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HOPF BIFURCATIONS AND SMALL AMPLITUDE LIMIT CYCLES IN RUCKLIDGE SYSTEMS

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Abstract. In this article we study Hopf bifurcations and small amplitude limit cycles in a family of quadratic systems in the three dimensional space called Rucklidge systems. Bifurcation analysis at the equilibria of Rucklidge system is pushed forward toward the calculation of the second Lyapunov coefficient, which makes possible the determination of the Lyapunov and higher order structural stability.

1. Introduction

In this article we study Hopf bifurcations and small amplitude limit cycles in the following family of quadratic systems, called Rucklidge system,

$$x' = -ax + by - yz, \quad y' = x, \quad z' = -z + y^2, \tag{1.1}$$

where $(x, y, z) \in \mathbb{R}^3$ are the state variables and $(a, b) \in \mathcal{W} = \mathbb{R}^2$ are real parameters. Despite the simplicity, system (1.1) has a rich local dynamical behavior and was widely analyzed (see [9] and references therein).

Quadratic systems in $\mathbb{R}^3$ are some of the simplest systems after linear ones and have been extensively studied in the last five decades. Examples of such systems are the Lorenz system, the Chen system, the Liu system, the Rössler system, the Rikitake system, the Lü system, the Genesio system among several others. See [2] and references therein.

An interesting problem related to quadratic systems defined in $\mathbb{R}^3$ is the determination of the number of their limit cycles. In $\mathbb{R}^2$ this number is finite [3, 5]. For quadratic systems in $\mathbb{R}^n$, $n \geq 3$ the scenario is very different. Recently Ferragut, Llibre and Pantazi [4] provided an example of quadratic vector field in $\mathbb{R}^3$ and an analytical proof that it has infinitely many limit cycles.

It is well known (see [9] and references therein) that system (1.1) has at most three equilibria $E_0 = (0, 0, 0)$ and $E_\pm = (0, \pm\sqrt{b}, b)$, when $b \geq 0$. In order to study the stability of $E_\pm$ it is sufficient only to study the stability of E_+ due to the symmetry $(x, y, z) \to (-x, -y, z)$ presented by system (1.1).

In general, to decide the stability of a non–hyperbolic equilibrium point of a system in $\mathbb{R}^3$ is very difficult even for quadratic systems. As far as we know, the

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stabilities of E_0 and $E_\pm$ were analyzed in [9]. But the studies of Hopf bifurcations presented in [9] are incomplete and are not correct.

Consider the subset $\mathcal{U} \subset \mathcal{W}$ of the parameter plane where $b \neq 0$. Write $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{H}_0$, where

$$\mathcal{U}_1 = \{a \in \mathbb{R},\ b > 0\}, \quad \mathcal{U}_2 = \{a < 0,\ b < 0\},$$
$$\mathcal{U}_3 = \{a > 0,\ b < 0\}, \quad \mathcal{H}_0 = \{a = a_c = 0,\ b < 0\}.$$

From the linear analysis of system (1.1) at E_0 the following statements hold: if $(a, b) \in \mathcal{U}_1 \cup \mathcal{U}_2$ then E_0 is unstable; if $(a, b) \in \mathcal{U}_3$ then E_0 is locally asymptotically stable; if $(a, b) \in \mathcal{H}_0$ then E_0 is a non–hyperbolic equilibrium of Hopf type, that is the Jacobian matrix of system (1.1) at E_0 has one negative real eigenvalue and a pair of purely imaginary eigenvalues

$$\theta_1 = -1 < 0, \quad \theta_{2,3} = \pm i\sqrt{-b}.$$

Now consider the subset $\mathcal{W}^+ \subset \mathcal{W}$ of the parameter plane where $b > 0$. Write $\mathcal{W}^+ = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3 \cup \mathcal{H}_+$, where

$$\mathcal{W}_1 = \{a \leq 0,\ b > 0\}, \quad \mathcal{W}_2 = \big\{a > 0,\ b > \frac{a(a+1)}{2}\big\},$$
$$\mathcal{W}_3 = \big\{a > 0,\ 0 < b < \frac{a(a+1)}{2}\big\}, \quad \mathcal{H}_+ = \big\{a > 0,\ b = b_c = \frac{a(a+1)}{2}\big\}.$$

From the linear analysis of system (1.1) at E_+ the following statements hold: if $(a, b) \in \mathcal{W}_1 \cup \mathcal{W}_2$ then E_+ is unstable; if $(a, b) \in \mathcal{W}_3$ then E_+ is locally asymptotically stable; if $(a, b) \in \mathcal{H}_+$ then E_+ is a non–hyperbolic equilibrium of Hopf type, that is the Jacobian matrix of system (1.1) at E_+ has one negative real eigenvalue and a pair of purely imaginary eigenvalues

$$\lambda_1 = -(a+1) < 0, \quad \lambda_{2,3} = \pm i\sqrt{a}.$$

The sets $\mathcal{H}_0$ and $\mathcal{H}_+$ are called the Hopf curves of the equilibria E_0 and E_+, respectively. From the Center Manifold Theorem, at a Hopf point a two dimensional center manifold is well–defined, it is invariant under the flow generated by (1.1) and can be continued with arbitrary high class of differentiability to nearby parameter values (see [6, p. 152]). These center manifolds are normally attracting since $\theta_1 < 0$ and $\lambda_1 < 0$. So it is enough to study the stability of E_0 and E_+ for the flow restricted to the family of parameter–dependent continuations of these center manifolds.

It is important to emphasize that the study the stability of E_0 and E_+ for the flow of system (1.1) restricted to the center manifolds is in fact the study of the center–focus problem in an extended version to systems in $\mathbb{R}^3$. Although this problem has a solution for quadratic systems in the plane [1] it remains open for quadratic systems in $\mathbb{R}^3$.

The study carried out in the present article may contribute to understand analytically the stability of the equilibria E_0 and E_+ of system (1.1). By using the classical projection method which allows us to calculate the first and the second Lyapunov coefficients associated to the Hopf points, we study the stability of E_0 and E_+ as well as the number of small amplitude limit cycles in system (1.1). More precisely, in this article we prove the following two theorems.

Theorem 1.1. *Consider system* (1.1) *with parameter values in* $\mathcal{H}_0$*; that is,* $a = a_c = 0$ *and* $b < 0$*. Then the first Lyapunov coefficient associated to* E_0 *is positive, so* E_0 *is an unstable equilibrium point.*

Theorem 1.2. *Consider system* (1.1) *with parameter values in* $\mathcal{H}_+$*. Define* $a_1 = 6 + \sqrt{37}$*. The following statements hold.*

(1) *If* $0 < a < a_1$ *and* $b = b_c$ *then the first Lyapunov coefficient associated to* E_+ *is negative, so* E_+ *is locally asymptotically stable.*
(2) *If* $a > a_1$ *and* $b = b_c$ *then the first Lyapunov coefficient associated to* E_+ *is positive, so* E_+ *is unstable.*
(3) *If* $a = a_1$ *and* $b = b_c$ *then the first Lyapunov coefficient associated to* E_+ *vanishes and the second Lyapunov coefficient is positive, so* E_+ *is unstable.*

The proofs of Theorems 1.1 and 1.2, and the study of the small amplitude limit cycles of system (1.1) are presented in Section 3. In Section 2, we present a review on the methods of Hopf bifurcation analysis. Some concluding remarks are presented in Section 4.

2. Review on Hopf bifurcation

In this section we present a review of the projection method described in [6] for the calculation of the first and second Lyapunov coefficients associated to Hopf bifurcations. This method was extended to the calculation of the third and fourth Lyapunov coefficients in [7] and [8], respectively.

Consider the differential equation

$$x' = f(x, \zeta), \tag{2.1}$$

where $x \in \mathbb{R}^3$ and $\zeta \in \mathbb{R}^2$ are respectively vectors representing phase variables and control parameters. Assume that f is of class C^∞ in $\mathbb{R}^3 \times \mathbb{R}^2$. Suppose that (2.1) has an equilibrium point $x = x_0$ at $\zeta = \zeta_0$ and, denoting the variable $x - x_0$ also by x, write

$$F(x) = f(x, \zeta_0) \tag{2.2}$$

as

$$\begin{aligned} F(x) = Ax &+ \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + \frac{1}{24}D(x,x,x,x) \\ &+ \frac{1}{120}E(x,x,x,x,x) + O(||x||^6), \end{aligned} \tag{2.3}$$

where $A = f_x(0, \zeta_0)$ and, for $i = 1, 2, 3$,

$$B_i(x,y) = \sum_{j,k=1}^{3} \frac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k}\Big|_{\xi=0} x_j\, y_k, \quad C_i(x,y,z) = \sum_{j,k,l=1}^{3} \frac{\partial^3 F_i(\xi)}{\partial \xi_j\, \partial \xi_k\, \partial \xi_l}\Big|_{\xi=0} x_j\, y_k\, z_l,$$

and so on for D_i and E_i.

Suppose that $(x_0, \zeta_0) = (0, \zeta_0)$ is an equilibrium point of (2.1) where the Jacobian matrix A has a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega_0$, $\omega_0 > 0$, and the other eigenvalue $\lambda_1 \neq 0$. Let T^c be the generalized eigenspace of A corresponding to $\lambda_{2,3}$. By this it is meant the largest subspace invariant by A on which the eigenvalues are $\lambda_{2,3}$. Let $p, q \in \mathbb{C}^3$ be vectors such that

$$Aq = i\omega_0 q, \quad A^\top p = -i\omega_0 p, \quad \langle p, q\rangle = \sum_{i=1}^{3} \bar{p}_i q_i = 1, \tag{2.4}$$

where $A^\top$ is the transpose of the matrix A. Any vector $y \in T^c$ can be represented as $y = wq + \bar{w}\bar{q}$, where $w = \langle p, y\rangle \in \mathbb{C}$. The two dimensional center manifold associated to the eigenvalues $\lambda_{2,3} = \pm i\omega_0$ can be parameterized by the variables w and $\bar{w}$ by means of an immersion of the form $x = H(w,\bar{w})$, where $H : \mathbb{C}^2 \to \mathbb{R}^3$ has a Taylor expansion of the form

$$H(w,\bar{w}) = wq + \bar{w}\bar{q} + \sum_{2\le j+k\le 5} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k + O(|w|^6), \tag{2.5}$$

with $h_{jk} \in \mathbb{C}^3$ and $h_{jk} = \bar{h}_{kj}$. Substituting this expression into (2.1) we obtain the following differential equation

$$H_w w' + H_{\bar{w}} \bar{w}' = F(H(w,\bar{w})), \tag{2.6}$$

where F is given by (2.2). The complex vectors h_{ij} are obtained solving the system of linear equations defined by the coefficients of (2.6), taking into account the coefficients of F (see Remark 3.1 of [7, p. 27]), so that system (2.6), on the chart w for a central manifold, writes as follows, with $G_{jk} \in \mathbb{C}$,

$$w' = i\omega_0 w + \frac{1}{2}\, G_{21} w|w|^2 + \frac{1}{12}\, G_{32} w|w|^4 + O(|w|^6).$$

The *first Lyapunov coefficient* l_1 is defined by

$$l_1 = \frac{1}{2} \operatorname{Re} G_{21}, \tag{2.7}$$

where $G_{21} = \langle p, \mathcal{H}_{21}\rangle$ and $\mathcal{H}_{21} = C(q,q,\bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11})$.

The *second Lyapunov coefficient* is defined by

$$l_2 = \frac{1}{12} \operatorname{Re} G_{32}, \tag{2.8}$$

where $G_{32} = \langle p, \mathcal{H}_{32}\rangle$ and

$$\begin{aligned}
\mathcal{H}_{32} &= 6B(h_{11}, h_{21}) + B(\bar{h}_{20}, h_{30}) + 3B(\bar{h}_{21}, h_{20}) + 3B(q, h_{22}) + 2B(\bar{q}, h_{31}) \\
&\quad + 6C(q, h_{11}, h_{11}) + 3C(q, \bar{h}_{20}, h_{20}) + 3C(q, q, \bar{h}_{21}) + 6C(q, \bar{q}, h_{21}) \\
&\quad + 6C(\bar{q}, h_{20}, h_{11}) + C(\bar{q}, \bar{q}, h_{30}) + D(q, q, q, \bar{h}_{20}) + 6D(q, q, \bar{q}, h_{11}) \\
&\quad + 3D(q, \bar{q}, \bar{q}, h_{20}) + E(q, q, q, \bar{q}, \bar{q}) - 6G_{21} h_{21} - 3\bar{G}_{21} h_{21},
\end{aligned}$$

A *Hopf point of codimension one* is an equilibrium point (x_0, ζ_0) such that linear part of the vector field f has eigenvalues λ_2 and $\lambda_3 = \overline{\lambda}$ with $\lambda = \lambda(\zeta) = \gamma(\zeta) + i\eta(\zeta)$, $\gamma(\zeta_0) = 0$, $\eta(\zeta_0) = \omega_0 > 0$, the other eigenvalue $\lambda_1 \neq 0$ and the first Lyapunov coefficient, $l_1(\zeta_0)$, is different from zero. A *transversal Hopf point of codimension one* is a Hopf point of codimension one for which the complex eigenvalues depending on the parameters cross the imaginary axis with nonzero derivative. When $l_1 < 0$ ($l_1 > 0$) one family of stable (unstable) periodic orbits can be found on the center manifold and its continuation, shrinking to the Hopf point.

Hopf point of codimension 2 is an equilibrium (x_0, ζ_0) of f that satisfies the definition of Hopf point of codimension one, except that $l_1(\zeta_0) = 0$, and an additional condition that the second Lyapunov coefficient, $l_2(\zeta_0)$, is nonzero. This point is *transversal* if the sets $\gamma^{-1}(0)$ and $l_1^{-1}(0)$ have transversal intersection, or equivalently, if the map $\zeta \mapsto (\gamma(\zeta), l_1(\zeta))$ is regular at $\zeta = \zeta_0$. The bifurcation diagrams for $l_2 \neq 0$ can be found in [6, p. 313]. In this bifurcation diagram two families of small amplitude limit cycles can be found.

3. Proofs of Theorems 1.1 and 1.2

3.1. **Proof of Theorem 1.1.** In this subsection we study Hopf bifurcations that occur at the equilibrium E_0 for parameters in the set $\mathcal{H}_0$.

Theorem 3.1. *Consider system* (1.1) *with parameter values in* $\mathcal{H}_0$. *Then the first Lyapunov coefficient at* E_0 *is given by*

$$l_1(a_c, b) = \frac{2}{1-4b} > 0, \tag{3.1}$$

since $b < 0$. *If* $\tau_0 = (a_c, b) \in \mathcal{H}_0$ *then system* (1.1) *has a transversal Hopf point at* E_0 *for the parameter vector* τ_0.

Proof. For parameters on the Hopf curve $\mathcal{H}_0$, the eigenvalues of the Jacobian matrix of system (1.1) at E_0 are $\theta_1 = -1 < 0$, $\theta_{2,3} = \pm i\omega_0$, $\omega_0 = \sqrt{-b}$, $b < 0$, the eigenvectors q and p defined in (2.4) are

$$q = (i\sqrt{-b}, 1, 0), \quad p = \Big(\frac{i}{2\sqrt{-b}}, \frac{1}{2}, 0\Big)$$

and the multilinear symmetric functions B and C can be written as

$$B(x, y) = (-(x_2y_3 + x_3y_2), 0, 2x_2y_2)\,, \quad C(x, y, z) = (0, 0, 0)\,.$$

The complex vectors h_{11} and h_{20} are given by

$$h_{11} = (0, 0, 2)\,, \quad h_{20} = \Big(0, 0, \frac{2}{1 + i2\sqrt{-b}}\Big).$$

By simple calculations, the first Lyapunov coefficient (2.7) is given by

$$l_1(a_c, b) = \frac{2}{1-4b},$$

which is positive, since $b < 0$. It remains only to verify the transversality condition of the Hopf bifurcation. In order to do so, consider the family of differential equations (1.1) regarded as dependent on the parameter a. The real part, $\gamma = \gamma(a)$, of the pair of complex eigenvalues at the critical parameter $a = a_c = 0$ verifies

$$\gamma'(a_c) = \operatorname{Re}\big\langle p, \frac{dA}{da}\Big|_{a=a_c} q\big\rangle = -\frac{1}{2} < 0.$$

In the above expression, A is the Jacobian matrix of system (1.1) at E_0. Therefore, the transversality condition at the Hopf point holds. $\square$

The proof of Theorem 1.1 follows from Theorem 3.1.

From Theorem 3.1, the sign of the first Lyapunov coefficient at E_0 is positive for parameters in $\mathcal{H}_0$. Thus the equilibrium E_0 is a weak repelling focus (for the flow of system (1.1) restricted to the center manifold) and there is one unstable limit cycle near the asymptotically stable equilibrium E_0 for suitable value of the parameters ($a > 0$). See the pertinent bifurcation diagram in [6, p. 89]. See Figures 1 and 2 where the stability of E_0 and small amplitude limit cycles are depicted.

3.2. **Proof of Theorem 1.2.** In this subsection we study Hopf bifurcations that occur at the equilibrium E_+ for parameters in the set $\mathcal{H}_+$.

Theorem 3.2. *Consider system* (1.1) *with parameter values in* $\mathcal{H}_+$. *Then the first Lyapunov coefficient at* E_+ *is given by*
$$l_1(a,b_c) = \frac{2(a^2-12a-1)}{a(a+1)(a(a+3)+1)(a(a+6)+1)}. \tag{3.2}$$
If $\zeta_0 = (a,b_c) \in \mathcal{H}_+$ *is such that* $a \neq a_1$ *then system* (1.1) *has a transversal Hopf point at* E_+ *for the parameter vector* ζ_0.

Proof. For parameters on the Hopf curve $\mathcal{H}_+$, the eigenvalues of the Jacobian matrix of system (1.1) at E_+ are $\lambda_1 = -(1+a) < 0$, $\lambda_{2,3} = \pm i\omega_0$, $\omega_0 = \sqrt{a}$, $a > 0$, the eigenvectors q and p defined in (2.4) are
$$q = \Big(-\frac{\omega_0 - i}{\sqrt{2}c}, \frac{i\omega_0+1}{\sqrt{2}c\omega_0}, 1\Big),$$
$$p = \Big(\frac{ic}{\sqrt{2}(c^2-i\omega_0)}, \frac{c(i\omega_0+1)\omega_0}{\sqrt{2}(c^2-i\omega_0)}, \frac{1}{2} - \frac{1}{2c^2-2i\omega_0}\Big)$$
where $c = \sqrt{1+a}$, and the multilinear symmetric functions B and C can be written as
$$B(x,y) = (-(x_2y_3 + x_3y_2), 0, 2x_2y_2), \quad C(x,y,z) = (0,0,0).$$
The complex vectors h_{11} and h_{20} are given by
$$h_{11} = \Big(0, -\frac{\omega_0^2+3}{\sqrt{2}c^3\omega_0^3}, -\frac{2}{c^2\omega_0^2}\Big),$$
$$\begin{aligned} h_{20} = \Big(&\frac{\sqrt{2}(5i\omega_0+3)(\omega_0-i)}{c\omega_0^2\left(c^2-2(\omega_0(2\omega_0+3i)+2)\right)}, \frac{\omega_0(5\omega_0-8i)-3}{\sqrt{2}c\omega_0^3\left(c^2-2(\omega_0(2\omega_0+3i)+2)\right)}, \\ &-\frac{2i(\omega_0-i)\left(c^2+\omega_0(\omega_0+i)+2\right)}{c^2\omega_0^2\left(c^2-2(\omega_0(2\omega_0+3i)+2)\right)}\Big). \end{aligned}$$
Therefore, the first Lyapunov coefficient (2.7) is
$$l_1 = \frac{D(c,\omega_0)}{2c^2\omega_0^4\left(c^4+\omega_0^2\right)\left(c^4-8\left(\omega_0^2+1\right)c^2+4\left(\omega_0^2+4\right)\left(4\omega_0^2+1\right)\right)},$$
where
$$\begin{aligned} D(c,\omega_0) = {}&\left(7\omega_0^2-9\right)c^6 + \left(-21\omega_0^4+76\omega_0^2+69\right)c^4 - 6\omega_0^4\left(4\omega_0^4+33\omega_0^2+66\right) \\ &+ 2\left(19\omega_0^6+75\omega_0^4-190\omega_0^2-78\right)c^2 + 306\omega_0^2 + 96. \end{aligned}$$
Substituting $\omega_0 = \sqrt{a}$ and $c = \sqrt{1+a}$ into the expression of l_1, it results (3.2).

It remains only to verify the transversality condition of the Hopf bifurcation. In order to do so, consider the family of differential equations (1.1) regarded as dependent on the parameter b. The real part, $\gamma = \gamma(b)$, of the pair of complex eigenvalues at the critical parameter $b = b_c$ verifies
$$\gamma'(b_c) = \operatorname{Re}\big\langle p, \frac{dA}{db}\Big|_{b=b_c} q\big\rangle = \frac{a+2}{a^3+4a^2+4a+1} > 0,$$
since $a > 0$. In the above expression A is the Jacobian matrix of system (1.1) at E_+. Therefore, the transversality condition at the Hopf point holds. $\square$

The sign of the first Lyapunov coefficient (3.2) is determined by the sign of the numerator of (3.2) since the denominator is positive. If $\zeta_0 = (a, b_c) \in \mathcal{H}_+$, $a \neq a_1$ then $l_1(\zeta_0) \neq 0$ and system (1.1) has a transversal Hopf point at E_+ for the parameter vector ζ_0. More specifically, if $\zeta_0 = (a, b_c) \in \mathcal{H}_+$, $0 < a < a_1$, then $l_1(\zeta_0) < 0$ and the Hopf point at E_+ is asymptotically stable (weak attracting focus for the flow of system (1.1) restricted to the center manifold) and for a suitable ζ close to ζ_0 there exists a stable limit cycle near the unstable equilibrium E_+; if $\zeta_0 = (a, b_c) \in \mathcal{H}_+$, $a > a_1$, then $l_1(\zeta_0) > 0$ and the Hopf point at E_+ is unstable (weak repelling focus for the flow of system (1.1) restricted to the center manifold) and for a suitable ζ close to ζ_0 there exists an unstable limit cycle near the asymptotically stable equilibrium E_+. See Figures 1 and 2 where the stability of E_+ and small amplitude limit cycles are depicted.

In the next theorem we study the stability of the equilibrium E_+ for the parameters in $\mathcal{H}_+$ when $a = a_1$.

Theorem 3.3. *Consider system* (1.1) *with parameters in $\mathcal{H}_+$, $a = a_1$. Then the second Lyapunov coefficient at E_+ is positive.*

Proof. Due to the quadratic nature of system (1.1), the multilinear symmetric functions D and E are $D(x, y, z, w) = E(x, y, z, w, r) = (0, 0, 0)$. The complex vectors h_{ij} are too long and will be omitted here. After a long calculation, it follows that the second Lyapunov coefficient (2.8) at E_+ is given by

$$l_2(a, b_c) = \frac{N(a)}{3a^3(1+a)^3(1+a(3+a))^3(1+a(6+a))^3(1+a\ (11+a))}, \tag{3.3}$$

where

$$\begin{aligned} N(a) = 20a^{13} &+ 3956a^{12} + 62848a^{11} + 394248a^{10} + 1125116a^9 \\ &20212a^8 - 8288340a^7 - 16285036a^6 - 11735384a^5 \\ &- 3575472a^4 - 523708a^3 - 44300a^2 - 2600a - 72. \end{aligned}$$

To study the real zeros of N we recall Descartes Theorem: the number of real positive roots of the real algebraic equation $N = 0$, counted with multiplicities, is at most the number of sign–changes of terms of N. It is easy to see that $N(a) = 0$ has at most one positive real root. Since

$$N(2) = -\frac{725431}{58852827} < 0 \quad \text{and} \quad N(3) = \frac{341087}{445944744} > 0,$$

the root of the equation $N = 0$ is in the open interval $(2, 3)$. Therefore $N(a_1) > 0$. It follows that the sign of the second Lyapunov coefficient is positive, since the denominator is positive. □

From Theorem 3.3, the sign of the second Lyapunov coefficient at E_+ is positive for parameters where $l_1 = 0$. Thus the equilibrium E_+ is a weak repelling focus (for the flow of system (1.1) restricted to the center manifold) and there are two limit cycles, one stable and the other unstable, near the equilibrium E_+ for suitable value of the parameters. See the pertinent bifurcation diagram in [6, p. 313]. See also Figures 1 and 2 where the stability of E_+ and small amplitude limit cycles are depicted.

The proof of Theorem 1.2 follows from Theorems 3.2 and 3.3.

4. Concluding remarks

This paper starts reviewing the stability analysis which accounts for the characterization, in the plane of parameters, of the structural as well as Lyapunov stability of the equilibria of system (1.1). It continues with the extension of the analysis to the first order, codimension one points, based on the calculation of the first Lyapunov coefficient for the equilibrium points E_0 and $E_\pm$. The bifurcation analysis at the equilibria $E_\pm$ of system (1.1) is pushed forward to the calculation of the second Lyapunov coefficient, which makes possible the determination of the Lyapunov as well as higher order structural stability.

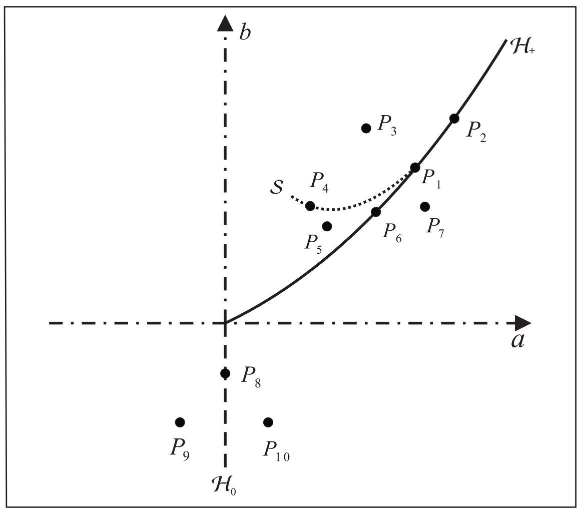

Figure 1. Bifurcation diagram of system (1.1). See also Figure 2

With the analytic data provided in the analysis performed here, the bifurcation diagrams of equilibria E_0 and E_+ are established and are put together in Figures 1 and 2, without danger of confusion. These figures provide a qualitative synthesis of the dynamical conclusions achieved at the parameter values where the system (1.1) has the most complex equilibrium points.

In Figure 1 the dashed (continuous) curve $\mathcal{H}_0$ ($\mathcal{H}_+$) is the Hopf curve of the equilibrium E_0 (E_+). The dotted curve $\mathcal{S}$ represents the curve of non–hyperbolic periodic orbits. The point P_1 has coordinates $a = a_1$ and $b = b_c$. The phase portraits for the flow of system (1.1) restricted to the center manifold and its continuations related to the points $P_1, \dots, P_{10}$ are illustrated in Figure 2 according to the following convention: linear repelling focus in (a) for the points P_3 (E_+) and P_9 (E_0); weak repelling focus in (b) for the points P_2 (E_+) and P_8 (E_0); linear attracting focus and one repelling hyperbolic cycle in (c) for the points P_7 (E_+) and P_{10} (E_0); weak attracting focus and one repelling hyperbolic cycle in (d) for the point P_6 (E_+); linear repelling focus and two hyperbolic cycles in (e) for the point P_5 (E_+), linear repelling focus and one non–hyperbolic cycle in (f) for the point P_4 (E_+); more weak repelling focus in (g) for the point P_1 (E_+).

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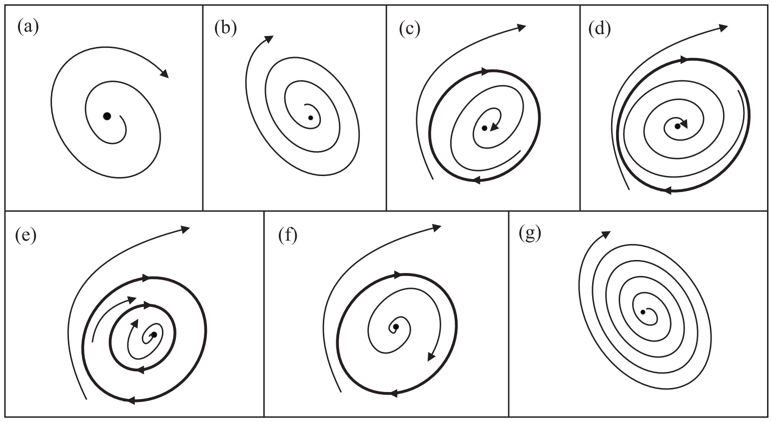

FIGURE 2. Sketch of the local phase portraits of system (1.1) related to the bifurcation diagram of Figure 1

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