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GLOBAL SOLUTIONS WITH $\mathcal{C}^k$-ESTIMATES FOR $\bar{\partial}$-EQUATIONS ON q-CONCAVE INTERSECTIONS

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Abstract. We construct a global solution to the $\bar{\partial}$-equation with $\mathcal{C}^k$-estimates on q-concave intersections in $\mathbb{C}^n$. Our main tools are integral formulas.

1. Introduction and statement of main result

This article is a continuation of [4] and concerns the study of the $\bar{\partial}$-equation on q-concave intersection in $\mathbb{C}^n$ from the viewpoint of $\mathcal{C}^k$-estimates by means of integral formulas. For this study we first solve the $\bar{\partial}$-equation with $\mathcal{C}^k$-estimates on local q-concave wedges in $\mathbb{C}^n$ and then we apply the pushing out method used by Kerzman [11].

We recall the notion of q-convexity in the sense of Andreotti-Grauert [4, 8, 17].

Definition 1.1. A bounded domain G of class $\mathcal{C}^2$ in $\mathbb{C}^n$ is called strictly q-convex if there exist an open neighborhood $\mathbb{U}$ of ∂G and a smooth $\mathcal{C}^2$-function $\rho : \mathbb{U} \to \mathbb{R}$ such that $G \cap \mathbb{U} = \{\zeta \in \mathbb{U} : \rho(\zeta) < 0\}$ and the Levi form

$$L_\rho(\zeta)t = \sum \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \bar{\zeta}_k} t_j \bar{t}_k, \quad t = (t_1, \dots, t_n) \in \mathbb{C}^n$$

has at least $q+1$ positive eigenvalues at each point $\zeta \in \mathbb{U}$.

A domain G in $\mathbb{C}^n$ is said to be strictly q-concave if G is in the form $G = G_1 \setminus \overline{G}_2$, where $G_2 \Subset G_1$ is strictly q-convex and $G_1 \Subset \mathbb{C}^n$ is strictly $(n-1)$-convex or compact. A point in ∂G_2, as a boundary point of G, is said to be strictly q-concave.

Applications to the tangential Cauchy-Riemann equations require that Definition 1.1 be extended to q-convex and q-concave domains with piecewise-smooth boundaries.

Definition 1.2. A bounded domain D in $\mathbb{C}^n$ is called a $\mathcal{C}^d$ q-convex intersection of order N, $d \geq 3$, if there exists a bounded neighborhood U of $\overline{D}$ and a finite number of real-valued $\mathcal{C}^d$ functions $\rho_1(z), \dots, \rho_N(z)$, $1 \leq N \leq n-1$, defined on U such that $D = \{z \in U : \rho_1(z) < 0, \dots, \rho_N(z) < 0\}$ and the following conditions are fulfilled:

(H1) For $1 \leq i_1 < i_2 < \cdots < i_\ell \leq N$ the 1-forms $d\rho_{i_1}, \dots, d\rho_{i_\ell}$ are $\mathbb{R}-$linearly independent on the set $\cap_{j=1}^{j=\ell} \{\rho_{i_j}(z) \leq 0\}$.

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(H2) For $1 \le i_1 < i_2 < \cdots < i_\ell \le N$, for every $z \in \cap_{j=1}^{j=\ell}\{\rho_{i_j}(z) \le 0\}$, if we set $I = (i_1, \dots, i_\ell)$, there exists a linear subspace T_z^I of $\mathbb{C}^n$ of complex dimension at least $q+1$ such that for $i \in I$ the Levi forms L_{ρ_i} restricted on T_z^I are positive definite.

A domain D in $\mathbb{C}^n$ is said to be a $\mathcal{C}^d$ q-concave intersection of order N if $D = D_1 \backslash \overline{D}_2$, where $D_2 \Subset D_1$ is a $\mathcal{C}^d$ q-convex intersection of order N and $D_1 \Subset \mathbb{C}^n$ is a $\mathcal{C}^d$ $(n-1)$-convex intersection. A point in ∂D_2, as a boundary point of D, is said to be strictly q-concave.

Condition (H2) was first introduced by Grauert [6], it implies that at every wedge the Levi forms of the corresponding $\{\rho_i\}$ have their positive eigenvalues along the same directions.

The study of the $\bar\partial$-equation on piecewise smooth intersections in $\mathbb{C}^n$ was initiated by Range and Siu [22] and then followed by many authors (see e.g., [1, 8, 12, 10, 18, 19, 20, 21]).

Motivated by the same problem, Laurent-Thiébaut and Leiterer [15] solved the $\bar\partial$-equation on piecewise smooth intersections of q-concave domains in $\mathbb{C}^n$ with uniform estimates for (n, s)-forms, $1 \le s \le q - N$; $q - N \ge 1$, where instead of condition (H2) they required the following Henkin's condition [1, 8]:

(H3) The Levi form of any nontrivial convex combination of $\{\rho_i\}_{i=1}^N$ has at least $q+1$ positive eigenvalues.

In addition, under slightly stronger hypotheses than those of [15], the authors extended their results in [16] to the case when $s = q - N + 1$.

Barkatou [2] obtained local solutions with $\mathcal{C}^k$-estimates for $\bar\partial$ on q-convex wedges in $\mathbb{C}^n$, his proof requires actually the following condition:

> there is a subdivision of the simplex $\triangle_N$ such that for every component $[a^1 \dots a^N]$ in this subdivision, the Leray maps of $\rho_{a^1}, \dots, \rho_{a^N}$ are $q+1$-holomorphic in the same directions with respect to the variable $z \in \mathbb{C}^n$, where for $a = (\lambda_1, \dots, \lambda_N)$, $\rho_a = \sum \lambda_i \rho_i$

which is weaker than condition (H2) and stronger than condition (H3).

Ricard [23] proved weaker $\mathcal{C}^k$-estimates than those obtained in [2] but for general q-convex (q-concave) wedges satisfying condition (3).

Recently, Barkatou and Khidr [4] constructed a global solution for $\bar\partial$ with $\mathcal{C}^k$-estimates with small loss of smoothness for $(0, s)$-forms, $n - q \le s \le n - 1$, on q-convex intersections in $\mathbb{C}^n$.

Let V be a bounded open set in $\mathbb{C}^n$. We use $\mathcal{C}^k_{r,s}(\overline{V})$, $k \in \mathbb{R}^+$, to denote the space of all continuous (r, s)-forms defined on $\overline{V}$ and having a continuous derivatives up to $[k]$ on $\overline{V}$ satisfying Hölder condition of order $k - [k]$. The corresponding norm is denoted by $\|\cdot\|_{k,V}$. Our main result is the following theorem.

Theorem 1.3. *Let D be a $\mathcal{C}^d$ q-concave intersection of order N in $\mathbb{C}^n$, $d \ge 3$, and let $f \in \mathcal{C}^0_{n,s}(D)$, $\bar\partial f = 0$, $1 \le s \le q - N$. Then there is a form $g \in \mathcal{C}^0_{n,s-1}(D)$ such that $\bar\partial g = f$ on D. If $f \in \mathcal{C}^k_{n,s}(\overline{D})$, $1 < k \le d - 2$; $\epsilon > 0$, then $g \in \mathcal{C}^{k-\epsilon}_{n,s-1}(\overline{D})$ and there exists a constant $C_{k,\epsilon} > 0$ such that*

$$\|g\|_{k-\epsilon,D} \le C_{k,\epsilon}\|f\|_{k,D}. \tag{1.1}$$

We note that for $q = n-1$ (i.e., the pseudoconvex case) this theorem was proved by Michel and Perotti [20] and for arbitrary q, but smooth ∂D, sharp $\mathcal{C}^k$ estimates were obtained by Lieb and Range [18].

The paper is organized in the following way: In section 1 we introduce the definition of a q-concave intersection in $\mathbb{C}^n$ and state our main result (Theorem 1.3). In section 2 we recall the generalized Koppelman lemma which plays a key role in the construction of the solution operators. Section 3 is devoted to the construction of the local solution operators with $\mathcal{C}^k$-estimates for $\bar{\partial}$. The main theorem is proved in section 4. The proof is based on pushing out the method of Kerzman [11].

2. Generalized Koppelman Lemma

In this section we recall a formal identity (the generalized Koppelman lemma) which is essential for our purposes. The exterior calculus we use here was developed by Harvey and Polking in [7] and Boggess [5].

Let D be an open set in $\mathbb{C}^n \times \mathbb{C}^n$. Let $G : D \to \mathbb{C}^n$ be a $\mathcal{C}^1$ map and write $G(\zeta, z) = (g_1(\zeta, z), \dots, g_n(\zeta, z))$. We define

$$\langle G(\zeta, z), \zeta - z\rangle = \sum_{j=1}^{n} g_j(\zeta, z)(\zeta_j - z_j)$$

$$\langle G(\zeta, z), d(\zeta - z)\rangle = \sum_{j=1}^{n} g_j(\zeta, z)d(\zeta_j - z_j)$$

$$\langle \bar{\partial}_{\zeta,z} G(\zeta, z), d(\zeta - z)\rangle = \sum_{j=1}^{n} \bar{\partial}_{\zeta,z} g_j(\zeta, z)d(\zeta_j - z_j),$$

where $\bar{\partial}_{\zeta,z} = \bar{\partial}_\zeta + \bar{\partial}_z$ (in the sense of distributions).

The Cauchy-Fantappiè form ω^G is defined by

$$\omega^G = \frac{\langle G(\zeta, z),\, d(\zeta - z)\rangle}{\langle G(\zeta, z),\, (\zeta - z)\rangle}$$

on the set where $\langle G(\zeta, z),\, (\zeta - z)\rangle \neq 0$.

Given m such maps, G^j, $1 \le j \le m$, the generalized Cauchy-Fantappiè kernel is given by

$$\begin{aligned}&\Omega(G^1, \dots, G^m)\\ &= (2\pi i)^{-n} \omega^{G^1} \wedge \dots \wedge \omega^{G^m} \wedge \sum_{\alpha_1+\dots+\alpha_m = n-m} (\bar{\partial}_{\zeta,z}\omega^{G^1})^{\alpha_1} \wedge \dots \wedge (\bar{\partial}_{\zeta,z}\omega^{G^m})^{\alpha_m}\end{aligned}$$

on the set where all the denominators are nonzero.

Lemma 2.1 (generalized Koppelman lemma)**.**

$$\bar{\partial}_{\zeta,z}\Omega(G^1, \dots, G^m) = \sum_{j=1}^{m} (-1)^j \Omega(G^1, \dots, \widehat{G}^j, \dots, G^m)$$

on the set where the denominators are nonzero.

If $\beta(\zeta, z) = (\overline{\zeta_1 - z_1}, \dots, \overline{\zeta_n - z_n})$, then $\Omega(\beta) = B(\zeta, z)$ is the usual Bochner-Martinelli-Koppelman kernel. Denote by $B_{r,s}(\zeta, z)$ the component of $B(\zeta, z)$ of type (r, s) in z and of type $(n - r, n - s - 1)$ in ζ. Then one has the following formula which is known as the Bochner-Martinelli-Koppelman formula (see e.g., [13, Theorem 1.7]).

Theorem 2.2. *Let $D \Subset \mathbb{C}^n$ be a bounded domain with $\mathcal{C}^1$-boundary, and let f be a continuous (r,s)-form on $\overline{D}$ such that $\bar\partial f$, in the sense of distributions, is also continuous on $\overline{D}$, $0 \le r, s \le n$. Then for any $z \in D$ we have*

$$(-1)^{r+s} f(z) = \int_{\zeta\in\partial D} f(\zeta)\wedge B_{r,s}(\zeta,z) - \int_{\zeta\in D} \bar\partial f(\zeta)\wedge B_{r,s}(\zeta,z) + \bar\partial_z \int_{\zeta\in D} f(\zeta)\wedge B_{r,s-1}(\zeta,z).$$

3. Solution operators on local q-concave wedges

In this section, we construct local solution operators T_s on the complement of a q-convex intersection. The plan of the construction is similar to that of Theorem 3.1 in [4]. The main differences are due to the fact that in this case the function ρ_{m+1} has convexity properties opposite to those of the functions $\rho_1,\dots,\rho_m$. Before we go further, we fix the following notation:

- Let $J=(j_1,\dots,j_\ell)$, $1\le \ell<\infty$, be an ordered collection of elements in $\mathbb{N}$. Then we write $|J|=\ell$, $J(\hat\nu)=(j_1,\dots,j_{\nu-1},j_{\nu+1},\dots,j_\ell)$ for $\nu=1,\dots,\ell$, and $j\in J$ if $j\in\{j_1,\dots,j_\ell\}$.
- Let $N\ge 1$ be an integer. Then we denote by $P(N)$ the set of all ordered collections $K=(k_1,\dots,k_\ell)$, $\ell\ge 1$, of integers with $1\le k_1,\dots,k_\ell\le N$. We call $P'(N)$ the subset of all $K=(k_1,\dots,k_\ell)$ with $k_1<\dots<k_\ell$.
- For $I=(j_1,\dots,j_\ell)\in P'(N)$ and $j\notin\{j_1,\dots,j_\ell\}$, we set $I_j=(k_1,\dots,k_{\ell+1})$ if $\{k_1,\dots,k_{\ell+1}\}\subset\{k_1,\dots,k_\ell,j\}$ and $k_1<\dots<k_{\ell+1}$.

Theorem 3.1. *Let D be a $\mathcal{C}^d$ $(d\ge 3)$ q-convex intersection of order n in $\mathbb{C}^n$. Then for each $\xi\in\partial D$, there is a radius $R>0$ such that on the set $\mathcal{W}=(U\setminus\overline{D})\cap\{z\in\mathbb{C}^n : |z-\xi|<R\}$ there are linear operators $T_s:\mathcal{C}^0_{n,s}(\mathcal{W})\to\mathcal{C}^0_{n,s-1}(\mathcal{W})$ such that $\bar\partial T_s f = f$ for all $f\in\mathcal{C}^0_{n,s}(\overline{\mathcal{W}})$, $1\le s\le q-N$, with $\bar\partial f=0$ (in the sense of distributions) on $\mathcal{W}$. If $f\in\mathcal{C}^k_{n,s}(\overline{\mathcal{W}})$, $1<k\le d-2$; $\epsilon>0$, then there exists a constant $C_{k,\epsilon}>0$ (independent of f) satisfying the estimates*

$$\|T_s f\|_{k-\epsilon,\mathcal{W}} \le C_{k,\epsilon}\|f\|_{k,\mathcal{W}}. \tag{3.1}$$

For $N=1$ (i.e., the case of local q-concave domains) this theorem was proved by Laurent-Thiébaut and Leiterer [14].

Proof. Let $D=\{z\in U|\rho_1(z)<0,\dots,\rho_N(z)<0\}\subset U$ be a q-convex intersection. We suppose for example that $E=\{\xi\in U|\rho_1(\xi)=\dots=\rho_m(\xi)=0\}$. If we set $\rho_{m+1}(\zeta)=|\zeta-\xi|^2-R^2$ for $R>0$, it follows from [15, Lemma 2.3] that $(E,(U\setminus\overline{D})\cap\{z\in\mathbb{C}^n : |z-\xi|<R\})$ is a local q-concave wedge.

Denote by $F_{\rho_i}(\zeta,.)$ the Levi polynomial of ρ_i at $\zeta\in U$. For $\zeta\in U$, $z\in\mathbb{C}^n$,

$$F_{\rho_i}(\zeta,z) = 2\sum_{j=1}^n \frac{\partial\rho_i(\zeta)}{\partial\zeta_j}(\zeta_j-z_j) - \sum_{j,k=1}^n \frac{\partial^2\rho_i}{\partial\zeta_j\partial\zeta_k}(\zeta_j-z_j)(\zeta_k-z_k).$$

By Definition 1.2, there exists an $(q+1)$-linear subspace T of $\mathbb{C}^n$ such that the Levi forms $L_{-\rho_i}$ at ξ are all positive definite on T.

Denote by P the orthogonal projection of $\mathbb{C}^n$ onto T and set $Q := Id-P$. Then it follows from Taylor's expansion theorem that there exist a number R and two positive constants A and B such that the following estimate holds:

$$-\operatorname{Re} F_{\rho_i}(\zeta,z) \ge \rho_i(z)-\rho_i(\zeta)+B|\zeta-z|^2 - A|Q(\zeta-z)|^2, \tag{3.2}$$

for every $i \in \{1, \dots, m\}$ and all $(z, \zeta) \in \mathbb{C}^n \times U$ with $|\xi - \zeta| < R$ and $|\xi - z| < R$.

Let $i \in \{1, \dots, m\}$. As ρ_i is of class $\mathcal{C}^2$ on U, we then can find $\mathcal{C}^\infty$ functions $a_i^{kj}(U)$, $j, k = 1, \dots, n$, such that for all $\zeta \in U$,

$$\big|a_i^{kj}(\zeta) - \frac{\partial^2 \rho(\zeta)}{\partial \zeta_k \partial \zeta_j}\big| < \frac{B}{2n^2}. \tag{3.3}$$

Denote by Q_{kj} the entries of the matrix Q; i.e.,

$$Q = (Q_{kj})_{k,j=1}^n \quad (k = \text{column index}).$$

We set, for $(z, \zeta) \in \mathbb{C}^n \times U$,

$$\begin{gathered} g_j^i(\zeta, z) = 2\frac{\partial \rho_i(\zeta)}{\partial \zeta_j} - \sum_{k=1}^n a_i^{kj}(\zeta)(\zeta_k - z_k) - A \sum_{k=1}^n \overline{Q_{kj}(\zeta_k - z_k)}, \\ G_i(\zeta, z) = (g_i^1(\zeta, z), \dots, g_i^n(\zeta, z)), \\ \Phi_i(\zeta, z) = \langle G_i(\zeta, z), \zeta - z \rangle. \end{gathered}$$

As Q is an orthogonal projection, we then have

$$\Phi_i(\zeta, z) = 2\sum_{j=1}^n \frac{\partial \rho_i(\zeta)}{\partial \zeta_j}(\zeta_j - z_j) - \sum_{k,j=1}^n a_i^{kj}(\zeta)(\zeta_k - z_k)(\zeta_j - z_j) - A|Q(\zeta - z)|^2.$$

The estimates (3.2) and (3.3) imply that

$$\operatorname{Re} \Phi_i(\zeta, z) \geq \rho_i(\zeta) - \rho_i(z) + \frac{B}{2}|\zeta - z|^2$$

for $(z, \zeta) \in \mathbb{C}^n \times U$ with $|z_0 - \zeta| \leq R$ and $|z_0 - z| \leq R$. $\square$

We recall that a map f defined on a complex manifold $\mathcal{X}$ is called k-holomorphic if, for each point $\xi \in \mathcal{X}$, there exist holomorphic coordinates $h_1, \dots, h_k$ in a neighborhood of ξ such that f is holomorphic with respect to $h_1, \dots, h_k$.

Lemma 3.2. *For every fixed $\zeta \in U$, the maps $G_i(\zeta, z)$ and the function $\Phi_i(\zeta, z)$ are $(q + 1)$-holomorphic in the same directions in $z \in \mathbb{C}^n$.*

Proof. Choose complex linear coordinates $h_1, \dots, h_n$ on $\mathbb{C}^n$ with

$$\{z \in \mathbb{C}^n : Q(z) = 0\} = \{z \in \mathbb{C}^n : h_{q+2}(z) = \dots = h_n(z) = 0\}.$$

The map $z \to \overline{Q(\zeta - z)}$ is then independent of $h_1, \dots, h_{q+1}$. This implies that the map $G_i(\zeta, z)$ is complex linear with respect to $h_1, \dots, h_{q+1}$ for all i, and the function $\Phi_i(\zeta, z)$ is a quadratic complex polynomial with respect to $h_1, \dots, h_{q+1}$. $\square$

Set

$$\begin{gathered} G_{m+1}(\zeta, z) = 2\Big(\frac{\partial \rho_{m+1}(\zeta)}{\partial \zeta_1}, \dots, \frac{\partial \rho_{m+1}(\zeta)}{\partial \zeta_n}\Big), \\ \Phi_{m+1}(\zeta, z) = \langle G_{m+1}(\zeta, z), (\zeta - z) \rangle. \end{gathered}$$

As $G_{m+1}(\zeta, z)$ and $\Phi_{m+1}(\zeta, z)$ are independent of R, we can choose $R_1 > 0$ such that for all $R \leq R_1$ there exists $\beta > 0$ satisfying

$$\operatorname{Re} \Phi_{m+1}(\zeta, z) \geq \rho_{m+1}(\zeta) - \rho_{m+1}(z) + \beta|\zeta - z|^2$$

for all $(z, \zeta) \in \mathbb{C}^n \times U$ with $|z_0 - \zeta| \leq R$ and $|z_0 - z| \leq R$. We define

$$\mathcal{W} = \{z \in U | \rho_j > 0 \text{ for } j = 1, \dots, m\} \cap \{z \in \mathbb{C}^n : |z - \xi| < R\}.$$

For $I = (j_1, \dots, j_\ell) \in P'(m+1)$, we define

$$\widetilde{\Omega}[I] := \Omega(G_{j_1}, \dots, G_{j_\ell}),$$
$$\widetilde{\Omega}[\partial I] := \sum_{k=1}^{\ell} (-1)^k \Omega(G_{j_1}, \dots, \widehat{G}_{j_k}, \dots, G_{j_\ell}).$$

Then, we can rewrite Lemma 2.1 in the following way.

Lemma 3.3. *For every $I \in P'(m+1)$, we have $\bar{\partial}_{\zeta,z}\widetilde{\Omega}[I] = \widetilde{\Omega}[\partial I]$ outside the singularities.*

For every $0 \le r \le n$, $0 \le s \le n-p$ ($p \ge 1$) and any I, we define $\widetilde{\Omega}_{r,s}[I]$ as the component of $\widetilde{\Omega}[I]$ which is of type (r,s) in z. One has the following lemma.

Lemma 3.4. *For any $I \in P'(m+1)$. For any $r \ge 0$ and $s \ge n-q$ we have*
 (i) $\widetilde{\Omega}_{r,s}(I) = 0$,
 (ii) $\bar{\partial}_z\widetilde{\Omega}_{r,n-q-1}(I) = 0$,

on the set where all the denominators are non-zero.

Proof. Statement (i) follows from Lemma 2.1 and the fact that the map $z \mapsto G_{m+1}(\zeta, z)$ is holomorphic. Lemmas 3.3 and 2.1 imply that

$$\bar{\partial}_z\widetilde{\Omega}_{r,s-1}(I) = -\bar{\partial}_\zeta\widetilde{\Omega}_{r,s}(I) + \widetilde{\Omega}_{r,s}(\partial I).$$

Statement (ii) follows from (i). □

Let $\beta(\zeta, z) = (\overline{\zeta_1 - z_1}, \dots, \overline{\zeta_n - z_n})$ be the classical section that defines the usual Bochner-Martinelli kernel in $\mathbb{C}^n$ and define

$$\widetilde{\Omega}_\beta[I] := \Omega(\beta, G_{j_1}, \dots, G_{j_\ell})$$

for any $I \in P'(m+1)$. Lemma 3.4 implies that

$$\bar{\partial}_{\zeta,z}\widetilde{\Omega}_\beta[I] = -\widetilde{\Omega}[I] - \widetilde{\Omega}_\beta[\partial I]$$

outside the singularities, where $\widetilde{\Omega}_\beta[\partial I] := \Omega(\beta)$ if $|I| = 1$. Define, for $|I| \ge 1$,

$$K^I(\zeta, z) = \widetilde{\Omega}_\beta[I](\zeta, z),$$
$$B^I(\zeta, z) = -\widetilde{\Omega}_\beta[\partial I](\zeta, z).$$

Then we obtain the following lemma.

Lemma 3.5. *For any $I \in P'(m+1)$,*

$$\bar{\partial}_{\zeta,z}K^I(\zeta, z) = B^I(\zeta, z) - \widetilde{\Omega}[I](\zeta, z)$$

outside the singularities.

Proof. For every $I = (j_1, \dots, j_\ell) \in P'(m+1)$, define

$$S_I = \{z \in \partial\mathcal{W} | \rho_{j_1}(z) = \dots = \rho_{j_\ell}(z) = 0\}$$

and choose the orientation of S_I such that the orientation is skew symmetric in the components of I and the following two equations hold when $\mathcal{W}$ is given the natural orientation:

$$\partial\mathcal{W} = \sum_{j=1}^{m+1} S_j, \quad \partial S_I = \sum_{j \notin I} S_{I_j}.$$

Denote by $K^I_{r,s}(\zeta,z)$ the component of $K^I(\zeta,z)$ which is of type (r,s) in z.

Then Lemmas 3.4 and 3.5, Theorem 2.2, and Stoke's theorem imply that the following formulas hold in the sense of distribution in $\mathcal{W}$ (see [3, Theorem 2.7]):

For any continuous (n,s)-form f on $\overline{\mathcal{W}}$, $1 \le s \le q-m$, such that $\bar\partial f$ is also continuous on $\overline{\mathcal{W}}$. We have

$$\begin{aligned}
&(-1)^{n+s} f(\zeta)\\
&= \sum_{I\in P'(m+1)} (-1)^{(n+s)|I|+\frac{|I|(|I|-1)}{2}} \quad \int_{z\in S_I} \bar\partial f(z)\wedge K^I_{0,n-|I|-s}(\zeta,z)\\
&\quad + \sum_{I\in P'(m+1)} (-1)^{(n+s)|I|+\frac{|I|(|I|+1)}{2}+1} \quad \bar\partial_\zeta \int_{z\in S_I} f(z)\wedge K^I_{0,n-|I|-s+1}(\zeta,z)\\
&\quad - \int_{z\in\mathcal{W}} \bar\partial f(z)\wedge B_{0,n-s-1}(\zeta,z) + \bar\partial_\zeta \int_{z\in\mathcal{W}} f(z)\wedge B_{0,n-s}(\zeta,z) + Lf(\zeta)
\end{aligned}$$

where Lf is a linear combination of the integrals $\int_{z\in S_{I_{m+1}}} f(z)\wedge\widetilde{\Omega}[I](\zeta,z)$, where $I\in P'(m)$.

It is easy to see that Lf is of class $\mathcal{C}^{d-2}$ in a neighborhood of ξ; moreover if $\bar\partial f = 0$, then $\bar\partial Lf = 0$. Let H be the Henkin operator for solving the $\bar\partial$-equation in a ball $B(\xi,R')$. From the smoothness properties of H, it follows that $H(Lf)$ is of class $\mathcal{C}^{d-2+\frac12}$. Note that

$$\begin{aligned}
T_s(f)(\zeta) = &\sum_{I\in P'(m+1)} (-1)^{(n+s)|I|+\frac{|I|(|I|+1)}{2}+1} \int_{z\in S_I} f(z)\wedge K^I_{0,n-|I|-s+1}(z,\zeta)\\
&+ (-1)^{n+s}\int_{z\in\mathcal{W}} f(z)\wedge B_{0,n-s}(z,\zeta) + H(Lf)(\zeta)
\end{aligned}$$

satisfies the equation $\bar\partial u = f$ on $\mathcal{W}\cap B(\xi,R')$ with $u = T_s(f)(\zeta)$.

The $\mathcal{C}^k$-estimates follows, as in [2], by using arguments similar to those in [18]. $\square$

4. Proof of Theorem 1.3

Theorem 3.1 yields the following continuation lemma which in turn enables us to complete the proof of Theorem 1.3.

Lemma 4.1 (An extension lemma with bounds). *Let D be a $\mathcal{C}^d$, $d\ge 3$, q-concave intersection of order N in $\mathbb{C}^n$. Then there exists another slightly larger q-concave intersection of order N, $\widetilde{D}\Subset\mathbb{C}^n$ such that $D\Subset\widetilde{D}$ and for any $f\in\mathcal{C}^0_{n,s}(D)$, $1\le s\le q-N$, with $\bar\partial f = 0$ there exist two linear operators N_1, N_2, a form $\tilde f = N_1 f\in\mathcal{C}^0_{n,s}(\widetilde{D})$ and a form $u = N_2 f\in\mathcal{C}^0_{n,s-1}(D)$ such that:*

- (i) *$\bar\partial\tilde f = 0$ in $\widetilde{D}$.*
- (ii) *$\tilde f = f - \bar\partial u$ in D.*
- (iii) *If $f\in\mathcal{C}^k_{n,s}(\overline{D})$, $1<k\le d-2$, $\epsilon>0$, then $\tilde f\in\mathcal{C}^{k-\epsilon}_{n,s}(\widetilde{D})$, $u\in\mathcal{C}^{k-\epsilon}_{n,s-1}(\overline{D})$ and we have the estimates:*

$$\|\tilde f\|_{k-\epsilon,\widetilde{D}} \le C_{k,\epsilon}\|f\|_{k,D}, \tag{4.1}$$

$$\|u\|_{k-\epsilon,D} \le C_{k,\epsilon}\|f\|_{k,D}. \tag{4.2}$$

Proof. As ∂D is compact, there are finitely many open neighborhoods $(B_{\xi_j})_{j=1,\dots,K}$ of ξ_j covering ∂D. Let $(\theta_j)_{j=1,\dots,K}$ be a partition of unity such that $\theta_j \in \mathcal{C}_0^\infty(B'_{\xi_j})$, $B'_{\xi_j} \Subset B_{\xi_j}$, $0 \le \theta_j \le 1$, and $\sum_{j=1}^K \theta_j = 1$ on a neighborhood V_0 of ∂D. We choose $V_1 \Subset V_0 \Subset U$. We enlarge D to $\widetilde{D}$ in K step as follows. For $\delta > 0$, sufficiently small to be chosen fixed later on, and for $j = 1, \dots, K$ we define

$$D_j = \Big\{ z \in D \cup V_1 : \rho_1(z) > -\delta \sum_{k=1}^{j} \theta_k(z), \dots, \rho_N(z) > -\delta \sum_{k=1}^{j} \theta_k(z) \Big\}.$$

We set $D_0 = D$ and $\widetilde{D} = D_K$. Clearly

$$D \subseteq D_j \subseteq D_{j+1} \subseteq \dots \subseteq \widetilde{D} = D_K.$$

Reducing δ if necessary, we see that all D_j, $j \in \{= 1, \dots, K\}$ (in particular $\widetilde{D}$) are $\mathcal{C}^d$ q-concave intersections.

Claim: For any $f_j \in \mathcal{C}^0_{n,s}(D_j)$ with $\bar\partial f_j = 0$, $j \in \{1, \dots, K-1\}$, there exist two forms $f_{j+1} \in \mathcal{C}^0_{n,s}(D_{j+1})$ and $u_j \in \mathcal{C}^0_{n,s-1}(D_j)$ such that (i), (ii) and (iii) of Lemma 4.1 hold when f, $\tilde f$, u, D and $\widetilde{D}$ are replaced by f_j, f_{j+1}, u_j, D_j and D_{j+1} respectively.

Proof. (see [11, p. 318]): Fix $\delta > 0$ so small that we can apply Theorem 3.1, we obtain a solution g_j of $\bar\partial g = f_j$ defined in $D_j \cap B_{\xi_{j+1}}$ and satisfies the estimates of the local theorem. Let $\eta_{j+1} \in C_0^\infty(B_{\xi_{j+1}})$, $\eta_{j+1} = 1$ in a neighborhood of the support of θ_{j+1}. We set

$$f_{j+1} = \begin{cases} f_j - \bar\partial u_j & \text{in } D_j, \\ 0 & \text{in } D_{j+1} \setminus D_j, \end{cases} \qquad u_j = \begin{cases} g_j \eta_{j+1} & \text{in } D_j \cap B_{\xi_{j+1}}, \\ 0 & \text{in } D_j \backslash B_{\xi_{j+1}}. \end{cases}$$

The estimates for f_{j+1} and u_j follow from those of the local theorem. The claim is proved. ☐

Using the above claim, we can now complete the proof of Lemma 4.1. Applying the claim K-times, starting with $D_0 = D$, $f_0 = f$ and ending with $D_K = \widetilde{D}$, $f_K = \tilde f$, yield $\tilde f = f - \bar\partial u$ in D, where we set $u = \sum_{j=0}^{K-1} u_j$. Collecting the estimates for f_{j+1} and u_j in each step, we obtain (4.1) and (4.2). Clearly $\tilde f$ and u are linear in f. ☐

Lemma 4.2. *There exists a strictly q-concave domain with smooth boundary $D' \Subset \mathbb{C}^n$ satisfying*

$$D \Subset D' \Subset \widetilde{D}.$$

Proof. Let V_2 be a neighborhood of D such that $V_2 \Subset V_1$ and for $\tau > 0$ we define $D_\tau := \{z \in D \cup V_2 |\, \rho_1(z) > \tau, \dots, \rho_N(z) > \tau\}$. Recall that D is defined by the $\mathcal{C}^d$-functions $\rho_1, \dots, \rho_N$. For each $\beta > 0$, let χ_β be a fixed non-negative real C^∞ function on $\mathbb{R}$ such that, for all $x \in \mathbb{R}$, $\chi_\beta(x) = \chi_\beta(-x)$, $|x| \le \chi_\beta(x) \le |x| + \beta$, $|\chi'_\beta| \le 1$, $\chi''_\beta \ge 0$ and $\chi_\beta(x) = |x|$ if $|x| \ge \frac{\beta}{2}$. Moreover, we assume that $\chi'_\beta(x) > 0$ if $x > 0$ and $\chi'_\beta(x) < 0$ if $x < 0$. We define as in [9, Definition 4.12] $\max_\beta(t, s) = \frac{t+s}{2} + \chi_\beta(\frac{t-s}{2}), t, s \in \mathbb{R}$, and $\varphi_1 = \rho_1$, $\varphi_2 = \max_\beta(\rho_1, \rho_2)$, $\dots$, $\varphi_N = \max_\beta(\varphi_{N-1}, \rho_N)$. Then it is easy to compute that the Levi form of φ_N has at least $q+1$ negative eigenvalues at each point in U. For $\tau > 0$ we can choose positive numbers $\beta = \frac{\tau}{2(N+1)}$, $\gamma = \frac{\tau}{2}$ small enough and $V_3 \Subset V_2$ such that

$$D \Subset D^* = \{z \in D \cup V_3 |\, \varphi_N(z) - \gamma > 0\} \Subset D_\tau.$$

then D^* is a strictly q-concave domain. According to [9, Theorem 6.6], there exists a strictly q-concave domain D' with smooth boundary such that $D \Subset D' \Subset D^*$. Choose τ small enough to get $D_\tau \Subset \widetilde{D}$. $\square$

Let $f \in \mathcal{C}^k_{n,s}(\overline{D})$ be a $\overline{\partial}$-closed form. Let $\widetilde{D}$, $\tilde{f}$ and u as in Lemma 4.1. Let D' be given as in Lemma 4.2 and set $f_1 = \tilde{f}|_{D'}$. It follows from [18, Theorem 2] that there exists $\eta \in \mathcal{C}^{k-\epsilon}_{n,s-1}(\overline{D})$ such that $\bar{\partial}\eta = f_1$ on D and $\|\eta\|_{k-\epsilon,\overline{D}} \leq C_{k,\epsilon}\|f_1\|_{k-\epsilon,D'}$. Then we have $f = \bar{\partial}(u+\eta)$. The form $g = u+\eta$ is a global solution that satisfies the $\mathcal{C}^k$-estimates (1.1) of Theorem 1.3.

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