H1)–(H3) *are satisfied. Then, the function* $\mathcal{R}(\cdot)$ *is a resolvent operator for system* (2.1)-(2.2)*. If* $\omega + r < 0$*, the function* $\mathcal{R}(\cdot)$ *is an exponentially stable resolvent operator for system* (2.1)-(2.2)*.*

In the next result we denote by $(-A)^\vartheta$ the fractional power of the operator $(-A)$, (see [35] for details).

Theorem 2.7 ([16, Corollary 3.2])**.** *Suppose that conditions* (H1)–(H3) *are satisfied. Then there exists a positive number* C *such that*

$$\|(-A)^\vartheta \mathcal{R}(t)\| \leq \begin{cases} Ce^{(r+\omega)t}, & t \geq 1, \\ Ce^{(r+\omega)t} t^{-\vartheta}, & t \in (0, 1), \end{cases} \tag{2.6}$$

for all $\vartheta \in (0,1)$. *If* $\omega + r < 0$ *and* $\vartheta \in (0,1)$*, then there exists* $\phi \in L^1([0,\infty))$ *such that*

$$\|(-A)^{\vartheta}\mathcal{R}(t)\| \le \phi(t). \tag{2.7}$$

In the remaining of this section we discuss the existence of solutions to

$$\frac{dx(t)}{dt} = Ax(t) + \int_0^t B(t-s)x(s)\,ds + f(t), \quad t \in [0,a], \tag{2.8}$$

$$x(0) = z \in X, \tag{2.9}$$

where $f \in L^1([0,a],X)$. In the sequel, $\mathcal{R}(\cdot)$ is the operator function defined by (2.5). We begin by introducing the following concept of classical solution.

Definition 2.8. A function $x : [0,b] \to X$, $0 < b \le a$, is called a classical solution of (2.8)-(2.9) on $[0,b]$ if $x \in C([0,b],[D(A)]) \cap C^1((0,b],X)$, the condition (2.9) holds and the equation (2.8) is satisfied on $[0,a]$.

Theorem 2.9 ([18, Theorem 2])**.** *Let* $z \in X$*. Assume that* $f \in C([0,a],X)$ *and* $x(\cdot)$ *is a classical solution of* (2.8)-(2.9) *on* $[0,a]$*. Then*

$$x(t) = \mathcal{R}(t)z + \int_0^t \mathcal{R}(t-s)f(s)\,ds, \quad t \in [0,a]. \tag{2.10}$$

Motivated by (2.10), we introduce the following concept.

Definition 2.10. A function $u \in C([0,a],X)$ is called a mild solution of (2.8)-(2.9) if

$$u(t) = \mathcal{R}(t)z + \int_0^t \mathcal{R}(t-s)f(s)\,ds, \quad t \in [0,a].$$

To establish our existence result, motivated by the previous facts, we introduce the following assumptions.

- (P1) There exists a Banach space $(Y, \|\cdot\|_Y)$ continuously included in X such that the following conditions are verified.
  - (a) For every $t \in (0,\infty)$, $\mathcal{R}(t) \in \mathcal{L}(X) \cap \mathcal{L}(Y,[D(A)])$ and $B(t) \in \mathcal{L}(Y,X)$. In addition, $A\mathcal{R}(\cdot)x, B(\cdot)x \in C((0,\infty),X)$ for every $x \in Y$.
  - (b) There are positive constants M, β such that
  $$\|\mathcal{R}(s)\| \le Me^{-\beta s}, \quad s \ge 0.$$
  - (c) There exists $\phi \in L^1([0,\infty))$ such that $\|A\mathcal{R}(t)\|_{\mathcal{L}(Y,X)} \le \phi(t)$, $t \ge 0$.
- (PF) $f : \mathbb{R} \times \mathcal{B} \to Y$ is a continuous function and there exists a continuous non decreasing function $L_f : [0,\infty) \to [0,\infty)$, such that
$$\|f(t,\psi_1) - f(t,\psi_2)\|_Y \le L_f(r)\|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad (t,\psi_j) \in \mathbb{R} \times B_r(0,\mathcal{B}).$$
- (PG) $g : \mathbb{R} \times \mathcal{B} \to X$ is a continuous function and there exists a continuous and non decreasing function $L_g : [0,\infty) \to [0,\infty)$ such that
$$\|g(t,\psi_1) - g(t,\psi_2)\| \le L_g(r)\|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad (t,\psi_j) \in \mathbb{R} \times B_r(0,\mathcal{B}).$$
- (P2)
$$\begin{aligned}&\sup_{r>0}\Big[\frac{r}{2\mathfrak{K}} - L_f(2\mathfrak{K}r)r\mu - \frac{M}{\beta}L_g(2\mathfrak{K}r)r\Big]\\ &\ge \frac{1}{2\mathfrak{K}}(M\|\varphi\|_{\mathcal{B}} + M\|f(0,\varphi)\| + \sup_{t\in[0,\infty)}\|f(t,0)\|_Y\mu + \frac{M}{\beta}\sup_{t\in[0,\infty)}\|g(t,0)\|),\end{aligned}$$

where $\mu = (\|i_c\|_{\mathcal{L}(Y,X)} + \|\phi\|_{L^1} + \frac{M}{\beta}\|b\|_{L^1})$.

Motivated by the theory of resolvent operator, we introduce the following concept of mild solution for (1.1)-(1.2).

Definition 2.11. A function $u : (-\infty, b] \to X$, $0 < b \leq a$, is called a mild solution of (1.1)-(1.2) on $[0,b]$, if $u_0 = \varphi \in \mathcal{B}$; $u|_{[0,b]} \in C([0,b] : X)$; the functions $\tau \mapsto A\mathcal{R}(t-\tau)f(\tau, u_\tau)$ and $\tau \mapsto \int_0^\tau B(\tau-\xi)f(\xi, u_\xi)d\xi$ are integrable on $[0,t)$ for every $t \in (0,b]$ and

$$\begin{aligned} u(t) &= \mathcal{R}(t)(\varphi(0) + f(0,\varphi)) - f(t,u_t) - \int_0^t A\mathcal{R}(t-s)f(s,u_s)ds \\ &\quad - \int_0^t \mathcal{R}(t-s)\int_0^s B(s-\xi)f(\xi,u_\xi)d\xi ds + \int_0^t \mathcal{R}(t-s)g(s,u_s)ds, \quad t \in [0,b]. \end{aligned}$$

Now, we need to introduce some concepts, definitions and technicalities on asymptotically almost periodical functions, S-asymptotically ω-periodic, asymptotically ω-periodic asymptotically and almost automorphic functions.

Definition 2.12. A function $f \in C(\mathbb{R}, Z)$ is almost periodic (a.p.) if for every $\varepsilon > 0$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{H}(\varepsilon, f, Z)$, such that

$$\|f(t+\xi) - f(t)\|_Z < \varepsilon, \quad t \in \mathbb{R},\ \xi \in \mathcal{H}(\varepsilon, f, Z).$$

Definition 2.13. A function $f \in C([0,\infty), Z)$ is asymptotically almost periodic (a.a.p.) if there exists an almost periodic function $g(\cdot)$ and $w \in C_0([0,\infty), Z)$ such that $f(\cdot) = g(\cdot) + w(\cdot)$.

In this paper, $AP(Z)$ and $AAP(Z)$ are the spaces

$$\begin{gathered} AP(Z) = \{f \in C(\mathbb{R}, Z) : f \text{ is a.p. }\}, \\ AAP(Z) = \{f \in C([0,\infty), Z) : f \text{ is a.a.p. }\}, \end{gathered}$$

endowed with the norm of the uniform convergence. We know from the result in [40] that $AP(Z)$ and $AAP(Z)$ are Banach spaces.

Definition 2.14. A function $u \in C_b([0,\infty), X)$ is said S-asymptotically ω-periodic if

$$\lim_{t\to\infty} (u(t+\omega) - u(t)) = 0.$$

In the rest of this paper, the notation $SAP_\omega(X)$ stands for the space

$$SAP_\omega(X) = \{f \in C_b(\mathbb{R}, X) : f \text{ is } S\text{-asymptotically } \omega\text{-periodic }\},$$

endowed with the norm of the uniform convergence. It is clear that $SAP_\omega(X)$ is a Banach space.

Definition 2.15. A continuous function $f : [0,\infty) \times Z \to W$ is said uniformly S-asymptotically ω-periodic on bounded sets if $f(\cdot, x)$ is bounded for each $x \in Z$, and for every $\varepsilon > 0$ and for all bounded set $K \subseteq Z$, there exists $L(K,\varepsilon) \geq 0$ such that $\|f(t,x) - f(t+\omega, x)\|_W \leq \varepsilon$ for every $t \geq L(K,\varepsilon)$ and all $x \in K$.

Definition 2.16. A continuous function $f : [0,\infty) \times Z \to W$ is said asymptotically uniformly continuous on bounded sets, if for every $\varepsilon > 0$ and for all bounded set $K \subseteq Z$ there exist constants $L(K,\varepsilon) \geq 0$ and $\delta = \delta(K,\varepsilon) > 0$ such that $\|f(t,x) - f(t,y)\|_W \leq \varepsilon$ for all $t \geq L(K,\varepsilon)$ and every $x, y \in K$ with $\|x-y\|_Z \leq \delta$.

Lemma 2.17 ([22, Lemma 4.1])**.** *Assume that $f : [0, \infty) \times Z \to W$ is a function uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Let $u \in SAP_\omega(Z)$, then the function $\theta : \mathbb{R} \to W$ defined by $\theta(t) = f(t, u(t))$ is S-asymptotically ω-periodic.*

By using a similar procedure to the proof of the [23, Lemma 3.5], we prove the next result.

Lemma 2.18. *Suppose that condition* (P1)(b) *holds and $f \in SAP_\omega(X)$. Let $F : [0, \infty) \to X$ be the function defined by*

$$F(t) := \int_0^t \mathcal{R}(t-s)f(s)ds.$$

Then $F \in SAP_\omega(X)$.

Lemma 2.19 ([23, Lemma 2.10])**.** *Assume that $\mathcal{B}$ is a fading memory space and $u \in C(\mathbb{R}, X)$ is such that $u_0 \in \mathcal{B}$ and $u|_{[0,\infty)} \in SAP_\omega(X)$, then $t \mapsto u_t \in SAP_\omega(\mathcal{B})$.*

Definition 2.20. A function $u \in C_b([0, \infty), X)$ is called asymptotically ω-periodic if there exists an ω-periodic function v and $w \in C_0([0, \infty), X)$ such that $u = v + w$.

Remark 2.21. In [23] the authors have shown that the set of the asymptotically ω-periodic functions is properly contained in $SAP_\omega(W)$.

Lemma 2.22 ([23, Remark 3.13])**.** *If $u \in C_b([0, \infty), X)$ is a function such that $\lim_{t\to\infty}(u(t + n\omega) - u(t)) = 0$, uniformly for $n \in \mathbb{N}$, then $u(\cdot)$ is asymptotically ω-periodic.*

In the rest of this paper, $S_\omega(X)$ stands for the space

$$S_\omega(X) = \{f \in C_b([0, \infty), X) : \lim_{t\to\infty} f(t + n\omega) - f(t) = 0, \text{ uniformly for } n \in \mathbb{N}\},$$

endowed with the norm of the uniform convergence.

Lemma 2.23 ([4, Lemma 2.3])**.** *Let $f : [0, \infty)\times Z \to W$ be asymptotically uniformly continuous on bounded sets. Suppose that for all bounded subset $K \subset Z$, the set $\{f(t, z) \geq 0, z \in K\}$ is bounded and $\lim_{t\to\infty} \|f(t + n\omega, z) - f(t, z)\| = 0$, uniformly for $z \in K$ and $n \in \mathbb{N}$. If $u \in S_\omega(Z)$, then $f(\cdot, u(\cdot)) \in S_\omega(W)$.*

Lemma 2.24. [4, Lemma 3.7] *Suppose that condition* (P1)(b) *holds and $f \in S_\omega(X)$. If F is the function defined by $F(t) := \int_0^t \mathcal{R}(t - s)f(s)ds$, $t \geq 0$, then $F \in S_\omega(X)$.*

We now introduce some notion of asymptotically almost automorphic.

Definition 2.25. A function $f \in C(\mathbb{R}, X)$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t - s_n)$$

for all $t \in \mathbb{R}$.

It is well known that the range of an almost automorphic function is relatively compact on X, and hence it is bounded. Moreover, the space of all almost automorphic functions, denoted by $AA(X)$, endowed with the norm of the uniform convergence is a Banach space [33].

Definition 2.26. A function $f \in C([0,\infty), Z)$ is said to be asymptotically almost automorphic if it can be written as $f = g + h$ where $g \in AA(Z)$ and $h \in C_0([0,\infty), Z)$. Denote by $AAA(Z)$ the set of all such functions.

Definition 2.27. A function $f \in C(\mathbb{R}, Z)$ is said to be compact almost automorphic if for every sequence of real numbers $(\sigma_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (\sigma_n)_{n\in\mathbb{N}}$ such that

$$\begin{aligned} g(t) &:= \lim_{n\to\infty} f(t+s_n), \\ f(t) &= \lim_{n\to\infty} g(t-s_n) \end{aligned}$$

uniformly on compact subsets of $\mathbb{R}$. The collection of those functions will be denoted by $AA_c(Z)$.

Definition 2.28. A function $f \in C(\mathbb{R} \times Z, W)$ is said to be compact almost automorphic in $t \in \mathbb{R}$, if for every sequence of real numbers $(\sigma_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (\sigma_n)_{n\in\mathbb{N}}$ such that

$$\begin{aligned} g(t,z) &:= \lim_{n\to\infty} f(t+s_n, z), \\ f(t,z) &= \lim_{n\to\infty} g(t-s_n, z), \end{aligned}$$

where the limits are uniform on compact subset of $\mathbb{R}$, for each $z \in Z$. The space of such functions will be denoted by $AA_c(Z, W)$.

Definition 2.29. A continuous function $f \in C([0,\infty), Z)$ is said to be compact asymptotically almost automorphic if it can be written as $f = g + h$ where $g \in AA_c(Z)$ and $h \in C_0(\mathbb{R}^+, Z)$. Denote by $AAA_c(Z)$ the set of all such functions.

Definition 2.30. Let $K \subset Z$ and $I \subset \mathbb{R}$. Let $C_K(I \times Z, W)$ denote the collection of functions $f : I \times Z \to W$ such that $f(t,\cdot)$ is uniformly continuous on K for every $t \in I \subseteq \mathbb{R}$.

Definition 2.31. A function $f \in C([0,\infty) \times Z, W)$ is said to be compact asymptotically almost automorphic if it can be written as $f = g + h$, where $g \in AA_c(Z, W)$ and $h \in C_0([0,\infty) \times Z, W)$. Denote by $AAA_c(Z, W)$ the set of all such functions.

Lemma 2.32 ([9, Lemma 3.3])**.** *Let $u \in AAA_c(Z)$ and $f \in AAA_c(Z, W) \cap C_R(\mathbb{R} \times Z, W)$, where $R = \overline{\{u(t) : t \in \mathbb{R}\}}$. Then the function $\Phi : \mathbb{R} \to W$ defined by $\Phi(t) = f(t, u(t)) \in AAA_c(W)$.*

Lemma 2.33 ([9, Lemma 3.4])**.** *Suppose that condition (P1)-(b) holds and $f \in AAA_c(X)$. If F is the function defined by*

$$F(t) := \int_0^t \mathcal{R}(t-s) f(s) ds, \quad t \geq 0,$$

then $F \in AAA_c(X)$.

Lemma 2.34 ([9, Lemma 3.5])**.** *If $u \in AA_c(X)$, then the function $s \mapsto u_s$ belongs to $AA_c(\mathcal{B})$. Moreover, if $\mathcal{B}$ is a fading memory space and $u \in C(\mathbb{R}, X)$ is such that $u_0 \in \mathcal{B}$ and $u|_{[0,\infty)} \in AAA_c(X)$, then $t \mapsto u_t \in AAA_c(\mathcal{B})$.*

3. Several types of periodicity of mild solutions

In this section we establish the existence of several type of periodicity for solutions to partial neutral integro-differential equations system (1.1)-(1.2). For that, we need to introduce a few preliminaries and important results. Following, we consider the problem of the existence of compact asymptotically almost automorphic solutions.

In the following, we let $\mathcal{A}(Z)$ stands for one of the spaces $AAA_c(Z)$, $SAP_\omega(Z)$ or $S_\omega(Z)$.

Lemma 3.1. *Assume the condition* (P1) *is fulfilled. Let $u \in \mathcal{A}(Y)$ and $G(\cdot) : [0,\infty) \to X$ be the function defined by*

$$G(t) = \int_0^t \mathcal{R}(t-s) \int_0^s B(s-\tau)u(\tau)\ d\tau ds, \quad t \geq 0.$$

Then $G(\cdot) \in \mathcal{A}(X)$.

Proof. First we consider the $AAA_c(Y)$ case. By Lemma 2.33 is sufficient to prove that $H(t) = \int_0^t B(t-s)u(s)ds \in AAA_c(Y)$. Suppose $u = k + h$ where $k \in AA_c(Y)$ and $h \in C_0([0,\infty), Y)$. Then

$$\begin{aligned} H(t) &= \int_{-\infty}^t B(t-s)k(s)ds - \int_{-\infty}^0 B(t-s)k(s)ds + \int_0^t B(t-s)h(s)ds \\ &= w(t) + q(t), \end{aligned}$$

where

$$w(t) = \int_{-\infty}^t B(t-s)k(s)ds,$$
$$q(t) = \int_0^t B(t-s)h(s)ds - \int_{-\infty}^0 B(t-s)k(s)ds.$$

For a given sequence $(\sigma_n)_{n\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$, and a continuous functions $v \in C_b(\mathbb{R}, Y)$ such that $k(t+s_n)$ converges to $v(t)$ in Y, and $v(t-s_n)$ converges to $k(t)$ in Y, uniformly on compact sets of $\mathbb{R}$.

From the Bochner's criterion related to integrable functions and the estimate

$$\|B(t-s)k(s)\| = \|B(t-s)\|_{\mathcal{L}(Y,X)}\|k(s)\|_Y \leq b(t-s)\|k(s)\|_Y \tag{3.1}$$

it follows that the function $s \mapsto B(t-s)k(s)$ is integrable over $(-\infty, t)$ for each $t \in \mathbb{R}$. Furthermore, since

$$w(t+s_n) = \int_{-\infty}^t B(t-s)k(s+s_n)ds, \quad t \in \mathbb{R},\ n \in \mathbb{N},$$

using the estimate (3.1) and the Lebesgue Dominated Convergence Theorem, it follows that $w(t+s_n)$ converges to $z(t) = \int_{-\infty}^t B(t-s)v(s)ds$ for each $t \in \mathbb{R}$.

The remaining task consists of showing that the convergence is uniform on all compact subsets of $\mathbb{R}$ and that $q(\cdot) \in C_0([0,\infty), X)$. Let $K \subset \mathbb{R}$ be an arbitrary compact and let $\varepsilon > 0$. Since $h \in C_0([0,\infty), Y)$ and $k(\cdot) \in AA_c(Y)$, there exists a constant L and N_ε such that $K \subset [\frac{-L}{2}, \frac{L}{2}]$ with

$$\int_{\frac{L}{2}}^\infty b(s)ds < \varepsilon,$$

$$\|k(s+s_n)-v(s)\|_Y \le \varepsilon, \quad n \ge N_\varepsilon,\ s \in [-L,L],$$
$$\|h(s)\|_Y \le \varepsilon, \quad s \ge L.$$

For each $t \in K$, one has

$$\begin{aligned}
&\|w(t+s_n)-z(t)\| \\
&\le \int_{-\infty}^{t} \|B(t-s)\|_{\mathcal{L}(Y,X)} \|k(s+s_n)-v(s)\|_Y ds \\
&\le \int_{-\infty}^{-L} b(t-s)\|k(s+s_n)-v(s)\|_Y ds + \int_{-L}^{t} b(t-s)\|k(s+s_n)-v(s)\|_Y ds \\
&\le 2\|k\|_{Y,\infty} \int_{t+L}^{\infty} b(s)ds + \varepsilon \int_0^\infty b(s)ds \\
&\le 2\|k\|_{Y,\infty} \int_{\frac{L}{2}}^{\infty} b(s)ds + \varepsilon \int_0^\infty b(s)ds \\
&\le \varepsilon\Big(2\|k\|_{Y,\infty} + \int_0^\infty b(s)ds\Big),
\end{aligned}$$

which proves that the convergence is uniform on K, from the fact that the last estimate is independent of $t \in K$. Proceeding as previously, one can similarly prove that $z(t-s_n)$ converges to w uniformly on all compact subsets of $\mathbb{R}$. Next, let us show that $q(\cdot) \in C_0([0,\infty),X)$. For all $t \ge 2L$ we obtain

$$\begin{aligned}
\|q(t)\| &\le \int_{-\infty}^{0} \|B(t-s)\|_{\mathcal{L}(Y,X)} \|k(s)\|_Y ds + \int_0^t \|B(t-s)\|_{\mathcal{L}(Y,X)} \|h(s)\|_Y ds \\
&\le \int_{-\infty}^{0} b(t-s)\|k(s)\|_Y ds + \int_{t/2}^{t} b(t-s)\|h(s)\|_Y ds + \int_0^{t/2} b(t-s)\|h(s)\|_Y ds \\
&\le \int_{\frac{L}{2}}^{\infty} b(s)ds\|k\|_{Y,\infty} + \varepsilon \int_{t/2}^{t} b(s)ds + \int_{\frac{L}{2}}^{\infty} b(s)ds\|h\|_{Y,\infty} \\
&\le \varepsilon(\|k\|_{Y,\infty} + \int_0^\infty b(s)ds + \|h\|_{Y,\infty}).
\end{aligned}$$

Now we consider the $SAP_\omega(Y)$ case. From Lemma 2.18 is sufficient to prove that

$$H(t) = \int_0^t B(t-s)u(s)ds$$

is $SAP_\omega(X)$. For all $t \ge 0$,

$$\begin{aligned}
\|H(t)\| &\le \int_0^t \|B(t-s)\|_{\mathcal{L}(Y,X)} \|u(s)\|_Y d\tau \\
&\le \int_0^t b(t-s)\|u(s)\|_Y ds \\
&\le \|u\|_{Y,\infty} \int_0^\infty b(s)ds.
\end{aligned}$$

This shows that $H \in C_b([0,\infty),X)$. Furthermore, for $\omega \ge 0$, we have for $t \ge L > 0$,

$$\begin{aligned}
&\|H(t+\omega)-H(t)\| \\
&= \| \int_0^{t+\omega} B(t+\omega-s)u(s)ds - \int_0^t B(t-s)u(s)ds\|
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^\omega b(t+\omega-s)\|u(s)\|_Y ds + \|\int_0^t B(t-s)u(s+\omega)ds - \int_0^t B(t-s)u(s)ds\| \\
&\le \|u\|_{Y,\infty}\int_0^\omega b(t+\omega-s)ds + \int_0^t \|B(t-s)(u(s+\omega)-u(s))\|ds \\
&\le \|u\|_{Y,\infty}\int_0^\omega b(t+\omega-s)ds + \int_0^L b(t-s)\|u(s+\omega)-u(s)\|_Y ds \\
&\quad + \int_L^t b(t-s)\|u(s+\omega)-u(s)\|_Y ds.
\end{aligned}
$$

For all $\varepsilon > 0$, we choose L sufficiently large such that $\|u(s+\omega)-u(s)\|_Y < \varepsilon$ for all $s \ge L$ and $\int_L^\infty b(s)ds < \varepsilon$. Hence, for $t \ge 2L$ we obtain

$$
\begin{aligned}
\|H(t+\omega)-H(t)\| &\le \|u\|_{Y,\infty}\int_t^{t+\omega} b(s)ds + 2\|u\|_{Y,\infty}\int_{t-L}^t b(s)ds + \varepsilon\int_0^{t-L} b(s)ds \\
&\le \|u\|_{Y,\infty}\varepsilon + 2\|u\|_{Y,\infty}\varepsilon + \varepsilon\int_0^{t-L} b(s)ds \\
&\le \varepsilon\Big(3\|u\|_{Y,\infty} + \int_0^\infty b(s)ds\Big).
\end{aligned}
$$

Finally, let us prove the $S_\omega(Y)$ case. From the Lemma 2.24 is sufficient prove that $\lim_{t\to\infty} H(t+n\omega)-H(t) = 0$, uniformly in $n \in \mathbb{N}$, where $H(t) = \int_0^t B(t-s)u(s)ds$. For all $\varepsilon > 0$, we choose L sufficiently large such that $\|u(s+n\omega)-u(s)\|_Y < \varepsilon$ for all $s \ge L$ and $\int_L^\infty b(s)ds < \varepsilon$. Hence, for $t \ge 2L$ we obtain

$$
\begin{aligned}
&\|H(t+n\omega)-H(t)\| \\
&\le \|\int_0^{t+n\omega} B(t+n\omega-s)u(s)ds - \int_0^t B(t-s)u(s)ds\| \\
&\le \|u\|_{Y,\infty}\int_0^{n\omega} b(t+n\omega-s)ds + \int_0^L b(t-s)\|u(s+n\omega)-u(s)\|_Y ds \\
&+ \quad \int_L^t b(t-s)\|u(s+n\omega)-u(s)\|_Y ds \\
&\le \|u\|_{Y,\infty}\int_t^{t+n\omega} b(s)ds + 2\|u\|_{Y,\infty}\int_{t-L}^t b(s)ds + \varepsilon\int_0^\infty b(s)ds \\
&\le \varepsilon(3\|u\|_{Y,\infty} + \int_0^\infty b(s)ds).
\end{aligned}
$$

This completes the proof. ☐

Lemma 3.2. *Let condition* (P1)(c) *hold and u be a function in $\mathcal{A}(Y)$. If $I : [0,\infty) \to X$ is the function defined by $I(t) = \int_0^t A\mathcal{R}(t-s)u(s)ds$, then $I(\cdot) \in \mathcal{A}(X)$.*

Proof. All the $AAA_c(Y)$, $SAP_\omega(Y)$ and $S_\omega(Y)$ cases require small modifications in the proof of Lemma 3.1. ☐

Theorem 3.3. *Let $f \in AAA_c([0,\infty)\times\mathcal{B}, Y)$ and $g \in AAA_c([0,\infty)\times\mathcal{B}, X)$. Assume that $\mathcal{B}$ is a fading memory space and* (P1), (P2), (PF), (PG) *hold. Then there exists $\varepsilon > 0$ such that for each $\varphi \in B_\varepsilon(0,\mathcal{B})$ there exists a unique mild solution $u(\cdot,\varphi) \in AAA_c(X)$ of* (1.1)-(1.2).

Proof. By the hypothesis there exists a constant $r > 0$ such that

$$
\begin{aligned}
&[r - L_f(2\mathfrak{K}r)2\mathfrak{K}r\mu - \frac{M}{\beta}L_g(2\mathfrak{K}r)2\mathfrak{K}r] \\
&\geq M\|\varphi\|_{\mathcal{B}} + M\|f(0,\varphi)\| + \sup_{t\in[0,\infty)}\|f(t,0)\|_Y\mu + \frac{M}{\beta}\sup_{t\in[0,\infty)}\|g(t,0)\|,
\end{aligned}
$$

where $\mathfrak{K}$ is the constant introduced in Remark 2.2. We affirm that the assertion holds for $\varepsilon \leq r$. Let $\varphi \in B_\varepsilon(0,\mathcal{B})$ and the space

$$
\mathfrak{D} = \{x \in AAA_c(X) : x(0) = \varphi(0), \|x(t)\| \leq r,\, t \geq 0\}
$$

endowed with the metric $d(u,v) = \|u - v\|_\infty$, we define the operator $\Gamma : \mathfrak{D} \to C([0,\infty);X)$ by

$$
\begin{aligned}
\Gamma u(t) &= \mathcal{R}(t)(\varphi(0) + f(0,\varphi)) - f(t,\widetilde{u}_t) - \int_0^t A\mathcal{R}(t-s)f(s,\widetilde{u}_s)ds \\
&\quad - \int_0^t \mathcal{R}(t-s)\int_0^s B(s-\xi)f(\xi,\widetilde{u}_\xi)d\xi ds + \int_0^t \mathcal{R}(t-s)g(s,\widetilde{u}_s)ds, \quad t \geq 0
\end{aligned}
$$

where $\widetilde{u} : \mathbb{R} \to X$ is the function defined by the relation $\widetilde{u}_0 = \varphi$ and $\widetilde{u} = u$ on $[0,\infty)$. From the hypothesis (P1) (PF) and (PG) we obtain that Γu is well defined and that $\Gamma u \in C([0,\infty);X)$. Moreover, from Lemma 2.34, we have that function $s \mapsto \widetilde{u}_s \in AAA_c(\mathcal{B})$. By Lemma 2.32, we conclude that $s \mapsto f(s,\widetilde{u}_s) \in AAA_c([0,\infty),Y)$ and $s \mapsto g(s,\widetilde{u}_s) \in AAA_c([0,\infty),X)$. From Lemmas 2.33, 3.1, 3.2 and $\lim_{t\to\infty}\|\mathcal{R}(t)(\varphi(0) + f(0,\varphi))\| = 0$, we obtain that $\Gamma u \in AAA_c(X)$.

Next, we prove that $\Gamma(\cdot)$ is a contraction from $\mathfrak{D}$ into $\mathfrak{D}$. If $u \in \mathfrak{D}$ and $t \geq 0$, we obtain

$$
\begin{aligned}
&\|\Gamma u(t)\| \\
&\leq \|\mathcal{R}(t)(\varphi(0) + f(0,\varphi))\| + \|i_c\|_{\mathcal{L}(Y,X)}(\|f(t,\widetilde{u}_t) - f(t,0)\|_Y + \|f(t,0)\|_Y) \\
&\quad + \int_0^t \|A\mathcal{R}(t-s)(f(s,\widetilde{u}_s) - f(s,0))\|ds + \int_0^t \|A\mathcal{R}(t-s)f(s,0)\|ds \\
&\quad + \int_0^t \|\mathcal{R}(t-s)\int_0^s B(s-\xi)(f(\xi,\widetilde{u}_\xi) - f(\xi,0))d\xi\|ds \\
&\quad + \int_0^t \|\mathcal{R}(t-s)\int_0^s B(s-\xi)f(\xi,0)d\xi\|ds \\
&\quad + \int_0^t \|\mathcal{R}(t-s)(g(s,\widetilde{u}_s) - g(s,0))\|ds + \int_0^t \|\mathcal{R}(t-s)g(s,0)\|ds \\
&\leq M\|\varphi\|_{\mathcal{B}} + M\|f(0,\varphi)\| + \|i_c\|_{\mathcal{L}(Y,X)}(L_f(\|\widetilde{u}_t\|_{\mathcal{B}})\|\widetilde{u}_t\|_{\mathcal{B}} + \sup_{t\in[0,\infty)}\|f(t,0)\|_Y) \\
&\quad + \int_0^t \phi(t-s)L_f(\|\widetilde{u}_s\|_{\mathcal{B}})\|\widetilde{u}_s\|_{\mathcal{B}}ds + \sup_{t\in[0,\infty)}\|f(t,0)\|_Y \int_0^t \phi(s)ds \\
&\quad + \int_0^t Me^{-\beta(t-s)}\int_0^s b(s-\xi)L_f(\|\widetilde{u}_\xi\|_{\mathcal{B}})\|\widetilde{u}_\xi\|_{\mathcal{B}}d\xi ds \\
&\quad + \sup_{t\in[0,\infty)}\|f(t,0)\|_Y \int_0^t Me^{-\beta(t-s)}\int_0^s b(s-\xi)d\xi ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t Me^{-\beta(t-s)}L_g(\|\widetilde{u}_s\|_{\mathcal{B}})\|\widetilde{u}_s\|_{\mathcal{B}}ds+\sup_{t\in[0,\infty)}\|g(t,0)\|\int_0^t Me^{-\beta(t-s)}ds\\
&\le M\|\varphi\|_{\mathcal{B}}+M\|f(0,\varphi)\|\\
&\quad+\sup_{t\in[0,\infty)}\|f(t,0)\|_Y(\|i_c\|_{\mathcal{L}(Y,X)}+\int_0^\infty\phi(s)ds+\frac{M}{\beta}\int_0^\infty b(s)ds)\\
&\quad+\frac{M}{\beta}\sup_{t\in[0,\infty)}\|g(t,0)\|\\
&\quad+L_f(\|\widetilde{u}_t\|_{\mathcal{B}})(\|i_c\|_{\mathcal{L}(Y,X)}+\int_0^\infty\phi(s)ds+\frac{M}{\beta}\int_0^\infty b(s)ds)\|\widetilde{u}_t\|_{\mathcal{B}}\\
&\quad+\frac{M}{\beta}L_g(\|\widetilde{u}_t\|_{\mathcal{B}})\|\widetilde{u}_t\|_{\mathcal{B}}\\
&\le M\|\varphi\|_{\mathcal{B}}+M\|f(0,\varphi)\|\\
&\quad+\sup_{t\in[0,\infty)}\|f(t,0)\|_Y(\|i_c\|_{\mathcal{L}(Y,X)}+\|\phi\|_{L^1}+\frac{M}{\beta}\|b\|_{L^1})\\
&\quad+\frac{M}{\beta}\sup_{t\in[0,\infty)}\|g(t,0)\|+L_f(2\mathfrak{K}r)(\|i_c\|_{\mathcal{L}(Y,X)}+\|\phi\|_{L^1}+\frac{M}{\beta}\|b\|_{L^1})2\mathfrak{K}r\\
&\quad+\frac{M}{\beta}L_g(2\mathfrak{K}r)2\mathfrak{K}r\le r
\end{aligned}
$$

where the inequality $\|\widetilde{u}_t\|\le 2\mathfrak{K}r$ has been used and $i_c: Y\to X$ represents the continuous inclusion of Y on X. Thus, $\Gamma(\mathfrak{D})\subset\mathfrak{D}$. On the other hand, for $u,v\in\mathfrak{D}$ we see that

$$
\begin{aligned}
&\|\Gamma u(t)-\Gamma v(t)\|\\
&\le\|i_c\|_{\mathcal{L}(Y,X)}\|f(t,\widetilde{u}_t)-f(t,\widetilde{v}_t)\|_Y\\
&\quad+\int_0^t\|A\mathcal{R}(t-s)\|_{\mathcal{L}(Y,X)}\|f(s,\widetilde{u}_s)-f(s,\widetilde{v}_s)\|_Y ds\\
&\quad+\int_0^t\|\mathcal{R}(t-s)\|(\int_0^s\|B(s-\xi)\|_{\mathcal{L}(Y,X)}\|f(\xi,\widetilde{u}_\xi)-f(\xi,\widetilde{v}_\xi)\|_Y d\xi)ds\\
&\quad+\int_0^t\|\mathcal{R}(t-s)\|\|g(s,\widetilde{u}_s)-g(s,\widetilde{v}_s)\|ds\\
&\le\Big(L_f(2\mathfrak{K}r)\mathfrak{K}\mu+L_g(2\mathfrak{K}r)\mathfrak{K}\frac{M}{\beta}\Big)\|u-v\|_\infty\\
&\le\Big(L_f(2\mathfrak{K}r)2\mathfrak{K}\mu+L_g(2\mathfrak{K}r)2\mathfrak{K}\frac{M}{\beta}\Big)\|u-v\|_\infty,
\end{aligned}
$$

we observe that $r-L_f(2\mathfrak{K}r)2\mathfrak{K}r\mu-\frac{M}{\beta}L_g(2\mathfrak{K}r)2\mathfrak{K}r>0$, this implies that

$$
L_f(2\mathfrak{K}r)2\mathfrak{K}\mu+\frac{M}{\beta}L_g(2\mathfrak{K}r)2\mathfrak{K}<1,
$$

which shows that $\Gamma(\cdot)$ is a contraction from $\mathfrak{D}$ into $\mathfrak{D}$. The assertion is now a consequence of the contraction mapping principle. The proof is complete. □

Remark 3.4. A similar result was obtained by Dos Santos et al. [16] for the existence of asymptotically almost periodic solutions for the system (1.1)-(1.2).

Proposition 3.5. *Let $f:[0,\infty)\times\mathcal{B}\to Y$ and $g:[0,\infty)\times\mathcal{B}\to X$ be uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Assume that $\mathcal{B}$ is a fading memory space and* (P1), (P2), (PF), (PG) *hold. Then there exists $\varepsilon>0$ such that for each $\varphi\in B_\varepsilon(0,\mathcal{B})$ there exists a unique mild solution $u(\cdot,\varphi)\in SAP_\omega(X)$ of* (1.1)-(1.2) *on $[0,\infty)$.*

Proof. Let the space

$$\mathfrak{D}_\omega=\{x\in SAP_\omega(X):x(0)=\varphi(0),\|x(t)\|\le r,\ t\ge 0\}$$

endowed with the metric $d(u,v)=\|u-v\|_\infty$, we define the operator $\Gamma:\mathfrak{D}_\omega\to C([0,\infty);X)$ by

$$\begin{aligned}\Gamma u(t)&=\mathcal{R}(t)(\varphi(0)+f(0,\varphi))-f(t,\widetilde{u}_t)-\int_0^t A\mathcal{R}(t-s)f(s,\widetilde{u}_s)ds\\&\quad-\int_0^t\mathcal{R}(t-s)\int_0^s B(s-\xi)f(\xi,\widetilde{u}_\xi)d\xi ds+\int_0^t\mathcal{R}(t-s)g(s,\widetilde{u}_s)ds,\quad t\ge 0,\end{aligned}$$

where $\widetilde{u}:\mathbb{R}\to X$ is the function defined by the relation $\widetilde{u}_0=\varphi$ and $\widetilde{u}=u$ on $[0,\infty)$. From the hypothesis (P1), (PF) and (PG) we obtain that Γu is well defined and that $\Gamma u\in C([0,\infty);X)$. Moreover, from Lemma 2.19, we have that function $s\mapsto\widetilde{u}_s\in SAP_\omega(\mathcal{B})$. By Lemma 2.17, we conclude that $s\mapsto f(s,\widetilde{u}_s)\in SAP_\omega([0,\infty),Y)$ and $s\mapsto g(s,\widetilde{u}_s)\in SAP_\omega([0,\infty),X)$. From Lemmas 2.18, 3.1 and 3.2 it follows that $\Gamma u\in SAP_\omega(X)$. Using the same argument of Theorem 3.3 proof, we obtain that $\Gamma(\mathfrak{D}_\omega)\subset\mathfrak{D}_\omega$ and Γ is a contraction. This completes the proof. □

Proposition 3.6. *Let $f:[0,\infty)\times\mathcal{B}\to Y$ and $g:[0,\infty)\times\mathcal{B}\to X$ be asymptotically uniformly continuous on bounded subset $K\subset\mathcal{B}$, and $\lim_{t\to\infty}\|f(t+n\omega,\psi)-f(t,\psi)\|_Y=0$, $\lim_{t\to\infty}\|g(t+n\omega,\psi)-g(t,\psi)\|=0$ uniformly for $\psi\in K$ and $n\in\mathbb{N}$. Assume that $\mathcal{B}$ is a fading memory space and (P1), (P2), (PF) and (PG) hold. Then there exists $\varepsilon>0$ such that for each $\varphi\in B_\varepsilon(0,\mathcal{B})$ there exists a unique asymptotically ω-periodic mild solution $u(\cdot,\varphi)$ of* (1.1)-(1.2) *on $[0,\infty)$.*

Proof. We define the space

$$\mathfrak{D}_0=\{x\in S_\omega(X):x(0)=\varphi(0),\|x(t)\|\le r,\ t\ge 0\}$$

endowed with the metric $d(u,v)=\|u-v\|_\infty$. It is easy see that $\mathfrak{D}_0$ is a closed subspace of S_ω. We define the operator $\Gamma:\mathfrak{D}_0\to C([0,\infty);X)$ by

$$\begin{aligned}\Gamma u(t)&=\mathcal{R}(t)(\varphi(0)+f(0,\varphi))-f(t,\widetilde{u}_t)-\int_0^t A\mathcal{R}(t-s)f(s,\widetilde{u}_s)ds\\&\quad-\int_0^t\mathcal{R}(t-s)\int_0^s B(s-\xi)f(\xi,\widetilde{u}_\xi)d\xi ds+\int_0^t\mathcal{R}(t-s)g(s,\widetilde{u}_s)ds,\quad t\ge 0,\end{aligned}$$

where $\widetilde{u}:\mathbb{R}\to X$ is the function defined by the relation $\widetilde{u}_0=\varphi$ and $\widetilde{u}=u$ on $[0,\infty)$. We observe that $\mathcal{R}(\cdot)(\varphi(0)+f(0,\varphi))\in C_b([0,\infty),X))$ and

$$\lim_{t\to\infty}(\mathcal{R}(t+n\omega)-R(t))(\varphi(0)+f(0,\varphi))=0,$$

uniformly in $n\in\mathbb{N}$. Moreover, from [36, Lemma 3.16] and Lemma 2.23, we obtain that $\lim_{t\to\infty}\|f(t+n\omega,\widetilde{u}_{t+n\omega})-f(t,\widetilde{u}_t)\|_Y=0$ and $\lim_{t\to\infty}\|g(t+n\omega,\widetilde{u}_{t+n\omega})-g(t,\widetilde{u}_t)\|=0$, uniformly in $n\in\mathbb{N}$. By Lemmas 2.24, 3.1 and 3.2 we have that

$$\lim_{t\to\infty}\Gamma x(t+n\omega)-\Gamma x(t)=0,$$

uniformly in $n \in \mathbb{N}$. From Lemma 2.22 and using the same argument of the Theorem 3.3 proof we conclude that $u = \Gamma u \in \mathfrak{D}_0$ and u is asymptotically ω-periodic. The proof is ended. $\square$

4. Applications

In this section we study the existence of several type of asymptotically periodicity solutions of the partial neutral integro-differential system

$$\begin{aligned} &\frac{\partial}{\partial t}\Big[u(t,\xi) + \int_{-\infty}^{t}\int_0^{\pi} b(s-t,\eta,\xi)u(s,\eta)d\eta ds\Big] \\ &= (\frac{\partial^2}{\partial \xi^2} + \nu)\Big[u(t,\xi) + \int_0^t e^{-\gamma(t-s)}u(s,\xi)ds\Big] + \int_{-\infty}^{t} a_0(s-t)u(s,\xi)ds, \end{aligned} \tag{4.1}$$

$$u(t,0) = u(t,\pi) = 0, \quad u(\theta,\xi) = \varphi(\theta,\xi), \tag{4.2}$$

for $(t,\xi) \in [0,a] \times [0,\pi]$, $\theta \le 0, \nu < 0$ and $\gamma > 0$. Moreover, we have identified $\varphi(\theta)(\xi) = \varphi(\theta,\xi)$.

To represent this system in the abstract form (1.1)-(1.2), we choose the spaces $X = L^2([0,\pi])$ and $\mathcal{B} = C_0 \times L^2(\rho, X)$, see Example 2.3 for details. We also consider the operators $A, B(t) : D(A) \subseteq X \to X$, $t \ge 0$, given by $Ax = x'' + \nu x$, $B(t)x = e^{-\gamma t}Ax$ for $x \in D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}$. Moreover, A has discrete spectrum, the eigenvalues are $-n^2 + \nu$, $n \in \mathbb{N}$, with corresponding eigenvectors $z_n(\xi) = (\frac{2}{\pi})^{1/2}\sin(n\xi)$, the set of functions $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X and $T(t)x = \sum_{n=1}^{\infty} e^{-(n^2-\nu)t}\langle x, z_n\rangle z_n$ for $x \in X$. For $\alpha \in (0,1)$, from [35] we can define the fractional power $(-A)^\alpha : D((-A)^\alpha) \subset X \to X$ of A is given by $(-A)^\alpha x = \sum_{n=1}^{\infty}(n^2-\nu)^\alpha\langle x, z_n\rangle z_n$, where $D((-A)^\alpha) = \{x \in X : (-A)^\alpha x \in X\}$. In the next Theorem we consider $Y = D((-A)^{1/2})$. We observe that $\rho(A) \supset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \ge \nu\}$ and $\|\lambda R(\lambda, A)\| \le M_1$ for $\mathrm{Re}(\lambda) \ge \nu$, from [31, Proposition 2.2.11] we obtain that A is a sectorial operator satisfying $\|R(\lambda, A)\| \le \frac{M}{|\lambda-\nu|}, M > 0$, therefore (H1) is satisfied. Moreover, it is easy to see that conditions (H2)–(H3) are satisfied with $b(t) = e^{-\gamma t}$, and $D = C_0^\infty([0,\pi])$ the space of infinitely differentiable functions that vanishes at $\xi = 0$ and $\xi = \pi$. Under the above conditions we can represent the system

$$\frac{\partial u(t,\xi)}{\partial t} = \Big(\frac{\partial^2}{\partial \xi^2} + \nu\Big)\Big[u(t,\xi) + \int_0^t e^{-\gamma(t-s)}u(s,\xi)ds\Big], \tag{4.3}$$

$$u(t,\pi) = u(t,0) = 0, \tag{4.4}$$

in the abstract for

$$\begin{gathered} \frac{dx(t)}{dt} = Ax(t) + \int_0^t B(t-s)x(s)ds, \\ x(0) = z \in X. \end{gathered}$$

We define the functions $f, g : \mathcal{B} \to X$ by

$$\begin{gathered} f(\psi)(\xi) = \int_{-\infty}^{0}\int_0^{\pi} b(s,\eta,\xi)\psi(s,\eta)d\eta ds, \\ g(\psi)(\xi) = \int_{-\infty}^{0} a_0(s)\psi(s,\xi)ds, \end{gathered}$$

where

(i) The function $a_0 : \mathbb{R} \to \mathbb{R}$ is continuous and $L_g := (\int_{-\infty}^0 \frac{(a_0(s))^2}{\rho(s)} ds)^{\frac{1}{2}} < \infty$.

(ii) The functions $b(\cdot)$, $\frac{\partial b(s,\eta,\xi)}{\partial \xi}$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ for all (s,η) and

$$L_f := \max\Big\{ \Big(\int_0^{\pi} \int_{-\infty}^0 \int_0^{\pi} \rho^{-1}(\theta) \Big(\frac{\partial^i}{\partial \xi^i} b(\theta,\eta,\xi) \Big)^2 d\eta d\theta d\xi \Big)^{1/2} : i = 0,1 \Big\} < \infty.$$

Moreover, f, g are bounded linear operators, $\|f\|_{\mathcal{L}(\mathcal{B},X)} \le L_f$, $\|g\|_{\mathcal{L}(\mathcal{B},X)} \le L_g$ and a straightforward estimation using (**ii**) shows that $f(I \times \mathcal{B}) \subset D((-A)^{\frac{1}{2}})$ and

$$\|(-A)^{\frac{1}{2}} f(t,\cdot)\|_{\mathcal{L}(\mathcal{B},X)} \le L_f$$

for all $t \in I$. This allows us to rewrite the system (4.1)-(4.2) in the abstract form (1.1)-(1.2) with $u_0 = \varphi \in \mathcal{B}$.

Theorem 4.1. *Assume that the previous conditions are verified. Let* $2 < K < \gamma$ *and* $\nu < 0$ *such that* $|\nu| > \max\{M(K+1+\gamma), \gamma\}$. *If* $\frac{1}{2\mathfrak{K}} \ge L_f \mu + \frac{M}{|r+\nu|} L_g$, *where*
$$\mu = (\|(-A)^{-\frac{1}{2}}\| + M(2 + \frac{e^{r+\nu}}{|r+\nu|} + \frac{1}{|r+\nu|\gamma})),$$
then there exists $R > 0$ *such that if* $\|\varphi\|_{\mathcal{B}} < R$,

(i) *there exists a unique mild solution* $u(\cdot) \in AAA_c(X)$ *of* (4.1)-(4.2).

(ii) *there exists a unique mild solution* $u(\cdot) \in SAP_\omega(X)$ *of* (4.1)-(4.2).

(iii) *there exists a unique asymptotically* ω*-periodic mild solution* $u(\cdot)$ *of* (4.1)-(4.2).

Proof. By using a similar procedure as in the proof of [16, Theorem 5.1] we obtain an exponentially stable resolvent operator for the system (4.3)-(4.4). From the previous facts, Theorem 2.6 and Theorem 2.7, the assumption (P1) is satisfied. Observing that

$$M\|\varphi\|_{\mathcal{B}}(1 + L_f) < +\infty,$$

since $\frac{r}{2\mathfrak{K}} \ge L_f\mu + \frac{M}{\beta} L_g$, there exists a constant r_0 such that if $R \ge r_0$, we have

$$\frac{R}{2\mathfrak{K}} - L_f \mu R - \frac{M}{\beta} L_g R > M\|\varphi\|_{\mathcal{B}}(1 + L_f).$$

Now, for $\|\varphi\|_{\mathcal{B}} < R$, from Theorem 3.3 we obtain that there exists a unique mild solution of (4.1)-(4.2) such that $u(\cdot) \in AAA_c(X)$. By Proposition 3.5 there exists a unique mild solution $u(\cdot) \in SAP_\omega(X)$ of (4.1)-(4.2) and from Proposition 3.6 it follows that there exists a unique asymptotically ω-periodic mild solution $u(\cdot)$ of (4.1)-(4.2). The proof is complete. □

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