Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 60, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

INTEGRO-DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER WITH NONLOCAL FRACTIONAL BOUNDARY CONDITIONS ASSOCIATED WITH FINANCIAL ASSET MODEL

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ABSTRACT. In this article, we discuss the existence of solutions for a boundary-value problem of integro-differential equations of fractional order with nonlocal fractional boundary conditions by means of some standard tools of fixed point theory. Our problem describes a more general form of fractional stochastic dynamic model for financial asset. An illustrative example is also presented.

1. Formulation and basic result

Fractional calculus, regarded as a branch of mathematical analysis dealing with derivatives and integrals of arbitrary order, has been extensively developed and applied to a variety of problems appearing in sciences and engineering. It is worthwhile to mention that this branch of mathematics has played a crucial role in exploring various characteristics of engineering materials such as viscoelastic polymers, foams, gels, and animal tissues, and their engineering and scientific applications. For a recent detailed survey of the activities involving fractional calculus, we refer a recent paper by Machado, Kiryakova and Mainardi [16]. Some recent work on the topic can be found in [1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 14, 17] and references therein.

The underlying dynamics of equity prices following a jump process or a Levy process provide a basis for modeling of financial assets. The CGMY, KoBoL and FMLS are examples of some interesting financial models involving the dynamics of stock prices. In [8], it is shown that the prices of financial derivatives are expressible in terms of fractional derivative.

In [15], the author described the dynamics of a financial asset by the fractional stochastic differential equation of order μ (representing the dynamical memory effects in the market stochastic evolution) with fractional boundary conditions. In the present paper, we study a more general model associated with financial asset.

2000 *Mathematics Subject Classification.* 34A08, 34B10, 34B15.
Key words and phrases. Fractional differential equations; integral boundary conditions; existence; fixed point theorems; financial asset.

Submitted December 6, 2012. Published February 26, 2013.

Precisely, we consider the following problem:

$$
\begin{gathered}
-D^{\alpha}x(t) = Af(t,x(t)) + BI^{\beta}g(t,x(t)), \quad (n-1) < \alpha \le n,\ t \in [0,1], \\
D^{\delta}x(0) = 0, \quad D^{\delta+1}x(0) = 0, \dots,\ D^{\delta+(n-2)}x(0) = 0, \quad D^{\delta}x(1) = \int_0^{\eta} D^{\delta}x(s)ds,
\end{gathered} \tag{1.1}
$$

where $0 < \delta \le 1$, $\alpha - \delta > n$, $0 < \beta < 1$, $0 < \eta < 1$, $D^{(\cdot)}$ denotes the Riemann-Lioville fractional derivative of order $(\cdot)$, f, g are given continuous function, and A, B are real constants.

We remark that the problem (1.1) also arises in real estate asset securitization modeling [18].

By the substitution $x(t) = I^{\delta}y(t) = D^{-\alpha}y(t)$, the problem (1.1) takes the form

$$
\begin{gathered}
-D^{\alpha-\delta}y(t) = Af(t, I^{\delta}y(t)) + BI^{\beta}g(t, I^{\delta}y(t)), \quad t \in [0,1], \\
y(0) = 0, \quad y'(0) = 0, \dots, y^{(n-2)}(0) = 0, \quad y(1) = \int_0^{\eta} y(s)ds.
\end{gathered} \tag{1.2}
$$

Lemma 1.1. *For any $h \in C(0,1) \cap L(0,1)$, the unique solution of the linear fractional boundary-value problem*

$$
\begin{gathered}
-D^{\alpha-\delta}y(t) = h(t), \quad t \in [0,1], \\
y(0) = 0, \quad y'(0) = 0, \dots, y^{(n-2)}(0) = 0, \quad y(1) = \int_0^{\eta} y(s)ds,
\end{gathered} \tag{1.3}
$$

is

$$
y(t) = -I^{\alpha-\delta}h(t) + \frac{(\alpha-\delta)t^{\alpha-\delta-1}}{\alpha-\delta-\eta^{\alpha-\delta}}\Big(I^{\alpha-\delta}h(1) - I^{\alpha-\delta+1}h(\eta)\Big),
$$

where $I^{(\cdot)}(\cdot)$ denotes Riemann-Liouville integral.

Proof. It is well known that the solutions of fractional differential equation in (1.1) can be written as

$$
y(t) = -I^{\alpha-\delta}h(t) + c_1 t^{\alpha-\delta-1} + c_2 t^{\alpha-\delta-2} + c_3 t^{\alpha-\delta-3} + \dots + c_n t^{\alpha-\delta-n}, \tag{1.4}
$$

where $c_1, c_2, \dots, c_n \in \mathbb{R}$ are arbitrary constants [12]. Using the given boundary conditions, we find that $c_2 = 0$, $c_3 = 0, \dots, c_n = 0$ and

$$
c_1 = \frac{\alpha-\delta}{\alpha-\delta-\eta^{\alpha-\delta}}\Big(I^{\alpha-\delta}h(1) - I^{\alpha-\delta+1}h(\eta)\Big).
$$

Substituting these values in (1.1) yields

$$
y(t) = -I^{\alpha-\delta}h(t) + \frac{(\alpha-\delta)t^{\alpha-\delta-1}}{\alpha-\delta-\eta^{\alpha-\delta}}\Big(I^{\alpha-\delta}h(1) - I^{\alpha-\delta+1}h(\eta)\Big).
$$

This completes the proof. □

Thus, the solution of the linear variant of the problem (1.1) can be written as

$$
\begin{aligned}
x(t) &= I^{\delta}y(t) \\
&= I^{\delta}\Big[-I^{\alpha-\delta}h(t) + \frac{(\alpha-\delta)t^{\alpha-\delta-1}}{\alpha-\delta-\eta^{\alpha-\delta}}\Big(I^{\alpha-\delta}h(1) - I^{\alpha-\delta+1}h(\eta)\Big)\Big] \\
&= -I^{\alpha}h(t) + \frac{(\alpha-\delta)}{\alpha-\delta-\eta^{\alpha-\delta}}\Big(I^{\alpha-\delta}h(1) - I^{\alpha-\delta+1}h(\eta)\Big)\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}s^{\alpha-\delta-1}ds
\end{aligned}
$$

$$
\begin{aligned}
&= -I^{\alpha}h(t) + \frac{(\alpha-\delta)}{\alpha-\delta-\eta^{\alpha-\delta}}\Big(I^{\alpha-\delta}h(1) - I^{\alpha-\delta+1}h(\eta)\Big)\times \\
&\quad \times \Big\{\frac{t^{\alpha-1}}{\Gamma(\delta)}\int_0^1 (1-\nu)^{\delta-1}\nu^{\alpha-\delta-1}d\nu\Big\},
\end{aligned}
$$

where we have used the substitution $s=\nu t$ in the integral of the last term. Using the relation for Beta function $B(\cdot,\cdot)$:

$$
B(\beta+1,\alpha) = \int_0^1 (1-u)^{\alpha-1}u^{\beta}du = \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)},
$$

we obtain

$$
x(t) = -I^{\alpha}h(t) + \frac{\Gamma(\alpha-\delta+1)t^{\alpha-1}}{(\alpha-\delta-\eta^{\alpha-\delta})\Gamma(\alpha)}\Big(I^{\alpha-\delta}h(1) - I^{\alpha-\delta+1}h(\eta)\Big). \tag{1.5}
$$

The solution of the original nonlinear problem (1.1) can be obtained by replacing h with the right hand side of the fractional equation of (1.1) in (1.5).

Let $\mathcal{C} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]\to\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t\in[0,1]\}$.

In relation to problem (1.1), we define an operator $\mathcal{U}:\mathcal{C}\to\mathcal{C}$ as

$$
\begin{aligned}
&(\mathcal{U}x)(t) \\
&= -A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds - B\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}g(s,x(s))ds \\
&\quad + Qt^{\alpha-1}\Big[A\int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}f(s,x(s))ds + B\int_0^1 \frac{(1-s)^{\alpha-\delta+\beta-1}}{\Gamma(\alpha-\delta+\beta)}g(s,x(s))ds \\
&\quad - A\int_0^{\eta} \frac{(\eta-s)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}f(s,x(s))ds - B\int_0^{\eta} \frac{(\eta-s)^{\alpha-\delta+\beta}}{\Gamma(\alpha-\delta+\beta+1)}g(s,x(s))ds\Big],
\end{aligned}
$$

where

$$
Q = \frac{\Gamma(\alpha-\delta+1)}{(\alpha-\delta-\eta^{\alpha-\delta})\Gamma(\alpha)}, \quad \alpha\neq\delta+\eta^{\alpha-\delta}.
$$

For the sake of convenience, we set

$$
\begin{aligned}
\Omega = \sup_{t\in[0,1]} &\Big\{|A|\Big[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + |Q|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+1)} + \frac{\eta^{\alpha-\delta+1}}{\Gamma(\alpha-\delta+2)}\Big)\Big] \\
&+ |B|\Big[\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |Q|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+\beta+1)} + \frac{\eta^{\alpha-\delta+\beta+1}}{\Gamma(\alpha-\delta+\beta+2)}\Big)\Big]\Big\}.
\end{aligned} \tag{1.6}
$$

1.1. **Existence results via Banach's fixed point theorem.**

Theorem 1.2. *Assume that $f,g:[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying the condition:*

- (A1) $|f(t,x)-f(t,y)|\le L_1|x-y|$, $|g(t,x)-g(t,y)|\le L_2|x-y|$, *for all* $t\in[0,1]$, $L_1,L_2>0$, $x,y\in\mathbb{R}$.

Then the boundary-value problem (1.1) *has a unique solution if* $L<1/\Omega$, *where* $L=\max\{L_1,L_2\}$ *and* Ω *is given by* (1.6).

Proof. Let us define $M = \max\{M_1, M_2\}$, where M_1, M_2 are finite numbers given by $\sup_{t\in[0,1]} |f(t,0)| = M_1$, $\sup_{t\in[0,1]} |g(t,0)| = M_2$. Selecting $r \geq \frac{\Omega M}{1-L\Omega}$, we show that $\mathcal{U}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. Using that $|f(s,x(s)) \leq |f(s,x(s)) - f(s,0)|+|f(s,0)| \leq L_1r+M_1$, $|g(s,x(s))| \leq |g(s,x(s))-g(s,0)|+|g(s,0)| \leq L_2r+M_2$ for $x \in B_r$ and (1.6), it can easily be shown that

$$
\begin{aligned}
&\|(\mathcal{U}x)\| \\
&\leq (Lr+M)\sup_{t\in[0,1]}\Big\{|A|\Big[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + |Q|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+1)} + \frac{\eta^{\alpha-\delta+1}}{\Gamma(\alpha-\delta+2)}\Big)\Big] \\
&\quad + |B|\Big[\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |Q|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+\beta+1)} + \frac{\eta^{\alpha-\delta+\beta+1}}{\Gamma(\alpha-\delta+\beta+2)}\Big)\Big]\Big\} \\
&= (Lr+M)\Omega \leq r,
\end{aligned}
$$

which implies that $\mathcal{U}B_r \subset B_r$. Now, for $x, y \in \mathcal{C}$ we obtain

$$
\begin{aligned}
&\|\mathcal{U}x - \mathcal{U}y\| \\
&\leq \sup_{t\in[0,1]}\Big\{|A|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s)) - f(s,y(s))|ds \\
&\quad + |B|\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|g(s,x(s)) - g(s,y(s))|ds \\
&\quad + |Q|t^{\alpha-1}\Big[|A|\int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}|f(s,x(s)) - f(s,y(s))|ds \\
&\quad + |B|\int_0^1 \frac{(1-s)^{\alpha-\delta+\beta-1}}{\Gamma(\alpha-\delta+\beta)}|g(s,x(s)) - g(s,y(s))|ds \\
&\quad + |A|\int_0^{\eta} \frac{(\eta-s)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}|f(s,x(s)) - f(s,y(s))|ds \\
&\quad + |B|\int_0^{\eta} \frac{(\eta-s)^{\alpha-\delta+\beta}}{\Gamma(\alpha-\delta+\beta+1)}|g(s,x(s)) - g(s,y(s))|ds\Big]\Big\} \\
&\leq L\sup_{t\in[0,1]}\Big\{|A|\Big[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + |Q|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+1)} + \frac{\eta^{\alpha-\delta+1}}{\Gamma(\alpha-\delta+2)}\Big)\Big] \\
&\quad + |B|\Big[\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |Q|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+\beta+1)} + \frac{\eta^{\alpha-\delta+\beta+1}}{\Gamma(\alpha-\delta+\beta+2)}\Big)\Big]\Big\} \\
&\quad \times \|x-y\| \\
&= L\Omega\|x-y\|.
\end{aligned}
$$

By the given assumption, $L < 1/\Omega$. Therefore $\mathcal{U}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). $\square$

Now we present another variant of existence-uniqueness result. This result is based on the Hölder's inequality.

Theorem 1.3. *Suppose that the continuous functions f and g satisfy the following assumptions:*

(H1) $|f(t,x) - f(t,y)| \leq m(t)|x-y|$, $|g(t,x) - g(t,y)| \leq n(t)|x-y|$, *for* $t \in [0,1]$, $x, y \in \mathbb{R}$, *and* $m, n \in L^{\frac{1}{\gamma}}([0,1], \mathbb{R}^+)$, $\gamma \in (0, \alpha-\delta-n)$.

(H2) $|A|\|m\|Z_1 + |B|\|n\|Z_2 < 1$, *where*

$$Z_1 = \frac{1}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\alpha-\gamma}\Big)^{1-\gamma} + \frac{|Q|}{\Gamma(\alpha-\delta)}\Big(\frac{1-\gamma}{\alpha-\delta-\gamma}\Big)^{1-\gamma} + \frac{|Q|}{\Gamma(\alpha-\delta+1)}\Big(\frac{1-\gamma}{\alpha-\delta+1-\gamma}\Big)^{1-\gamma}\eta^{\alpha-\delta+1-\gamma},$$

$$Z_2 = \frac{1}{\Gamma(\alpha+\beta)}\Big(\frac{1-\gamma}{\alpha+\beta-\gamma}\Big)^{1-\gamma} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta)}\Big(\frac{1-\gamma}{\alpha-\delta+\beta-\gamma}\Big)^{1-\gamma} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta+1)}\Big(\frac{1-\gamma}{\alpha-\delta+\beta+1-\gamma}\Big)^{1-\gamma}\eta^{\alpha-\delta+\beta+1-\gamma},$$

and $\|\mu\| = \big(\int_0^1 |\mu(s)|^{\frac{1}{\gamma}}ds\big)^{\gamma}$, $\mu = m, n$. *Then the boundary value problem* (1.1) *has a unique solution.*

Proof. For $x, y \in \mathbb{R}$ and for each $t \in [0,1]$, by Hölder inequality, we have

$$\begin{aligned}
&\|\mathcal{U}x - \mathcal{U}y\| \\
&\le \sup_{t\in[0,1]}\Big\{|A|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)|x(s)-y(s)|ds \\
&\quad + |B|\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}n(s)|x(s)-y(s)|ds \\
&\quad + |Q|\Big[|A|\int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}m(s)|x(s)-y(s)|ds \\
&\quad + |B|\int_0^1 \frac{(1-s)^{\alpha-\delta+\beta-1}}{\Gamma(\alpha-\delta+\beta)}n(s)|x(s)-y(s)|ds \\
&\quad + |A|\int_0^\eta \frac{(\eta-s)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}m(s)|x(s)-y(s)|ds \\
&\quad + |B|\int_0^\eta \frac{(\eta-s)^{\alpha-\delta+\beta}}{\Gamma(\alpha-\delta+\beta+1)}n(s)|x(s)-y(s)|ds\Big]\Big\} \\
&\le \sup_{t\in[0,1]}\Big\{\frac{|A|\|m\|}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\alpha-\gamma}\Big)^{1-\gamma}t^{\alpha-\gamma} + \frac{|B|\|n\|}{\Gamma(\alpha+\beta)}\Big(\frac{1-\gamma}{\alpha+\beta-\gamma}\Big)^{1-\gamma}t^{\alpha+\beta-\gamma} \\
&\quad + |Q|\Big[\frac{|A|\|m\|}{\Gamma(\alpha-\delta)}\Big(\frac{1-\gamma}{\alpha-\delta-\gamma}\Big)^{1-\gamma} + \frac{|B|\|n\|}{\Gamma(\alpha-\delta+\beta)}\Big(\frac{1-\gamma}{\alpha-\delta+\beta-\gamma}\Big)^{1-\gamma} \\
&\quad + \frac{|A|\|m\|}{\Gamma(\alpha-\delta+1)}\Big(\frac{1-\gamma}{\alpha-\delta+1-\gamma}\Big)^{1-\gamma}\eta^{\alpha-\delta+1-\gamma} \\
&\quad + \frac{|B|\|n\|}{\Gamma(\alpha-\delta+\beta+1)}\Big(\frac{1-\gamma}{\alpha-\delta+\beta+1-\gamma}\Big)^{1-\gamma}\eta^{\alpha-\delta+\beta+1-\gamma}\Big]\Big\}\|x-y\| \\
&\le |A|\|m\|\Big[\frac{1}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\alpha-\gamma}\Big)^{1-\gamma} + \frac{|Q|}{\Gamma(\alpha-\delta)}\Big(\frac{1-\gamma}{\alpha-\delta-\gamma}\Big)^{1-\gamma} \\
&\quad + \frac{|Q|}{\Gamma(\alpha-\delta+1)}\Big(\frac{1-\gamma}{\alpha-\delta+1-\gamma}\Big)^{1-\gamma}\Big]\|x-y\| \\
&\quad + |B|\|n\|\Big[\frac{1}{\Gamma(\alpha+\beta)}\Big(\frac{1-\gamma}{\alpha+\beta-\gamma}\Big)^{1-\gamma} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta)}\Big(\frac{1-\gamma}{\alpha-\delta+\beta-\gamma}\Big)^{1-\gamma}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{|Q|}{\Gamma(\alpha-\delta+\beta+1)}\Big(\frac{1-\gamma}{\alpha-\delta+\beta+1-\gamma}\Big)^{1-\gamma}\eta^{\alpha-\delta+\beta+1-\gamma}\Big]\|x-y\| \\
&= [|A|\|m\|Z_1 + |B|\|n\|Z_2]\|x-y\|.
\end{aligned}
$$

In view of condition (H2), it follows that $\mathcal{U}$ is a contraction mapping. Hence, Banach's fixed point theorem applies and $\mathcal{U}$ has a unique fixed point which is the unique solution of problem (1.1). This completes the proof. □

1.2. **Existence result via Leray-Schauder Alternative.**

Lemma 1.4 (Nonlinear alternative for single valued maps [11]). *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

- (i) *F has a fixed point in $\overline{U}$, or*
- (ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Theorem 1.5. *Assume that $f, g : [0,1]\times\mathbb{R} \to \mathbb{R}$ are continuous functions. Assume that:*

- (A3) *There exist functions $p_1, p_2 \in L^1([0,1], \mathbb{R}^+)$, and nondecreasing functions $\psi_1, \psi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that*
$$|f(t,x)| \le p_1(t)\psi_1(\|x\|), \quad |g(t,x)| \le p_2(t)\psi_2(\|x\|),$$
for all $(t,x) \in [0,1]\times\mathbb{R}$.
- (A4) *There exists a constant $M > 0$ such that*
$$\frac{M}{|A|\Lambda_1\psi_1(M)\|p_1\|_{L^1} + |B|\Lambda_1\psi_2(M)\|p_2\|_{L^1}} > 1,$$
where
$$\Lambda_1 = \frac{1}{\Gamma(\alpha+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+2)},$$
$$\Lambda_2 = \frac{1}{\Gamma(\alpha+\beta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta+2)}.$$

Then the boundary-value problem (1.1) has at least one solution on $[0,1]$.

Proof. Consider the operator $\mathcal{U} : \mathcal{C} \to \mathcal{C}$ with $x = \mathcal{U}x$, where

$$
\begin{aligned}
&(\mathcal{U}x)(t) \\
&= -A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds - B\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s,x(s))ds \\
&\quad + Qt^{\alpha-1}\Big[A\int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} f(s,x(s))ds + B\int_0^1 \frac{(1-s)^{\alpha-\delta+\beta-1}}{\Gamma(\alpha-\delta+\beta)} g(s,x(s))ds \\
&\quad - A\int_0^\eta \frac{(\eta-s)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} f(s,x(s))ds - B\int_0^\eta \frac{(\eta-s)^{\alpha-\delta+\beta}}{\Gamma(\alpha-\delta+\beta+1)} g(s,x(s))ds\Big].
\end{aligned}
$$

We show that F maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then

$$|(\mathcal{U}x)(t)|$$

$$
\begin{aligned}
&\le |A|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_1(s)\psi_1(\|x\|)ds + |B|\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}p_2(s)\psi_2(\|x\|)ds\\
&\quad + |Q|t^{\alpha-1}\Big[|A|\int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}p_1(s)\psi_1(\|x\|)ds\\
&\quad + |B|\int_0^1 \frac{(1-s)^{\alpha-\delta+\beta-1}}{\Gamma(\alpha-\delta+\beta)}p_2(s)\psi_2(\|x\|)ds\\
&\quad + |A|\int_0^\eta \frac{(\eta-s)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}p_1(s)\psi_1(\|x\|)ds\\
&\quad + |B|\int_0^\eta \frac{(\eta-s)^{\alpha-\delta+\beta}}{\Gamma(\alpha-\delta+\beta+1)}p_2(s)\psi_2(\|x\|)ds\Big]\\
&\le |A|\psi_1(r)\|p_1\|_{L^1}\Big\{\frac{1}{\Gamma(\alpha+1)}+\frac{|Q|}{\Gamma(\alpha-\delta+1)}+\frac{|Q|}{\Gamma(\alpha-\delta+2)}\Big\}\\
&\quad + |B|\psi_2(r)\|p_2\|_{L^1}\Big\{\frac{1}{\Gamma(\alpha+\beta+1)}+\frac{|Q|}{\Gamma(\alpha-\delta+\beta+1)}+\frac{|Q|}{\Gamma(\alpha-\delta+\beta+2)}\Big\}.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&\|\mathcal{U}x\|\\
&\le |A|\psi_1(r)\|p_1\|_{L^1}\Big\{\frac{1}{\Gamma(\alpha+1)}+\frac{|Q|}{\Gamma(\alpha-\delta+1)}+\frac{|Q|}{\Gamma(\alpha-\delta+2)}\Big\}\\
&\quad + |B|\psi_2(r)\|p_2\|_{L^1}\Big\{\frac{1}{\Gamma(\alpha+\beta+1)}+\frac{|Q|}{\Gamma(\alpha-\delta+\beta+1)}+\frac{|Q|}{\Gamma(\alpha-\delta+\beta+2)}\Big\}.
\end{aligned}
$$

Next we show that F maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $x\in B_r$, where B_r is a bounded set of $C([0,1],\mathbb{R})$. Then we obtain

$$
\begin{aligned}
&\|(\mathcal{U}x)(t_2)-(\mathcal{U}x)(t_1)\|\\
&\le \Big\|\frac{|A|}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]f(s,x(s))ds\\
&\quad + \frac{|A|}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,x(s))ds\\
&\quad + \frac{|B|}{\Gamma(\alpha+\beta)}\int_0^{t_1}[(t_2-s)^{\alpha+\beta-1}-(t_1-s)^{\alpha+\beta-1}]g(s,x(s))ds\\
&\quad + \frac{|B|}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}g(s,x(s))ds\\
&\quad + |Q|[(t_2)^{\alpha-1}-(t_1)^{\alpha-1}]\Big[|A|\int_0^1\frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}|f(s,x(s))|ds\\
&\quad + |B|\int_0^1\frac{(1-s)^{\alpha-\delta+\beta-1}}{\Gamma(\alpha-\delta+\beta)}|g(s,x(s))|ds\\
&\quad + |A|\int_0^\eta\frac{(\eta-s)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}|f(s,x(s))|ds + |B|\int_0^\eta\frac{(\eta-s)^{\alpha-\delta+\beta}}{\Gamma(\alpha-\delta+\beta+1)}|g(s,x(s))|ds\Big]\Big\|\\
&\le \Big\|\frac{|A|}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]p_1(s)\psi_1(r)ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|A|}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} p_1(s)\psi_1(r)ds \\
&+ \frac{|B|}{\Gamma(\alpha+\beta)} \int_0^{t_1} [(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}] p_2(s)\psi_2(r)ds \\
&+ \frac{|B|}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} (t_2-s)^{\alpha+\beta-1} p_2(s)\psi_2(r)ds \\
&+ |Q|[(t_2)^{\alpha-1} - (t_1)^{\alpha-1}]\Big[|A| \int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} p_1(s)\psi_1(r)ds \\
&+ |B| \int_0^1 \frac{(1-s)^{\alpha-\delta+\beta-1}}{\Gamma(\alpha-\delta+\beta)} p_2(s)\psi_2(r)ds \\
&+ |A| \int_0^\eta \frac{(\eta-s)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} p_1(s)\psi_1(r)ds + |B| \int_0^\eta \frac{(\eta-s)^{\alpha-\delta+\beta}}{\Gamma(\alpha-\delta+\beta+1)} p_2(s)\psi_2(r)ds\Big]\Big\|.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. As $\mathcal{U}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{U} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Lemma 1.4) once we have proved the boundendness of the set of all solutions to equations $x = \lambda\mathcal{U}x$ for $\lambda \in [0,1]$.

Let x be a solution. Then, for $t \in [0,1]$, and using the computations in proving that $\mathcal{U}$ is bounded, we have

$$
\begin{aligned}
&|x(t)| \\
&= |\lambda(\mathcal{U}x)(t)| \le |A|\psi_1(\|x\|)\|p_1\|_{L^1}\Big\{\frac{1}{\Gamma(\alpha+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+2)}\Big\} \\
&\quad + |B|\psi_2(\|x\|)\|p_2\|_{L^1}\Big\{\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta+2)}\Big\}.
\end{aligned}
$$

Consequently,

$$
\frac{\|x\|}{|A|\Lambda_1\psi_1(\|x\|)\|p_1\|_{L^1} + |B|\Lambda_1\psi_2(\|x\|)\|p_2\|_{L^1}} \le 1.
$$

In view of (A4), there exists M such that $\|x\| \ne M$. Let us set

$$
U = \{x \in C([0,1],X) : \|x\| < M\}.
$$

Note that the operator $\mathcal{U} : \overline{U} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda\mathcal{U}(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 1.4), we deduce that $\mathcal{U}$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

Example. Consider a boundary-value problem of integro-differential equations of fractional order with nonlocal fractional boundary conditions given by

$$
\begin{gathered}
-D^{5/2}x(t) = Af(t,x(t)) + BI^\beta g(t,x(t)), \quad t \in [0,1], \\
D^{1/4}x(0) = 0, \quad D^{5/4}x(0) = 0, \quad D^{1/4}x(1) = \int_0^\eta D^{1/4}x(s)ds,
\end{gathered} \tag{1.7}
$$

where $n = 3$, $A = B = 1$, $\beta = 3/4$, $\eta = 2/3$, $f(t,x) = \frac{3|x|(2+|x|)}{8(1+|x|)} + 4t$, $g(t,x) = \frac{1}{2}\tan^{-1}x + \sin^2 t$. With the given data, we find that

$$Q = \frac{\Gamma(\alpha-\delta+1)}{(\alpha-\delta-\eta^{\alpha-\delta})\Gamma(\alpha)} = 1.037485,$$

and

$$\begin{aligned}\Omega = \sup_{t\in[0,1]} \Big\{ &|A|\Big[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + |Q|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+1)} + \frac{\eta^{\alpha-\delta+1}}{\Gamma(\alpha-\delta+2)}\Big)\Big]\\ &+ |B|\Big[\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |Q|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+\beta+1)} + \frac{\eta^{\alpha-\delta+\beta+1}}{\Gamma(\alpha-\delta+\beta+2)}\Big)\Big]\Big\}\\ = 1.043555,&\end{aligned}$$

and $L_1 = 3/4$, $L_2 = 1/2$ as $|f(t,x)-f(t,y)| \le \frac{3}{4}|x-y|$, $|g(t,x)-g(t,y)| \le \frac{1}{2}|x-y|$. Clearly $L = \max\{L_1, L_2\} = 3/4$ and $L < 1/\Omega$. Thus all the assumptions of Theorem 1.2 are satisfied. Hence, by the conclusion of Theorem 1.2, the problem (1.7) has a unique solution.

Acknowledgments. The authors gratefully acknowledge the editor for his constructive comments.

References

[1] R. P. Agarwal, B. Ahmad; Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions, *Comput. Math. Appl.* **62** (2011), 1200-1214.

[2] A. Aghajani, Y. Jalilian, J. J. Trujillo; On the existence of solutions of fractional integro-differential equations, *Fract. Calc. Appl. Anal.* **15**, No 2 (2012), 44-69.

[3] B. Ahmad, J.J. Nieto; Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions, *Bound. Value Probl.* **2011:36** (2011), 9 pages.

[4] B. Ahmad, J. J. Nieto, A. Alsaedi, M. El-Shahed; A study of nonlinear Langevin equation involving two fractional orders in different intervals, *Nonlinear Anal. Real World Appl.* **13** (2012), 599-606.

[5] B. Ahmad, S.K. Ntouyas; A note on fractional differential equations with fractional separated boundary conditions, *Abstr. Appl. Anal. 2012, Article ID 818703, 11 pages.*

[6] B. Ahmad, S.K. Ntouyas; Nonlinear fractional differential equations and inclusions of arbitrary order and multi-strip boundary conditions, *Electron. J. Diff. Equ.* No. **98** (2012), pp. 1-22.

[7] M. Blasik, M. Klimek; On application of the contraction principle to solve the two-term fractional differential equations, *Acta Mechanica et Automatica* **5** (2011), No.2, 5-10.

[8] A. Carteaa, D. del-Castillo-Negrete; Fractional diffusion models of option prices in markets with jumps, *Physica A* **374** (2007) 749–763.

[9] C. Cuevas, H. Soto, A. Sepulveda; Almost periodic and pseudo-almost periodic solutions to fractional differential and integrodifferential equations, *Appl. Math. Comput.* **218** (2011), 1735-1745.

[10] N. J. Ford, M. L. Morgado; Fractional boundary value problems: Analysis and numerical methods, *Fract. Calc. Appl. Anal.* **14**, No 4 (2011), 554-567.

[11] A. Granas, J. Dugundji; *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[12] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[13] M. Kirane, S.A. Malik; The profile of blowing-up solutions to a nonlinear system of fractional differential equations, *Nonlinear Anal.* **73** (2010), 3723-3736.

[14] A. Z. Fino, M. Kirane; Qualitative properties of solutions to a time-space fractional evolution equation, *Quart. Appl. Math.* **70** (2012), 133-157.

[15] N. Laskin; Fractional market dynamics, *Physica A* **287** (2000) 482–492.

[16] J. T. Machado, V. Kiryakova, F. Mainardi; Recent history of fractional calculus, *Commun. Nonlinear Sci. Numer. Simulat.* **16** (2011) 1140–1153.
[17] W. Sudsutad, J. Tariboon; Existence results of fractional integro-differential equations with m-point multi-term fractional order integral boundary conditions, *Bound. Value Probl*, 2012, **2012**:94.
[18] H. Tao, M. Fu, R. Qian; Positive Solutions for Fractional Differential Equations from Real Estate Asset Securitization via New Fixed Point Theorem, *Abstr. Appl. Anal. 2012 (2012), Article ID 842358, 11 pages.*

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