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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR NONLOCAL ELLIPTIC PROBLEMS

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Abstract. This article concerns the existence and multiplicity solutions for a class of p-Kirchhoff type equations with Neumann boundary conditions. Our technical approach is based on variational methods.

1. Introduction

In this work, we study the existence and multiplicity of solutions for the nonlocal elliptic problem under Neumann boundary condition:

$$\begin{gathered}\Big[M\Big(\int_\Omega (|\nabla u|^p + a(x)|u|^p)dx\Big)\Big]^{p-1}\Big(-\Delta_p u + a(x)|u|^{p-2}u\Big) = \lambda f(x,u) \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $p > N$, Ω is a nonempty bounded open subset of $\mathbb{R}^N$ with a boundary of class C^1, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative, $a \in L^\infty(\Omega)$, with $\operatorname{ess\,inf}_\Omega a \geq 0$, $a \neq 0$, $\lambda \in (0,\infty)$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$ are two functions that satisfy conditions which will be stated later.

The problem (1.1)) is related to the stationary problem of a model introduced by Kirchhoff [9]. More precisely, Kirchhoff introduced a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Latter (1.2) was developed to form

$$u_{tt} - M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u) \quad \text{in } \Omega. \tag{1.3}$$

After that, many authors studied the following nonlocal elliptic boundary value problem

$$-M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

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Problems like (1.4) can be used for modeling several physical and biological systems where u describes a process which depends on the average of it self, such as the population density , see [1]. Many interesting results for problems of Kirchhoff type were obtained and we refer to [1, 2, 3, 7, 8, 10] and references therein for an overview on these subjects.

The main purpose of the present paper is to establish the existence of at least one solution and, as a consequence, existence results of two and three solutions for the nonlocal problem (1.1), by adopting the framework of Bonanno and Sciammetta [4].

2. Preliminaries and basic notation

Our main tools are two consequences of a local minimum theorem [5, Theorem 3.1] which are recalled below. Given X a set and two functionals $\Phi, \Psi : X \to \mathbb{R}$, put

$$\beta(r_1, r_2) = \inf_{v\in\Phi^{-1}((r_1,r_2))} \frac{\sup_{u\in\Phi^{-1}((r_1,r_2))} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}, \tag{2.1}$$

$$\rho_2(r_1, r_2) = \sup_{v\in\Phi^{-1}((r_1,r_2))} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}((-\infty,r_1])} \Psi(u)}{\Phi(v) - r_1}, \tag{2.2}$$

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, and

$$\rho(r) = \sup_{v\in\Phi^{-1}((r,+\infty))} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u)}{\Phi(v) - r}, \tag{2.3}$$

for all $r \in \mathbb{R}$.

Theorem 2.1 ([5, Theorem 5.1]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho_2(r_1, r_2), \tag{2.4}$$

where β and ρ_2 are given by (2.1) and (2.2). Then, for each $\lambda \in \big(\frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\big)$ there is $u_{0,\lambda} \in \Phi^{-1}((r_1, r_2))$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}((r_1, r_2))$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.2 ([5, Theorem 5.3]). *Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that*

$$\rho(r) > 0, \tag{2.5}$$

where ρ is given by (2.3) and for each $\lambda > 1/\rho(r)$ the function $I_\lambda = \Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda > 1/\rho(r)$ there is $u_{0,\lambda} \in \Phi^{-1}((r, +\infty))$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}((r, +\infty))$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorems 2.1 and 2.2 are consequences of a local minimum theorem [5, Theorem 3.1] which is a more general version of the Ricceri Variational Principle (see [14]).

Let X be the Sobolev space $W^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \Big(\int_\Omega (|\nabla u|^p + a(x)|u|^p)dx \Big)^{1/p}.$$

Let

$$k := \sup_{u \in X \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u(x)|}{\|u\|}. \tag{2.6}$$

Since $p > N$, X is compactly embedded in $C^0(\overline{\Omega})$, so that $k < \infty$. We have

$$|u(x)| \le k\|u\| \quad \text{for all } x \in \Omega,\ u \in X. \tag{2.7}$$

Therefore, taking $u \equiv 1$ in (2.7),

$$k^p \|a\|_1 \ge 1, \quad \text{where } \|a\|_1 = \int_\Omega |a(x)|dx.$$

We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is L^1-Carathéodory; that is, $x \mapsto f(x,t)$ is measurable for every $t \in \mathbb{R}$, $t \mapsto f(x,t)$ is continuous for almost every $x \in \Omega$ and for all $s > 0$ there is $l_s \in L^1(\Omega)$ such that

$$\sup_{|t| \le s} |f(x,t)| \le l_s(x) \quad \text{for a.e. } x \in \Omega,$$

and $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function with the following condition:

(M0) $m_0 := \inf_{t \ge 0} M(t) > 0$.

We say that $u \in X$ is a weak solution of problem (1.1) if

$$[M(\|u\|^p)]^{p-1} \int_\Omega \big(|\nabla u|^{p-2}\nabla u \nabla v + a(x)|u|^{p-2}uv\big)\,dx - \lambda \int_\Omega f(x,u)v dx = 0,$$

for all $v \in X$.

We introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$, defined by

$$\Phi(u) = \frac{1}{p}\widehat{M}(\|u\|^p), \quad \Psi(u) = \int_\Omega F(x,u)dx, \tag{2.8}$$

for all $u \in X$, where

$$\widehat{M}(t) = \int_0^t [M(s)]^{p-1} ds \ \text{ for all } t \ge 0,$$

$$F(x,\xi) = \int_0^\xi f(x,s)ds \ \text{ for all } (x,\xi) \in \Omega \times \mathbb{R}.$$

It is well known that Φ and Ψ are well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at point $u \in X$ are given by

$$\langle \Phi'(u), v\rangle = [M(\|u\|^p)]^{p-1} \int_\Omega \big(|\nabla u|^{p-2}\nabla u \nabla v + a(x)|u|^{p-2}uv\big)\,dx$$

$$\langle \Psi'(u), v\rangle = \int_\Omega f(x,u)v dx,$$

for all $v \in X$. Moreover, Ψ' is compact.

Proposition 2.3. *Assume that* (M0) *holds. Then*

(i) Φ *is sequentially weakly lower semicontinuous;*

(ii) Φ *is coercive;*
(iii) $\Phi' : X \to X^*$ *is strictly monotone;*
(iv) Φ' *is of type* (S_+), *i.e. if* $u_n \rightharpoonup u$ *in* X *and*

$$\overline{\lim}_{n\to+\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle = 0,$$

then $u_n \to u$ *in* X*;*
(v) Φ' *admits a continuous inverse on* X^*.

Proof. (i) Let $u_n \rightharpoonup u$ weakly in X. By the weakly lower semicontinuity of norm, it follows that

$$\|u\| \le \liminf_{n\to+\infty} \|u_n\|.$$

In view of the continuity and monotonicity of $\widehat{M}$, we deduce that

$$\widehat{M}\left(\|u\|^p\right) \le \widehat{M}\Big(\liminf_{n\to+\infty}\|u_n\|^p\Big) \le \liminf_{n\to+\infty} \widehat{M}\left(\|u_n\|^p\right),$$

and hence Φ is sequentially weakly lower semicontinuous.

(*ii*) Thanks to (M0), we have

$$\Phi(u) = \frac{1}{p}\widehat{M}\left(\|u\|^p\right) \ge \frac{m_0^{p-1}}{p}\|u\|^p. \tag{2.9}$$

So, Φ is coercive.

(iii) Consider the functional $T : X \to \mathbb{R}$, defined by

$$T(u) = \int_\Omega \left(|\nabla u|^p + a(x)|u|^p\right) dx \quad \text{for all } u \in X,$$

whose Gâteaux derivative at point $u \in X$ is given by

$$\langle T'(u), v\rangle = p\int_\Omega \left(|\nabla u|^{p-2}\nabla u\nabla v + a(x)|u|^{p-2}uv\right) dx, \quad \text{for all } v \in X,$$

Taking into account [15, (2.2)] for $p > 1$ there exists a positive constant C_p such that

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \ge \begin{cases} C_p|x-y|^p & \text{if } p \ge 2\\ C_p\frac{|x-y|^2}{(|x|+|y|)^{p-2}},\ (x,y) \ne (0,0) & \text{if } 1 < p < 2, \end{cases} \tag{2.10}$$

for all $x, y \in \mathbb{R}^N$. Therefore,

$$\begin{aligned}\langle T'(u) - T'(v), u - v\rangle &\ge \begin{cases} C_p\int_\Omega \left(|\nabla u - \nabla v|^p + a(x)|u-v|^p\right) dx & \text{if } p \ge 2\\ C_p\int_\Omega \left(\frac{|\nabla u-\nabla v|^2}{(|\nabla u|+|\nabla v|)^{2-p}} + \frac{a(x)|u-v|^2}{(|u|+|v|)^{2-p}}\right) dx & \text{if } 1 < p < 2 \end{cases}\\ &> 0,\end{aligned}$$

for all $u \ne v \in X$, which means that T' is strictly monotone. So, by [16, Prop. 25.10], T is strictly convex. Moreover, since M is nondecreasing, $\widehat{M}$ is convex in $[0, +\infty[$. Thus, for every $u, v \in X$ with $u \ne v$, and every $s, t \in (0, 1)$ with $s + t = 1$, one has

$$\widehat{M}(T(su + tv)) < \widehat{M}(sT(u) + tT(v)) \le s\widehat{M}(T(u)) + t\widehat{M}(T(v)).$$

This shows Φ is strictly convex, and, as already said, that Φ' is strictly monotone.

(iv) From (iii), if $u_n \rightharpoonup u$ in X and $\limsup_{n\to+\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle = 0$, then

$$\lim_{n\to+\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle = 0,$$

and so,

$$\lim_{n\to+\infty}\langle\Phi'(u_n),u_n-u\rangle=0, \tag{2.11}$$

that is,

$$\lim_{n\to+\infty}[M(\|u_n\|^p)]^{p-1}\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)+a(x)|u_n|^{p-2}u_n(u_n-u)dx=0. \tag{2.12}$$

Since (u_n) is bounded in X and M is continuous, up to subsequence, there is $t_0\geq 0$ such that

$$M(\|u_n\|^p)\to M(t_0^p)\geq m_0,\ \text{ as } n\to+\infty.$$

This and (2.12) imply

$$\lim_{n\to+\infty}\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)+a(x)|u_n|^{p-2}u_n(u_n-u)dx=0. \tag{2.13}$$

In a same way,

$$\lim_{n\to+\infty}\int_\Omega|\nabla u|^{p-2}\nabla u\nabla(u_n-u)+a(x)|u|^{p-2}u(u_n-u)dx=0. \tag{2.14}$$

Now, by using again inequality (2.10), we obtain by (2.13) and (2.14),

$$\begin{aligned}o_n(1)&=\int_\Omega\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\right)\nabla(u_n-u)dx\\&\quad+\int_\Omega a(x)\left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)(u_n-u)dx\\&\geq\begin{cases}C_p\int_\Omega\left(|\nabla u_n-\nabla u|^p+a(x)|u_n-u|^p\right)dx & \text{if } p\geq 2\\ C_p\int_\Omega\left(\frac{|\nabla u_n-\nabla u|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}}+\frac{a(x)|u_n-u|^2}{(|u_n|+|u|)^{2-p}}\right)dx & \text{if } 1<p<2.\end{cases}\end{aligned} \tag{2.15}$$

If $p\geq 2$, we have

$$\lim_{n\to+\infty}\int_\Omega\left(|\nabla u_n-\nabla u|^p+a(x)|u_n-u|^p\right)dx=0.$$

If $1<p<2$, by Hölder's inequality, it follows that By applying Hölder's inequality, we obtain

$$\begin{aligned}\int_\Omega|\nabla u_n-\nabla u|^pdx&\leq\Big(\int_\Omega\frac{|\nabla u_n-\nabla u|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}}dx\Big)^{p/2}\Big(\int_\Omega(|\nabla u_n|+|\nabla u|)^pdx\Big)^{\frac{2-p}{2}}\\&\leq\Big(\int_\Omega\frac{|\nabla u_n-\nabla u|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}}dx\Big)^{p/2}(\|u_n\|+\|u\|)^{\frac{2-p}{2}p}\\&\leq C\Big(\int_\Omega\frac{|\nabla u_n-\nabla u|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}}dx\Big)^{p/2}.\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\int_\Omega a(x)|u_n-u|^pdx&\leq\Big(\int_\Omega\frac{a(x)|u_n-u|^2}{(|u_n|+|u|)^{2-p}}dx\Big)^{p/2}\Big(\int_\Omega a(x)\left(|u_n|+|u|\right)^p dx\Big)^{\frac{2-p}{2}}\\&\leq\Big(\int_\Omega\frac{a(x)|u_n-u|^2}{(|u_n|+|u|)^{2-p}}dx\Big)^{p/2}(\|u_n\|+\|u\|)^{\frac{2-p}{2}p}\\&\leq C\Big(\int_\Omega\frac{a(x)|u_n-u|^2}{(|u_n|+|u|)^{2-p}}dx\Big)^{p/2}.\end{aligned} \tag{2.17}$$

From (2.15)-(2.17), it follows that

$$\begin{aligned}\|u_n-u\|^2&=\Big(\int_\Omega(|\nabla u_n-\nabla u|^p+a(x)|u_n-u|^p)\,dx\Big)^{2/p}\\&\le C'\Big[\Big(\int_\Omega|\nabla u_n-\nabla u|^pdx\Big)^{2/p}+\Big(\int_\Omega a(x)|u_n-u|^pdx\Big)^{2/p}\Big]\\&\le C'C^{2/p}\int_\Omega\Big(\frac{|\nabla u_n-\nabla u|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}}+\frac{a(x)|u_n-u|^2}{(|u_n|+|u|)^{2-p}}\Big)dx\\&\le o_n(1).\end{aligned}$$

Therefore, in both cases we have

$$\lim_{n\to+\infty}\|u_n-u\|=0,$$

and this completes the proof of (iv).

(v) Since Φ' is a strictly monotone operator in X, Φ' is an injection. For $u\in X$ with $\|u\|>1$, we have

$$\frac{\langle\Phi'(u),u\rangle}{\|u\|}=\frac{[M(\|u\|^p)]^{p-1}\|u\|^p}{\|u\|}\ge m_0^{p-1}\|u\|^{p-1},$$

therefore, Φ' is coercive. Clearly Φ' is also demicontinuous. On account of the well-known Minty-Browder theorem [16, Theorem 26A], the operator Φ' is a surjection, and hence the inverse $(\Phi')^{-1}:X^*\to X$ of Φ' exists. It suffices then to show the continuity of $(\Phi')^{-1}$. Let (g_n) be a sequence of X^* such that $g_n\to g$ in X^*. Let $u_n=(\Phi')^{-1}(g_n)$, $u=(\Phi')^{-1}(g)$, then $\Phi'(u_n)=g_n$, $\Phi'(u)=g$. By the coercivity of Φ', we deduces that (u_n) is bounded in X, up to subsequence, we can assume that $u_n\rightharpoonup u$. Since $g_n\to g$,

$$\lim_{n\to+\infty}\langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle=\lim_{n\to+\infty}\langle g_n-g,u_n-u\rangle=0.$$

Since Φ' is of type (S_+), $u_n\to u$, so $(\Phi')^{-1}$ is continuous. □

3. Main results

In this section we present our main results. To be precise, we establish an existence result of at least one solution, Theorem 3.1, which is based on Theorem 2.1, and we point out some consequences, Theorems 3.2, 3.3 and 3.4. Finally, we present an other existence result of at least one solution, Theorem 3.6, which is based in turn on Theorem 2.2.

Given two nonnegative constants c,d with $c\neq k\|a\|^{1/p}d$, put

$$\gamma(c):=\frac{\int_\Omega\max_{|\xi|\le\sigma(c)}F(x,\xi)dx-\int_\Omega F(x,d)dx}{\widehat{M}(\frac{c^p}{k^p})-\widehat{M}(d^p\|a\|_1)},$$

where

$$\sigma(c):=k\Big(\frac{1}{m_0^{p-1}}\widehat{M}\big(\frac{c^p}{k^p}\big)\Big)^{1/p}.$$

Theorem 3.1. *Assume that there exist three constants c_1,c_2,d with $0\le c_1<k\|a\|^{1/p}d<c_2$, such that*

$$\gamma(c_2)<\gamma(c_1).$$

Then, for each $\lambda\in\big(\frac{1}{p\gamma(c_1)},\frac{1}{p\gamma(c_2)}\big)$, problem (1.1) admits at least one nontrivial weak solution $\overline{u}$ such that $\frac{c_1}{k}<\|\overline{u}\|<\frac{c_2}{k}$.

Proof. Let Φ, Ψ be the functionals defined in Section 2. It is well known that they satisfy all regularity assumptions requested in Theorem 2.1 and that the critical points of the functional $\Phi - \lambda\Psi$ in X are exactly the weak solutions of problem (1.1). So, our aim is to verify condition (2.4) of Theorem 2.1. To this end, put

$$r_1 = \frac{1}{p}\widehat{M}\Big(\frac{c_1^p}{k^p}\Big), \quad r_2 = \frac{1}{p}\widehat{M}\Big(\frac{c_2^p}{k^p}\Big), \quad u_0(x) = d \quad \text{for all } x \in \Omega.$$

Clearly $u_0 \in X$,

$$\Phi(u_0) = \frac{1}{p}\widehat{M}(\|u_0\|^p) = \frac{1}{p}\widehat{M}(\|a\|_1 d^p),$$

and

$$\Psi(u_0) = \int_\Omega F(x, u_0)dx = \int_\Omega F(x, d)dx. \tag{3.1}$$

It follows from $c_1 < k\|a\|^{1/p}d < c_2$ and the strict monotonicity of $\widehat{M}$ that

$$\widehat{M}\Big(\frac{c_1^p}{k^p}\Big) < \widehat{M}(\|a\|_1 d^p) < \widehat{M}\Big(\frac{c_2^p}{k^p}\Big),$$

and so

$$r_1 < \Phi(u_0) < r_2. \tag{3.2}$$

Let $u \in X$ such that $u \in \Phi^{-1}((-\infty, r_2))$. By (2.9), one has

$$\frac{m_0^{p-1}}{p}\|u\|^p \le \Phi(u) < r_2.$$

Therefore,

$$\|u\| < \Big(\frac{pr_2}{m_0^{p-1}}\Big)^{1/p}$$

This together with (2.7), yields

$$|u(x)| \le k\|u\| < k\Big(\frac{pr_2}{m_0^{p-1}}\Big)^{1/p} = \sigma(c_2) \quad \text{for all } x \in \Omega. \tag{3.3}$$

So

$$\Psi(u) = \int_\Omega F(x, u)dx \le \int_\Omega \max_{|\xi|\le\sigma(c_2)} F(x, \xi)dx,$$

for all $u \in X$ such that $u \in \Phi^{-1}((-\infty, r_2))$. Thus

$$\sup_{u\in\Phi^{-1}((-\infty,r_2))} \Psi(u) \le \int_\Omega \max_{|\xi|\le\sigma(c_2)} F(x, \xi)dx. \tag{3.4}$$

On the other hand, arguing as before we obtain

$$\sup_{u\in\Phi^{-1}((-\infty,r_1))} \Psi(u) \le \int_\Omega \max_{|\xi|\le\sigma(c_1)} F(x, \xi)dx. \tag{3.5}$$

In view of (3.1)-(3.2) and (3.4)-(3.5), one has

$$\begin{aligned}
\beta(r_1, r_2) &\le \frac{\sup_{u\in\Phi^{-1}((-\infty,r_2))} \Psi(u) - \Psi(u_0)}{r_2 - \Phi(u_0)} \\
&\le p\frac{\int_\Omega \max_{|\xi|\le\sigma(c_2)} F(x, \xi)dx - \int_\Omega F(x, d)dx}{\widehat{M}(\frac{c_2^p}{k^p}) - \widehat{M}(\|a\|_1 d^p)} \\
&= p\gamma(c_2)
\end{aligned}$$

and

$$\begin{aligned}\rho_2(r_1,r_2) &\geq \frac{\Psi(u_0)-\sup_{u\in\Phi^{-1}((-\infty,r_1])}\Psi(u)}{\Phi(u_0)-r_1}\\ &\geq p\frac{\int_\Omega \max_{|\xi|\leq\sigma(c_1)}F(x,\xi)dx-\int_\Omega F(x,d)dx}{\widehat{M}(\frac{c_1^p}{k^p})-\widehat{M}(\|a\|_1 d^p)}\\ &= p\gamma(c_1).\end{aligned}$$

So, by our assumption it follows that

$$\beta(r_1,r_2)<\rho_2(r_1,r_2).$$

Hence, from Theorem 2.1 for each $\lambda\in\big(\frac{1}{p\gamma(c_1)},\frac{1}{p\gamma(c_2)}\big)\subset\big(\frac{1}{\rho_2(r_1,r_2)},\frac{1}{\beta(r_1,r_2)}\big)$, $I_\lambda:=\Phi-\lambda\Psi$ admits at least one critical point $\overline{u}$ such that

$$\widehat{M}\big(\frac{c_1^p}{k^p}\big)<\widehat{M}\big(\|\overline{u}\|^p\big)<\widehat{M}\big(\|\overline{u}\|^p\big).$$

Taking in to account that the function $\widehat{M}$ is increasing, it follows that

$$\frac{c_1}{k}<\|\overline{u}\|<\frac{c_2}{k},$$

and the proof of Theorem 3.1 is achieved. □

Now we point out the following consequence of Theorem 3.1.

Theorem 3.2. *Assume that there exist two positive constants c,d, with $k\|a\|^{1/p}d<c$, such that*

$$\frac{\int_\Omega \max_{|\xi|\leq\sigma(c)}F(x,\xi)dx}{\widehat{M}\big(\frac{c^p}{k^p}\big)}<\frac{\int_\Omega F(x,d)dx}{\widehat{M}(\|a\|_1 d^p)}. \tag{3.6}$$

Then, for each

$$\lambda\in\big(\frac{\widehat{M}(\|a\|_1 d^p)}{p\int_\Omega F(x,d)dx},\frac{\widehat{M}\big(\frac{c^p}{k^p}\big)}{p\int_\Omega \max_{|\xi|\leq\sigma(c)}F(x,\xi)dx}\big),$$

problem (1.1) *admits at least one nontrivial weak solution $\overline{u}$ such that $|\overline{u}(x)|<c$ for all $x\in\Omega$.*

Proof. Our aim is to apply Theorem 3.1. To this end we pick $c_1=0$ and $c_2=c$. From (3.6), one has

$$\begin{aligned}\gamma(c)&=\frac{\int_\Omega \max_{|\xi|\leq\sigma(c)}F(x,\xi)dx-\int_\Omega F(x,d)dx}{\widehat{M}(\frac{c^p}{k^p})-\widehat{M}(\|a\|_1 d^p)}\\ &<\frac{\int_\Omega \max_{|\xi|\leq\sigma(c)}F(x,\xi)dx-\frac{\widehat{M}(\|a\|_1 d^p)}{\widehat{M}(\frac{c^p}{k^p})}\int_\Omega \max_{|\xi|\leq\sigma(c)}F(x,\xi)dx}{\widehat{M}(\frac{c^p}{k^p})-\widehat{M}(\|a\|_1 d^p)}\\ &=\frac{\int_\Omega \max_{|\xi|\leq\sigma(c)}F(x,\xi)dx}{\widehat{M}(\frac{c^p}{k^p})}\\ &<\frac{\int_\Omega F(x,d)dx}{\widehat{M}(\|a\|_1 d^p)}=\gamma(0).\end{aligned}$$

Hence, Theorem (3.1) ensures the existence of weak solution $\overline{u}$ of problem (1.1), such that $\|\overline{u}\|<\frac{c}{k}$, and clearly by (2.7), $|\overline{u}(x)|<c$ for all $x\in\Omega$. □

Now, we point out a previous result when the nonlinear term has separable variables. To be precise, let $\alpha \in L^1(\Omega)$ such that $\alpha(x) \geq 0$ a.e. $x \in \Omega, \alpha \neq 0$, and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Consider the Neumann boundary-value problem

$$\begin{gathered}\Big[M\Big(\int_\Omega \big(|\nabla u|^p + a(x)|u|^p\big)dx\Big)\Big]^{p-1}\big(-\Delta_p u + a(x)|u|^{p-2}u\big) = \lambda\alpha(x)g(u) \quad \text{in } \Omega\\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{3.7}$$

Let

$$G(\xi) := \int_0^\xi g(t)dt \quad \text{for all } \xi \in \mathbb{R}.$$

Theorem 3.3. *Assume that g is nonnegative and there exist two positive constants c, d, with $k\|a\|^{1/p}d < c$, such that*

$$\frac{G(\sigma(c))}{\widehat{M}(\frac{c^p}{k^p})} < \frac{G(d)}{\widehat{M}(\|a\|_1 d^p)}. \tag{3.8}$$

Then, for each $\lambda \in \big(\frac{1}{p\|\alpha\|_1}\frac{\widehat{M}(\|a\|_1 d^p)}{G(d)}, \frac{1}{p\|\alpha\|_1}\frac{\widehat{M}(\frac{c^p}{k^p})}{G(\sigma(c))}\big)$, problem (3.7) admits at least one positive weak solution $\overline{u}$ such that $\overline{u}(x) < c$ for all $x \in \Omega$.

Proof. Put $f(x,t) = \alpha(x)g(t)$ for all $(x,t) \in \Omega \times \mathbb{R}$, thus $F(x,\xi) = \alpha(x)G(\xi)$ for all $(x,\xi) \in \Omega \times \mathbb{R}$. Therefore taking into account that G is nondecreasing, Theorem 3.2 ensures the existence of a nontrivial weak solution $\overline{u}$. We claim that $\overline{u}$ is nonnegative. In fact, let $\overline{u}_- := \max\{-\overline{u}, 0\}$ and setting

$$\Omega^- = \{x \in \Omega : \overline{u}(x) < 0\}.$$

So, taking into account that $\overline{u}$ is a weak solution and $\overline{u}_- \in X$, we have

$$[M(\|\overline{u}\|^p)]^{p-1}\int_{\Omega^-}(|\nabla\overline{u}|^p + a(x)|\overline{u}|^p)dx = \lambda\int_{\Omega^-} f(x,\overline{u})\overline{u}dx \leq 0.$$

Therefore,

$$\int_{\Omega^-}(|\nabla\overline{u}|^p + a(x)|\overline{u}|^p)\,dx = 0.$$

It follows that $\Omega^- = \emptyset$, and hence $u \geq 0$ in Ω. By the strong maximum principle (see, for instance, [12, Theorem 11.1]) the weak solution $\overline{u}$, being nontrivial, is positive and the conclusion is achieved. □

Theorem 3.4. *Assume that g is nonnegative such that*

$$\lim_{t\to 0^+}\frac{g(t)}{t^{p-1}} = +\infty, \tag{3.9}$$

and put

$$\lambda^* = \frac{1}{p\|\alpha\|_1}\sup_{c>0}\frac{\widehat{M}(\frac{c^p}{k^p})}{G(\sigma(c))}.$$

Then, for each $\lambda \in (0, \lambda^*)$, problem (3.7) admits at least one positive weak solution.

Proof. Fix $\lambda \in (0, \lambda^*)$. Then, there exists $c > 0$ such that $\lambda < \frac{1}{p\|\alpha\|_1} \frac{\widehat{M}(\frac{c^p}{k^p})}{G(\sigma(c))}$. By (3.9), one has

$$\lim_{t\to 0^+} \frac{g(t)}{t^{p-1}[M(t^p\|a\|_1)]^{p-1}} = +\infty,$$

and hence, there exists $0 < d < \frac{c}{k\|a\|_1^{1/p}}$ such that

$$\frac{\|a\|_1}{\lambda\|\alpha\|_1} < \frac{g(t)}{t^{p-1}[M(t^p\|a\|_1)]^{p-1}} \quad \text{for all } t \in (0, d).$$

Thus

$$\frac{\|a\|_1}{\lambda\|\alpha\|_1} \int_0^d t^{p-1} \left[M\left(t^p\|a\|_1\right)\right]^{p-1} dt < \int_0^d g(t)dt.$$

Using the change of variables $s = \|a\|_1 t^p$, we get

$$\frac{1}{\lambda p\|\alpha\|_1} \int_0^{\|a\|_1 d^p} [M(s)]^{p-1} ds < G(d),$$

that is,

$$\frac{1}{p\|\alpha\|_1} \frac{\widehat{M}(\|a\|_1 d^p)}{G(d)} < \lambda.$$

Hence, Theorem 3.3 ensures the conclusion. □

Remark 3.5. Let $g : \mathbb{R} \to \mathbb{R}$ such (3.9) holds (that is, without any assumption of sign). By (3.9), there is $\delta > 0$ such that $g(t) > 0$ for all $t \in (0, \delta)$. Then Put

$$\overline{\lambda}_0 := \frac{1}{p\|\alpha\|_1} \sup_{c\in(0,\delta)} \frac{\widehat{M}\left(\frac{c^p}{k^p}\right)}{G(\sigma(c))}.$$

Clearly $\overline{\lambda}_0 \le \lambda^*$, if g is nonnegative. Now, fixed $\lambda \in (0, \overline{\lambda}_0)$ and arguing as in the proof of Theorem 3.4, there are $c \in (0, \delta)$ and $0 < d < \frac{c}{k\|a\|_1^{1/p}}$ such that

$$\frac{1}{p\|\alpha\|_1} \frac{\widehat{M}(\|a\|_1 d^p)}{G(d)} < \lambda < \frac{1}{p\|\alpha\|_1} \frac{\widehat{M}\left(\frac{c^p}{k^p}\right)}{G(\sigma(c))}.$$

Hence, Theorem 3.3 ensures that, for each $\lambda \in (0, \overline{\lambda}_0)$, problem (3.7) admits at least one positive weak solution $\overline{u}(x) < \delta$ for all $x \in \Omega$.

Finally, we also give an application of Theorem 2.2 which we will use in next section to obtain multiple solutions.

Theorem 3.6. *Assume that there exist two constants $\overline{c}, \overline{d}$, with $0 < \overline{c} < k\|a\|_1^{1/p}\overline{d}$, such that*

$$\int_\Omega \max_{|\xi|\le\sigma(\overline{c})} F(x,\xi)dx < \int_\Omega F(x,\overline{d})dx, \tag{3.10}$$

$$\limsup_{|\xi|\to+\infty} \frac{F(x,\xi)}{|\xi|^p} \le 0 \quad \textit{uniformly in } x. \tag{3.11}$$

Then, for each $\lambda > \overline{\lambda}$, where

$$\overline{\lambda} = \frac{\widehat{M}\left(\frac{\overline{c}^p}{k^p}\right) - \widehat{M}(\|a\|_1\overline{d}^p)}{p\left(\int_\Omega \max_{|\xi|\le\sigma(\overline{c})} F(x,\xi)dx - \int_\Omega F(x,\overline{d})dx\right)},$$

problem (1.1) admits at least one nontrivial weak solution $\overline{u}$ such that $\|u\| > \overline{c}/k$.

Proof. The functionals Φ and Ψ given by (2.8) satisfy all regularity assumptions requested in Theorem 2.2. By (3.11), for every $\varepsilon > 0$ one has

$$F(x,\xi) \le \varepsilon|\xi|^p + l_\varepsilon(x) \quad \text{for all } (x,\xi) \in \Omega \times \mathbb{R},$$

where $l_\varepsilon \in L^1(\Omega)$. This implies that

$$\int_\Omega F(x,u)dx \le \varepsilon C_1 \|u\|^p + \int_\Omega l_\varepsilon(x)dx \quad \text{for all } u \in X,$$

where C_1 is a Sobolev constant. Therefore,

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u) \ge \Big(\frac{m_0^{p-1}}{p} - C_1\varepsilon\Big)\|u\|^p - \int_\Omega l_\varepsilon(x)dx.$$

So, choosing ε small enough we deduce that I_λ is coercive. To apply Theorem 2.2, it suffices to verify condition (2.5). Indeed, put

$$r = \frac{1}{p}\widehat{M}\Big(\frac{\overline{c}^p}{k^p}\Big), \quad u_0(x) = \overline{d} \quad \text{for all } x \in \Omega.$$

Arguing as in the proof of Theorem 3.1, we obtain

$$\rho(r) \ge p\frac{\int_\Omega \max_{|\xi|\le\sigma(\overline{c})} F(x,\xi)dx - \int_\Omega F(x,\overline{d})dx}{\widehat{M}(\frac{\overline{c}^p}{k^p}) - \widehat{M}\big(\|a\|_1\overline{d}^p\big)}.$$

So, from our assumption it follows that $\rho(r) > 0$. Hence, in view of Theorem 2.2 for each $\lambda > \overline{\lambda}$, I_λ admits at least one local minimum $\overline{u}$ such that

$$\widehat{M}\Big(\frac{\overline{c}^p}{k^p}\Big) < \widehat{M}\big(\|\overline{u}\|^p\big).$$

Therefore,

$$\frac{\overline{c}}{k} < \|\overline{u}\|,$$

and our conclusion is achieved. □

4. Applications

The main aim of this section is to present multiplicity results. First, as a consequence of Theorems 3.2, and 3.6 the following theorem of the existence of three solutions is obtained and its consequence for the nonlinearity with separable variables is presented.

Theorem 4.1. *Assume that* (3.11) *holds. Moreover, assume that there exist four positive constants $c, d, \overline{c}, \overline{d}$, with $k\|a\|_1^{1/p}d < c \le \overline{c} < k\|a\|_1^{1/p}\overline{d}$, such that* (3.6), (3.10) *and*

$$\frac{\int_\Omega \max_{|\xi|\le\sigma(c)} F(x,\xi)dx}{\widehat{M}\left(\frac{c^p}{k^p}\right)} < \frac{\int_\Omega \max_{|\xi|\le\sigma(\overline{c})} F(x,\xi)dx - \int_\Omega F(x,\overline{d})dx}{\widehat{M}\left(\frac{\overline{c}^p}{k^p}\right) - \widehat{M}\big(\|a\|_1\overline{d}^p\big)} \tag{4.1}$$

are satisfied. Then, for each

$$\lambda \in \Lambda := \Big(\max\Big\{\overline{\lambda}, \frac{\widehat{M}(\|a\|_1 d^p)}{p\int_\Omega F(x,d)dx}\Big\}, \frac{\widehat{M}(\frac{c^p}{k^p})}{p\int_\Omega \max_{|\xi|\le\sigma(c)} F(x,\xi)dx}\Big),$$

with $\overline{\lambda}$ is given in Theorem 3.6, problem (1.1) *admits at least three weak solutions.*

Proof. By assumptions, we see that $\Lambda \neq \emptyset$. Fix $\lambda \in \Lambda$. Theorem 3.2 ensures a nontrivial weak solution $\overline{u}_1$ such that $\|\overline{u}_1\| < \frac{c}{k}$ which is a local minimum for I_λ, as well as Theorem 3.6 guarantees a nontrivial weak solution $\overline{u}_2$ such that $\|\overline{u}_2\| > \frac{\overline{c}}{k}$ which is a local minimum for I_λ. Hence I_λ has two different local minimum points. By standard arguments, we see that I_λ satisfies the Palais-Smale condition. Hence, the theorem given by Pucci and Serrin [11, Corollary 1] ensures the third weak solution and the proof is achieved. $\square$

Theorem 4.2. *Assume that g is a nonnegative function such that*

$$\limsup_{\xi \to 0^+} \frac{G(\xi)}{\widehat{M}\big(\|a\|_1 \xi^p\big)} = +\infty, \tag{4.2}$$

$$\limsup_{\xi \to +\infty} \frac{G(\xi)}{\xi^p} = 0. \tag{4.3}$$

Further, assume that there exist two positive constants $\overline{c}, \overline{d}$, with $\overline{c} < k\left(\|a\|\right)^{1/p} \overline{d}$, such that

$$\frac{G(\sigma(\overline{c}))}{\widehat{M}\left(\frac{\overline{c}^p}{k^p}\right)} < \frac{G(\overline{d})}{\widehat{M}\big(\|a\|_1 \overline{d}^p\big)}. \tag{4.4}$$

Then, for each $\lambda \in \Big(\frac{1}{p\|\alpha\|_1} \frac{\widehat{M}\big(\|a\|_1 \overline{d}^p\big)}{G(\overline{d})}, \frac{1}{p\|\alpha\|_1} \frac{\widehat{M}\big(\frac{\overline{c}^p}{k^p}\big)}{G(\sigma(\overline{c}))}\Big)$, problem (3.7) admits at least three nonnegative weak solutions.

Proof. Put $f(x,t) = \alpha(x) g(t)$ for all $(x,t) \in \Omega \times \mathbb{R}$, then $F(x,\xi) = \alpha(x) G(\xi)$ for all $(x,\xi) \in \Omega \times \mathbb{R}$. By (4.3) and taking into account that g is nonnegative, it is easy to verify condition (3.11). Choosing $c = \overline{c}$, condition (4.2) ensures the existence of positive constant d, with $d < \frac{c}{k\|a\|_1^{1/p}}$ such that

$$\frac{G(\sigma(c))}{\widehat{M}\left(\frac{c^p}{k^p}\right)} < \frac{G(\overline{d})}{\widehat{M}\big(\|a\|_1 \overline{d}^p\big)} < \frac{G(d)}{\widehat{M}(\|a\|_1 d^p)}. \tag{4.5}$$

This implies (3.6). Since $\frac{\overline{c}^p}{k^p} < \|a\|_1 \overline{d}^p$ and the function $\widehat{M}$ is increasing,

$$\widehat{M}\Big(\frac{\overline{c}^p}{k^p}\Big) < \widehat{M}\big(\|a\|_1 \overline{d}^p\big).$$

Therefore, from (4.4), we deduce $G(\sigma(\overline{c})) < G(\overline{d})$, and hence (3.10) follows. Using again (4.4), one has

$$\begin{aligned}
\frac{\int_\Omega \max_{|\xi| \le \sigma(\overline{c})} F(x,\xi)dx - \int_\Omega F(x,\overline{d})dx}{\widehat{M}\left(\frac{\overline{c}^p}{k^p}\right) - \widehat{M}\big(\|a\|_1 \overline{d}^p\big)}
&= \|\alpha\|_1 \frac{G(\sigma(\overline{c})) - G(\overline{d}))}{\widehat{M}\left(\frac{\overline{c}^p}{k^p}\right) - \widehat{M}\big(\|a\|_1 \overline{d}^p\big)} \\
&> \|\alpha\|_1 \frac{G(\sigma(\overline{c}))\Big(1 - \frac{\widehat{M}\big(\|a\|_1 \overline{d}^p\big)}{\widehat{M}\big(\frac{\overline{c}^p}{k^p}\big)}\Big)}{\widehat{M}\left(\frac{\overline{c}^p}{k^p}\right) - \widehat{M}\big(\|a\|_1 \overline{d}^p\big)} \\
&= \|\alpha\|_1 \frac{G(\sigma(\overline{c}))}{\widehat{M}\left(\frac{\overline{c}^p}{k^p}\right)} \\
&= \frac{\int_\Omega \max_{|\xi| \le \sigma(\overline{c})} F(x,\xi)dx}{\widehat{M}\left(\frac{\overline{c}^p}{k^p}\right)}
\end{aligned}$$

$$= \frac{\int_\Omega \max_{|\xi|\le\sigma(c)} F(x,\xi)dx}{\widehat{M}\left(\frac{c^p}{k^p}\right)},$$

so, (4.1) holds. Also, by (4.5) one has

$$\begin{aligned}
\overline{\lambda} &= \frac{\widehat{M}\left(\frac{\overline{c}^p}{k^p}\right) - \widehat{M}\big(\|a\|_1 \overline{d}^p\big)}{p\left(\int_\Omega \max_{|\xi|\le\sigma(\overline{c})} F(x,\xi)dx - \int_\Omega F(x,\overline{d})dx\right)} \\
&< \frac{1}{p\|\alpha\|_1} \frac{\widehat{M}\big(\|a\|_1 \overline{d}^p\big)}{G(\overline{d})}.
\end{aligned}$$

Therefore,

$$\max\Big\{\overline{\lambda}, \frac{\widehat{M}(\|a\|_1 d^p)}{p\|\alpha\|_1 G(d)}\Big\} < \frac{1}{p\|\alpha\|_1} \frac{\widehat{M}\big(\|a\|_1 \overline{d}^p\big)}{G(\overline{d})},$$

thus $\left(\frac{1}{p\|\alpha\|_1} \frac{\widehat{M}\big(\|a\|_1 \overline{d}^p\big)}{G(\overline{d})}, \frac{1}{p\|\alpha\|_1} \frac{\widehat{M}\left(\frac{\overline{c}^p}{k^p}\right)}{G(\sigma(\overline{c}))}\right) \subset \Lambda$, and hence, Theorem 4.1 ensures three nonnegative weak solutions. □

Remark 4.3. If $g(0) \neq 0$, Theorem 4.2 ensures three positive weak solutions (see proof of Theorem 3.3).

Remark 4.4. In applying Theorem 3.4, it is enough to known an explicit upper bound for constant k defined in (2.6)). If Ω is convex, we have the following estimate (see [6, Remark 1])

$$k \le 2^{\frac{p-1}{p}} \max\Big\{\frac{1}{\|a\|_1^{1/p}}, \frac{\operatorname{diam}(\Omega)}{N^{1/p}} \Big(\frac{p-1}{p-N} \operatorname{meas}(\Omega)\Big)^{\frac{p-1}{p}} \frac{\|a\|_\infty}{\|a\|_1}\Big\}. \tag{4.6}$$

Example. Let $b_0, b_1 > 0$. Due to Theorem 3.4, for each

$$\lambda \in \Big(0, \frac{1}{2} \frac{b_0 + \frac{b_1}{2}}{\frac{1}{3}\big(2 + \frac{b_1}{b_0}\big)^{3/2} + \big(2 + \frac{b_1}{b_0}\big)^{1/2}}\Big),$$

the Neumann problem

$$\begin{gathered}
\Big(b_0 + b_1 \int_0^1 (|\nabla u|^2 + |u|^2)dx\Big)(-u'' + u) = \lambda\left(u^2 + 1\right) \quad \text{in } (0,1) \\
u'(0) = u'(1),
\end{gathered} \tag{4.7}$$

admits at least one positive weak solution. In fact, set $M(t) = b_0 + b_1 t$ for all $t \ge 0$, then $\widehat{M}(t) = b_0 t + \frac{b_1}{2} t^2$ for all $t \ge 0$ and (M0) holds. Observe that

$$\lim_{u\to 0^+} \frac{g(u)}{u} = \lim_{u\to 0^+} \frac{u^2+1}{u} = +\infty.$$

Moreover, one has

$$\sigma(k) = k\Big(\frac{1}{b_0}\widehat{M}(1)\Big)^{1/2} = k\Big(1 + \frac{b_1}{2b_0}\Big)^{1/2}$$

Therefore,

$$\begin{aligned}
\lambda^* &= \frac{1}{p\|\alpha\|_1} \sup_{c>0} \frac{\widehat{M}\left(\frac{c^p}{k^p}\right)}{G(\sigma(c))} \\
&\ge \frac{1}{p\|\alpha\|_1} \frac{\widehat{M}(1)}{G(\sigma(k))}
\end{aligned}$$

$$= \frac{1}{2} \frac{b_0 + \frac{b_1}{2}}{G\big(k\big(1 + \frac{b_1}{2b_0}\big)^{1/2}\big)}$$

Taking into account that estimate (4.6) implies $k \leq \sqrt{2}$, we deduce that

$$\lambda^* \geq \frac{1}{2} \frac{b_0 + \frac{b_1}{2}}{G\big(\big(2 + \frac{b_1}{b_0}\big)^{1/2}\big)} = \frac{1}{2} \frac{b_0 + \frac{b_1}{2}}{\frac{1}{3}\big(2 + \frac{b_1}{b_0}\big)^{3/2} + \big(2 + \frac{b_1}{b_0}\big)^{1/2}},$$

and Theorem 3.4 ensures the conclusion.

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References

[1] C. O. Alves, F. J. S. A. Corrêa, T. F. MA; *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl, 49 (2005), 85-93.
[2] G. Anello; *A uniqueness result for a nonlocal equation of Kirchhoff type and some related open problems*, J. Math. Anal. Appl. 373 (2011) 248-251.
[3] G. Autuori, P. Pucci, M. C. Salvatori; *Asymptotic Stability for anisotropic Kirchhoff system*, J. Math. Anal. Appl. 352 (2009) 149-165.
[4] G. Bonanno, A. Sciammetta; *Existence and multiplicity results to Neumann problems for elliptic equations involving the p-Laplacian*, J. Math. Anal. Appl. 390 (2012) 59-67.
[5] G. Bonanno; *A critical point theorem via the Ekeland variational principle*, Nonlinear Anal. 75 (2012) 2992-3007.
[6] G. Bonanno, P. Candito; *Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian*, Arch. Math. (Basel) 80 (2003) 424-429.
[7] M. Massar, A. Hamydy, N. Tsouli; *Existence of solutions for p-Kirchhoff type problems with critical exponent*, Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 105, pp. 1-8.
[8] X. He, W. Zou; *Infinitely many positive solutions for Kirchhoff-type problems*, Nonlinear Analysis. Theory, Methods and Applications, 70, 3, (2009), 1407-1414.
[9] G. Kirchhoff; *Mechanik*, Teubner, leipzig, Germany, 1883.
[10] A. Mao, Z. Zhang; *Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition*, Nonlinear Analysis. Theory, Methods Applications, 70, 3, (2009), 1275-1287.
[11] P. Pucci, J. Serrin; *A mountain pass theorem*, J. Differential Equations 63 (1985) 142-149.
[12] P. Pucci, J. Serrin; *The strong maximum principle revisited*, J. Differential Equations 196 (2004) 1-66.
[13] B. Ricceri; *On an elliptic Kirchhoff-type problem depending on two parameters*, J. Global Optim. 46 (2010) 543-549.
[14] B. Ricceri; *A general variational principle and some of its applications*, J. Comput. Appl. Math. 113 (2000) 401410.
[15] J. Simon; *Régularité de la solution d'une équationnon linéaire dans* $\mathbb{R}^N$, in Journées d'Analyse Non Linéaire (Proc. Conf., Besanon, 1977), in: Lecture Notes in Math., vol.665, Springer, Berlin, 1978, pp. 205-227.
[16] E. Zeidler; *Nonlinear Functional Analysis and its Applications*, vol. II/B, Berlin, Heidelberg, New York, 1985.

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