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GREEN'S FUNCTION FOR TWO-INTERVAL STURM-LIOUVILLE PROBLEMS

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Dedicated to John W. Neuberger on his 80th birthday

Abstract. We construct the Green's function and the characteristic function for two-interval regular Sturm-Liouville problems with separated and coupled, self-adjoint and non-self-adjoint, boundary conditions. In the self-adjoint case these problems may have boundary conditions requiring jump discontinuities of the eigenfunctions or their derivatives. Such conditions are known by various names including transmission and interface conditions and have been studied by many authors in the recent literature.

1. Introduction

We construct the Green's function for two-interval regular self-adjoint and non-self-adjoint Sturm-Liouville problems. The two intervals may be disjoint, overlap, or be identical.

In recent years Sturm-Liouville problems with boundary conditions requiring discontinuous eigenfunctions or discontinuous derivatives of eigenfunctions have been studied by many authors. Such conditions are known by various names including: transmission conditions [1, 2, 9, 10, 11, 27, 28], interface conditions [8, 25, 32], discontinuous conditions [5, 6, 10, 14, 15], multi-point conditions [7, 21, 31], point interactions (in the Physics literature), conditions on trees, graphs or networks [4, 13, 23, 24], etc. For an informative survey of such problems arising in applications including an extensive bibliography and historical notes, see Pokornyi-Borovskikh [23] and Prokornyi-Pryadiev [24].

As a special case our construction applies to such problems. It is modeled on a construction of Neuberger [12] for the one interval case. Neuberger's construction differs from the usual one found in textbooks and in most of the literature, in that the discontinuity of the derivative of the Green's function along the diagonal occurs naturally, in contrast to the usual construction as found, for example, in Coddington and Levinson [3] where this discontinuity is assigned a priori as part of the construction.

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2. Basic theory and notation

In this section we briefly review the basic theory and make some definitions. Although we only use the results for the second order case $n = 2$ we state them for general n since this does not introduce any additional complexity or length. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and denote by $L(J, \mathbb{C})$ the set of complex valued Lebesgue integrable functions on compact interval J of the real line and $AC(J)$ denotes the absolutely continuous complex valued functions on J.

Let $M_{n\times m}(\mathbb{C})$ denote the set of $n \times m$ matrices with complex entries. If $n = m$ we write $M_n(\mathbb{C}) = M_{n\times n}(\mathbb{C})$. Let $M_n(S)$ be the n by n matrices with entries from an arbitrary set S.

We start with some definitions and preliminary lemmas.

Lemma 2.1 (Existence and Uniqueness)**.** *Let $n, m \in \mathbb{N}$. If*

$$P \in M_n(L(J, \mathbb{C})), \tag{2.1}$$

$$F \in M_{n,m}(L(J, \mathbb{C})) \tag{2.2}$$

then every initial value problem

$$Y' = PY + F, \tag{2.3}$$

$$Y(u) = C, \quad u \in J, \quad C \in M_{n,m}(\mathbb{C}) \tag{2.4}$$

has a unique solution defined on all of J. Furthermore, if C, P, F are all real-valued, then there is a unique real valued solution.

Proofs of the two lemmas above can be found in [33]. Let $P \in M_n(L(J))$. From this Lemma we know that for each point u of J there is exactly one matrix solution X of

$$Y' = PY \quad \text{on } J \tag{2.5}$$

satisfying $X(u) = I_n$ where I_n denotes the n by n identity matrix.

Definition 2.2 (Primary fundamental matrix)**.** For each fixed $u \in J$ let $\Phi(\cdot, u)$ be the fundamental matrix of (2.5) satisfying $\Phi(u, u) = I_n$. Note that for each fixed u in J, $\Phi(\cdot, u)$ belongs to $M_n(AC_{loc}(J))$. Furthermore, if J is compact and $P \in M_n(L(J, \mathbb{C}))$ then u can be an endpoint of J and $\Phi(\cdot, u)$ belongs to $M_n(AC(J))$. We note that $\Phi(t, u)$ is invertible for each t, $u \in J$ and $\Phi(t, u) = Y(t)Y^{-1}(u)$ for any fundamental matrix Y of (2.5).

We call Φ the primary fundamental matrix of (2.5). Note that for any constant $n\times m$ matrix C, ΦC is also a solution of $Y' = PY$. If C is a constant nonsingular $n\times n$ matrix then ΦC is a fundamental matrix solution and every fundamental matrix solution has this form. For these and other basic facts, notation and terminology see Chapter 1 in [33].

The next lemma is fundamental in the theory of linear differential equations.

Lemma 2.3 (Variation of Parameters Formula, see [33])**.** *Let J be any compact interval, $P \in M_n(L(J, \mathbb{C}))$ and let $\Phi = \Phi(\cdot, \cdot, P)$ be the primary fundamental matrix of $Y' = PY$ on J. Let $F \in M_{n,m}(L(J, \mathbb{C}))$, $u \in J$ and $C \in M_{n,m}(\mathbb{C})$. Then*

$$Y(t) = \Phi(t, u, P)C + \int_u^t \Phi(t, s, P)F(s)\, ds, \quad t \in J \tag{2.6}$$

is the solution of (2.3), (2.4). Note that $Y \in M_{n,m}(AC(J))$.

3. The Characteristic Function

Next we study the two-interval characteristic function with general, not necessarily self-adjoint, boundary conditions. Let

$$J_r = (a_r, b_r), \quad -\infty < a_r < b_r < \infty,\ r = 1, 2,$$

and assume the coefficients and weight functions satisfy

$$p_r^{-1} = \frac{1}{p_r}, \quad q_r, w_r \in L(J_r, \mathbb{C}),\ r = 1, 2. \tag{3.1}$$

Define differential expressions M_r by

$$M_r y = -(p_r y')' + q_r y \quad \text{on } J_r,\ r = 1, 2. \tag{3.2}$$

Below we use the notation with a subscipt r to denote the $r - th$ interval. The subscript r is sometimes omitted when it is clear from the context. We consider the second order scalar differential equations

$$-(p_r y')' + q_r y = \lambda w_r y \quad \text{on } J_r, \quad r = 1, 2,\ \lambda \in \mathbb{C}, \tag{3.3}$$

together with boundary conditions

$$A_1 Y_1(a_1) + B_1 Y_1(b_1) + A_2 Y_2(a_2) + B_2 Y_2(b_2) = 0, \quad Y_r = \begin{bmatrix} y_r \\ (p_r y_r') \end{bmatrix},\ r = 1, 2. \tag{3.4}$$

Here $A_r, B_r \in M_{4\times 2}(\mathbb{C})$, $r = 1, 2$. From (3.1) and the basic theory of linear ordinary differential equations the boundary condition (3.4) is well defined. Next We comment on the assumption (3.1), equations (3.2) and condition (3.4).

Remark 3.1. It follows from the basic theory that, under condition (3.1), every solution y_r and its quasi-derivative py_r' are continuous on J_r but, $p_r(t)$ and $y_r'(t)$ may not exist for some t in J so we use the notation (py') to indicate that this is a continuous function which cannot, in general, be separated into $p(t)y'(t)$ for all t in J.

Remark 3.2. Note that each of $\frac{1}{p_r}, q_r, w_r$ can be zero not only at some points of J but on subintervals and even the whole interval. If q_r is zero on J then we simply have a restricted class of problems. If $\frac{1}{p_r} = 0$ or $w_r = 0$ on J, then we have a degenerate and uninteresting equation. In the latter case there is no λ dependence and so no need for a Green's function. Kong, Wu and Zettl and Volkmer, Kong, and Zettl [20] found a class of S-L problems where each of $\frac{1}{p}, q, w$ is identically zero on certain subintervals of J and whose spectrum has n eigenvalues for any $n = 1, 2, 3, \dots$. It is for this reason that we do not want to place any unnecessary restrictions on the coefficients. In the classical one-interval self-adjoint case the coefficients $\frac{1}{p}, q, w$ are assumed to be in $L(J, \mathbb{R})$ with $p, w > 0$ a.e. in J and the spectral properties are studied in the Hilbert space $L^2(J, w)$.

Below we will construct the characteristic function whose zeros are precisely the eigenvalues of the two-interval SLP. Let

$$P_r = \begin{bmatrix} 0 & \frac{1}{p_r} \\ q_r & 0 \end{bmatrix}, \quad W_r = \begin{bmatrix} 0 & 0 \\ w_r & 0 \end{bmatrix}. \tag{3.5}$$

Then the scalar equation (3.3) is equivalent to the first-order system

$$Y' = (P_r - \lambda W_r) Y = \begin{bmatrix} 0 & \frac{1}{p_r} \\ q_r - \lambda w_r & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} y \\ p_r y' \end{bmatrix}. \tag{3.6}$$

Note that, given any scalar solution y_r of $-(p_r y')' + q_r y = \lambda w_r y$ on J_r the vector Y_r defined by (3.4) is a solution of the system $Y' = (P_r - \lambda W_r)Y$ on J_r. Conversely, given any vector solution Y_r of system $Y' = (P_r - \lambda W_r)Y$ its top component y_r is a solution of $-(p_r y')' + q_r y = \lambda w_r y$.

Let $\Phi_r(\cdot, u_r, P_r, w_r, \lambda)$ be the primary fundamental matrix of (3.6) and we have

$$\Phi_r' = (P_r - \lambda W_r)\Phi_r \text{ on } J_r, \quad \Phi_r(u_r, u_r, \lambda) = I, \ a_r \le u_r \le b_r, \ \lambda \in \mathbb{C}, \tag{3.7}$$

where I denotes 2 by 2 identity matrix.

Here we use the notation $\Phi_r = \Phi_r(\cdot, u_r, P_r, w_r, \lambda)$ to indicate the dependence of the primary fundamental matrix on these quantities. Since P_r, w_r are fixed here, we simplify it to $\Phi_r(\cdot, u_r, \lambda)$. By (3.1), we have $\Phi(b_r, a_r, \lambda)$ exists.

Define the characteristic function Δ by

$$\begin{aligned} \Delta(\lambda) &= \Delta(a_1, b_1, a_2, b_2, A_1, B_1, A_2, B_2, P_1, P_2, w_1, w_2, \lambda) \\ &= \det[(A_1 + B_1\Phi_1(b_1, a_1, \lambda) \mid A_2 + B_2\Phi_2(b_2, a_2, \lambda))], \quad \lambda \in \mathbb{C}, \end{aligned} \tag{3.8}$$

where $(A_1 + B_1\Phi_1(b_1, a_1, \lambda) \mid A_2 + B_2\Phi_2(b_2, a_2, \lambda))$ denote the 4 by 4 complex matrix whose first two columns are those of $A_1 + B_1\Phi_1(b_1, a_1, \lambda)$, and the second two columns are those of $A_2 + B_2\Phi_2(b_2, a_2, \lambda)$.

Definition 3.3. By a trivial solution of equation $M_r y = \lambda w_r y$ on some interval I_r we mean a solution y_r which is identical zero on I_r and whose quasi-derivative $(p_r y_r')$ is also identically zero on I_r. (I_r may be a subinterval of J_r or it may be the whole interval J_r.) Note that, under the assumptions (3.1), solution y_r might be identically zero on I_r but its quasi-derivative $(p_r y_r')$ might not be identically zero on I_r.

Definition 3.4. By a trivial solution of the two-interval Sturm-Liouville equations ((3.3) we mean a solution $\mathbf{y} = \{y_1, y_2\}$ each of whose components y_r is a trivial solution of equation $M_r y = \lambda w_r y$ on J_r, $r = 1, 2$ i.e. y_r and $(p_r y_r')$ both are identically zero on J_r, $r = 1, 2$.

Definition 3.5. Let (3.1) hold. A complex number λ is called an eigenvalue of the two-interval S-L boundary value problems (BVP) consisting of (3.3) and (3.4) if the two-interval S-L equations (3.3) have a nontrivial solution $\mathbf{y}$ satisfying the boundary conditions (3.4). Such a solution $\mathbf{y}$ is called an eigenfunction of λ. Any multiple of an eigenfunction is also an eigenfunction.

Theorem 3.6. *Let* (3.1) *hold. Then a complex number* λ *is an eigenvalue of the boundary value problems* (3.3), (3.4) *if and only if* $\Delta(\lambda) = 0$.

Proof. If λ is an eigenvalue and $\mathbf{y} = \{y_1, y_2\}$ an eigenfunction of λ, then there exist $C_r \in M_{2\times 1}(\mathbb{C})$, $r = 1, 2$ and at least one of the vectors C_1 and C_2 is nonzero, such that

$$Y_r(t) = \Phi_r(t, a_r, \lambda)C_r. \tag{3.9}$$

Note that $\Phi_r(a_r, a_r, \lambda) = I$, $r = 1, 2$. Substituting (3.9) into the boundary conditions (3.4), we obtain

$$A_1C_1 + B_1\Phi_1(b_1, a_1, \lambda)C_1 + A_2C_2 + B_2\Phi_2(b_2, a_2, \lambda)C_2 = 0. \tag{3.10}$$

Set $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$. Therefore (3.10) can be written as

$$(A_1 + B_1\Phi_1(b_1, a_1, \lambda) \mid A_2 + B_2\Phi_2(b_2, a_2, \lambda))C = 0. \tag{3.11}$$

Since $C \neq 0$, and λ is an eigenvalue of BVP (3.3), (3.4) by assumption, it then follows that

$$\det[(A_1 + B_1\Phi_1(b_1, a_1, \lambda) \mid A_2 + B_2\Phi_2(b_2, a_2, \lambda))] = 0;$$

i.e., $\Delta(\lambda) = 0$.

Conversely, suppose $\Delta(\lambda) = 0$. Then (3.11) has a nontrivial vector $C \in M_{4\times 1}(\mathbb{C})$. We use the notation $C_1 \in M_{2\times 1}(\mathbb{C})$ denotes the vector whose rows are the first two rows of C, and $C_2 \in M_{2\times 1}(\mathbb{C})$ denotes the vector whose rows are the last two rows of C. At least one of the vectors C_1 and C_2 is nontrivial. Solve the initial value problems

$$Y' = (P_r - \lambda W_r)Y \text{ on } J_r, \quad Y_r(a_r) = C_r,\ r = 1, 2.$$

Then

$$Y_r(b_r) = \Phi_r(b_r, a_r, \lambda)Y_r(a_r),$$
$$(A_1 + B_1\Phi_1(b_1, a_1, \lambda))Y_1(a_1) + (A_2 + B_2\Phi_2(b_2, a_2, \lambda))Y_2(a_2) = 0.$$

Therefore, we have that $\mathbf{y} = \{y_1, y_2\}$ is an eigenfunction of the BVP (3.3),(3.4), where y_r is the top component of Y_r, $r = 1, 2$. This shows that λ is an eigenvalue of this BVP. □

4. The Green's Function

Since, as mentioned above, our method of constructing the Green's function - even in the one interval case - is not the standard one generally found in the literature and in textbooks we make it self-contained by presenting the basic theory used in the construction for the benefit of the reader.

Let p_r^{-1}, q_r, w_r satisfy (3.1) and $f_r \in L(J_r, \mathbb{C})$. We consider the two-interval boundary-value problem

$$-(p_r y')' + q_r y = \lambda w_r y + f_r \quad \text{on } J_r = (a_r, b_r),\ r = 1, 2,\ \lambda \in \mathbb{C}, \tag{4.1}$$

$$A_1Y_1(a_1) + B_1Y_1(b_1) + A_2Y_2(a_2) + B_2Y_2(b_2) = 0, \quad Y_r = \begin{bmatrix} y_r \\ p_r y_r' \end{bmatrix},\ r = 1, 2. \tag{4.2}$$

This boundary-value problem is equivalent to the system boundary-value problem

$$Y' = (P_r - \lambda W_r)Y + F_r, \quad A_1Y_1(a_1) + B_1Y_1(b_1) + A_2Y_2(a_2) + B_2Y_2(b_2) = 0, \tag{4.3}$$

where P_r, W_r are defined by (3.5) and

$$F_r = \begin{bmatrix} 0 \\ -f_r \end{bmatrix}$$

Let $\Phi_r = \Phi_r(\cdot, \cdot, \lambda)$ be the primary fundamental matrix of the homogeneous system

$$Y' = (P_r - \lambda W_r)Y. \tag{4.4}$$

Note that

$$\Phi_r(t, u_r, \lambda) = \Phi_r(t, a_r, \lambda)\,\Phi_r(a_r, u_r, \lambda)$$

for $a_r \le t, u_r \le b_r$.

The next theorem is a special case of the well known Fredholm alternative.

Theorem 4.1. *Let* (3.1), (4.1)–(4.4) *hold. Let* $\lambda \in \mathbb{C}$. *Then the following three statements are equivalent:*

(1) *when* $\mathbf{f} = \{f_1, f_2\} = 0$, *i.e.* $f_r = 0$ *on* J_r, $r = 1, 2$, *the two-interval BVP* (4.1)-(4.2) *(and consequently also* (4.3)*) has only the trivial solution.*

(2) *The matrix* $[A_1 + B_1\Phi_1(b_1, a_1, \lambda)|A_2 + B_2\Phi_2(b_2, a_2, \lambda)]$ *has an inverse.*
(3) *For every* $\mathbf{f} = \{f_1, f_2\}$, $f_r \in L(J_r, \mathbb{C})$, $r = 1, 2$, *each of the problems* (4.1)-(4.2) *and* (4.3) *has a unique solution.*

Proof. We know that Y_r is a solution of

$$Y' = (P_r - \lambda W_r)Y + F_r \quad \text{on} \quad J_r \tag{4.5}$$

if and only if y_r is a solution of

$$-(p_r y')' + q_r y = \lambda w_r y + f_r \quad \text{on } J_r, \tag{4.6}$$

where $Y_r = \begin{bmatrix} y_r \\ p_r y_r' \end{bmatrix}$. For $C_r = \begin{bmatrix} c_{r1} \\ c_{r2} \end{bmatrix}$, $c_{r1}, c_{r2} \in \mathbb{C}$, $r = 1, 2$, determine a solution Y_r of (4.5) on J_r by the initial condition

$$Y_r(a_r, \lambda) = C_r.$$

Then y_r is a solution of (4.6) determined by the initial conditions $y_r(a_r, \lambda) = c_{r1}$, $(p_r y_r')(a_r, \lambda) = c_{r2}$.

By the variation of parameters formula, we have

$$Y_r(t, \lambda) = \Phi_r(t, a_r, \lambda)C_r + \int_{a_r}^{t} \Phi_r(t, s, \lambda)F_r(s)ds, \quad a_r \le t \le b_r. \tag{4.7}$$

In particular,

$$Y_r(b_r, \lambda) = \Phi_r(b_r, a_r, \lambda)C_r + \int_{a_r}^{b_r} \Phi_r(b_r, s, \lambda)F_r(s)ds.$$

Let $D(\lambda) = (A_1 + B_1\Phi_1(b_1, a_1, \lambda) \mid A_2 + B_2\Phi_2(b_2, a_2, \lambda))$ and $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$, Then

$$\begin{aligned} &A_1Y_1(a_1, \lambda) + B_1Y_1(b_1, \lambda) + A_2Y_2(a_2, \lambda) + B_2Y_2(b_2, \lambda) \\ &= D(\lambda)C + B_1 \int_{a_1}^{b_1} \Phi_1(b_1, s, \lambda)F_1(s)\,\mathrm{d}s + B_2 \int_{a_2}^{b_2} \Phi_2(b_2, s, \lambda)F_2(s)\,\mathrm{d}s. \end{aligned} \tag{4.8}$$

When $f_r = 0$ on $J_r (r = 1, 2)$, $\mathbf{Y} = \{Y_1, Y_2\}$ and $\mathbf{y} = \{y_1, y_2\}$ are nontrivial solutions if and only if C is not the zero vector. By (4.8), we have that when $f_r = 0$ on $J_r (r = 1, 2)$, there is a nontrivial solution $\{Y_1, Y_2\}$ (and a nontrivial solution $\{y_1, y_2\}$ of (4.1)) satisfying the boundary conditions

$$A_1Y_1(a_1) + B_1Y_1(b_1) + A_2Y_2(a_2) + B_2Y_2(b_2) = 0$$

if and only if $D(\lambda)$ is singular. It also follows from (4.8) that there is a unique solution $\{Y_1, Y_2\}$ satisfying the boundary conditions (4.2) for every $f_r \in L(J_r, \mathbb{C}), r = 1, 2$, if and only if $D(\lambda)$ is nonsingular. Similarly there is a unique solution $\mathbf{y} = \{y_1, y_2\}$ satisfying the boundary conditions (4.2) for every $\mathbf{f} = \{f_1, f_2\}$, $f_r \in L(J_r, \mathbb{C})$,$r = 1, 2$ if and only if $D(\lambda)$ is nonsingular. □

Next we construct the Green's function for two-interval boundary-value problems. Assume that

$$D(\lambda) = (A_1 + B_1\Phi_1(b_1, a_1, \lambda) \mid A_2 + B_2\Phi_2(b_2, a_2, \lambda))$$

is nonsingular. We use the notation $D_1(\lambda)$ denotes the 2 by 4 matrix whose rows are the first two rows of $D^{-1}(\lambda)$, and $D_2(\lambda)$ denotes the 2 by 4 matrix whose rows are the last two rows of $D^{-1}(\lambda)$. Let

$$G_1(t,s,\lambda) = \begin{cases} -\Phi_1(t,a_1,\lambda)D_1(\lambda)B_1\Phi_1(b_1,s,\lambda), & a_1 \le t < s \le b_1, \\ -\Phi_1(t,a_1,\lambda)D_1(\lambda)B_1\Phi_1(b_1,s,\lambda) + \Phi_1(t,s,\lambda), & a_1 \le s \le t \le b_1, \end{cases}$$

$$\widetilde{G}_1(t,s,\lambda) = -\Phi_1(t,a_1,\lambda)D_1(\lambda)B_2\Phi_2(b_2,s,\lambda), \quad a_1 \le t \le b_1,\ a_2 \le s \le b_2.$$

$$G_2(t,s,\lambda) = -\Phi_2(t,a_2,\lambda)D_2(\lambda)B_1\Phi_1(b_1,s,\lambda), \quad a_2 \le t \le b_2,\ a_1 \le s \le b_1,$$

$$\widetilde{G}_2(t,s,\lambda) = \begin{cases} -\Phi_2(t,a_2,\lambda)D_2(\lambda)B_2\Phi_2(b_2,s,\lambda), & a_2 \le t < s \le b_2, \\ -\Phi_2(t,a_2,\lambda)D_2(\lambda)B_2\Phi_2(b_2,s,\lambda) + \Phi_2(t,s,\lambda), & a_2 \le s \le t \le b_2. \end{cases}$$

Theorem 4.2. *Assume $D(\lambda)$ is nonsingular; i.e., $[A_1 + B_1\Phi_1(b_1,a_1,\lambda) \mid A_2 + B_2\Phi_2(b_2,a_2,\lambda)]^{-1}$ exists, then for any $\mathbf{f} = \{f_1,f_2\}$, $f_r \in L(J,\mathbb{C}), r = 1,2$, the unique solution $\mathbf{y} = \{y_1,y_2\}$ of (4.1)-(4.2) and the unique solution $\mathbf{Y} = \{Y_1,Y_2\}$ of (4.3), respectively, are given by*

$$y_1(t) = -\int_{a_1}^{b_1} G_{1,(12)}(t,s,\lambda)f_1(s)\,\mathrm{d}s - \int_{a_2}^{b_2} \widetilde{G}_{1,(12)}(t,s,\lambda)f_2(s)\,\mathrm{d}s, \quad a_1 \le t \le b_1, \tag{4.9}$$

$$y_2(t) = -\int_{a_1}^{b_1} G_{2,(12)}(t,s,\lambda)f_1(s)\,\mathrm{d}s - \int_{a_2}^{b_2} \widetilde{G}_{2,(12)}(t,s,\lambda)f_2(s)\,\mathrm{d}s, \quad a_2 \le t \le b_2, \tag{4.10}$$

$$Y_1(t) = \int_{a_1}^{b_1} G_1(t,s,\lambda)F_1(s)\,\mathrm{d}s + \int_{a_2}^{b_2} \widetilde{G}_1(t,s,\lambda)F_2(s)\,\mathrm{d}s, \quad a_1 \le t \le b_1, \tag{4.11}$$

$$Y_2(t) = \int_{a_1}^{b_1} G_2(t,s,\lambda)F_1(s)\,\mathrm{d}s + \int_{a_2}^{b_2} \widetilde{G}_2(t,s,\lambda)F_2(s)\,\mathrm{d}s, \quad a_2 \le t \le b_2. \tag{4.12}$$

Set $\mathbf{K}(t,s,\lambda) = \{K_1(t,s,\lambda), K_2(t,s,\lambda)\}$, where

$$K_1(t,s,\lambda) = \begin{cases} G_1(t,s,\lambda) & a_1 \le s \le b_1, \\ \widetilde{G}_1(t,s,\lambda), & a_2 \le s \le b_2, \end{cases} \quad a_1 \le t \le b_1,$$

$$K_2(t,s,\lambda) = \begin{cases} G_2(t,s,\lambda) & a_1 \le s \le b_1, \\ \widetilde{G}_2(t,s,\lambda), & a_2 \le s \le b_2, \end{cases} \quad a_2 \le t \le b_2.$$

We call $\mathbf{K}(t,s,\lambda) = \mathbf{K}(t,s,\lambda,P_1,P_2,W_1,W_2,A_1,A_2,B_1,B_2)$ (Here we use the complete notation to highlight the dependence of $\mathbf{K}$ on these quantities.) the Green's matrix of the regular boundary value problem (3.6), (3.4). And we call $\mathbf{K}_{12} = \{K_{1,(12)}, K_{2(12)}\}$ the Green's function of two-interval boundary value problem (3.3), (3.4).

Proof. Let

$$C = D^{-1}(\lambda)(-B_1\int_{a_1}^{b_1} \Phi_1(b_1,s,\lambda)F_1(s)\,\mathrm{d}s - B_2\int_{a_2}^{b_2} \Phi_2(b_2,s,\lambda)F_2(s)\,\mathrm{d}s).$$

By (4.8), we have

$$A_1Y_1(a_1) + B_1Y_1(b_1) + A_2Y_2(a_2) + B_2Y_2(b_2) = 0.$$

Recall the notation $D_1(\lambda)$ and $D_2(\lambda)$, we have

$$C_1 = D_1(\lambda)(-B_1 \int_{a_1}^{b_1} \Phi_1(b_1, s, \lambda) F_1(s)\,\mathrm{d}s - B_2 \int_{a_2}^{b_2} \Phi_2(b_2, s, \lambda) F_2(s)\,\mathrm{d}s),$$
$$C_2 = D_2(\lambda)(-B_1 \int_{a_1}^{b_1} \Phi_1(b_1, s, \lambda) F_1(s)\,\mathrm{d}s - B_2 \int_{a_2}^{b_2} \Phi_2(b_2, s, \lambda) F_2(s)\,\mathrm{d}s).$$

From (4.7), we obtain that

$$\begin{aligned} Y_1(t) =& \Phi_1(t, a_1, \lambda) D_1(\lambda)(-B_1 \int_{a_1}^{b_1} \Phi_1(b_1, s, \lambda) F_1(s)\,\mathrm{d}s \\ & - B_2 \int_{a_2}^{b_2} \Phi_2(b_2, s, \lambda) F_2(s)\,\mathrm{d}s) + \int_{a_1}^{t} \Phi_1(t, s, \lambda) F_1(s)\,\mathrm{d}s \\ =& \int_{a_1}^{b_1} [\Phi_1(t, a_1, \lambda) D_1(\lambda)(-B_1 \Phi_1(b_1, s, \lambda) F_1(s))]\,\mathrm{d}s + \int_{a_1}^{t} \Phi_1(t, s, \lambda) F_1(s)\,\mathrm{d}s \\ & + \int_{a_2}^{b_2} [\Phi_1(t, a_1, \lambda) D_1(\lambda)(-B_2 \Phi_2(b_2, s, \lambda) F_2(s))]\,\mathrm{d}s \\ =& \int_{a_1}^{b_1} G_1(t, s, \lambda) F_1(s)\,\mathrm{d}s + \int_{a_2}^{b_2} \widetilde{G}_1(t, s, \lambda) F_2(s)\,\mathrm{d}s, \quad a_1 \le t \le b_1. \end{aligned} \tag{4.13}$$

$$\begin{aligned} Y_2(t) =& \Phi_2(t, a_2, \lambda) D_2(\lambda)(-B_1 \int_{a_1}^{b_1} \Phi_1(b_1, s, \lambda) F_1(s)\,\mathrm{d}s \\ & - B_2 \int_{a_2}^{b_2} \Phi_2(b_2, s, \lambda) F_2(s)\,\mathrm{d}s) + \int_{a_2}^{t} \Phi_2(t, s, \lambda) F_2(s)\,\mathrm{d}s \\ =& \int_{a_1}^{b_1} [\Phi_2(t, a_2, \lambda) D_2(\lambda)(-B_1 \Phi_1(b_1, s, \lambda) F_1(s))]\,\mathrm{d}s + \int_{a_2}^{t} \Phi_2(t, s, \lambda) F_2(s)\,\mathrm{d}s \\ & + \int_{a_2}^{b_2} [\Phi_2(t, a_2, \lambda) D_2(\lambda)(-B_2 \Phi_2(b_2, s, \lambda) F_2(s))]\,\mathrm{d}s \\ =& \int_{a_1}^{b_1} G_2(t, s, \lambda) F_1(s)\,\mathrm{d}s + \int_{a_2}^{b_2} \widetilde{G}_2(t, s, \lambda) F_2(s)\,\mathrm{d}s, \quad a_2 \le t \le b_2. \end{aligned} \tag{4.14}$$

Note that (4.9) and (4.10), respectively, follow from the identities (4.13) and (4.14) by taking the upper right component; i.e.,

$$y_1(t) = -\int_{a_1}^{b_1} G_{1,(12)}(t, s, \lambda) f_1(s)\,\mathrm{d}s - \int_{a_2}^{b_2} \widetilde{G}_{1,(12)}(t, s, \lambda) f_2(s)\,\mathrm{d}s, \quad a_1 \le t \le b_1,$$

$$y_2(t) = -\int_{a_1}^{b_1} G_{2,(12)}(t, s, \lambda) f_1(s)\,\mathrm{d}s - \int_{a_2}^{b_2} \widetilde{G}_{2,(12)}(t, s, \lambda) f_2(s)\,\mathrm{d}s, \quad a_2 \le t \le b_2.$$

□

Remark 4.3. Note that the above construction of the Green's function and the characteristic function does not assume any symmetry or self-adjointness of the problem. The coefficients p_r, q_r, w_r may be complex valued and the boundary conditions need not be self-adjoint. If w_r is identically zero on the whole interval

J_r there is no λ dependence and the problem becomes degenerate. Similarly the case when $1/p_r$ is identically zero on J_r the problem can be considered degenerate.

Remark 4.4. If p_r, q_r, w_r are real valued and $w_r > 0$ on J_r, $r = 1, 2$, the self-adjoint operators in the separate Hilbert spaces $H_1 = L^2(J_1, w_1)$, $H_2 = L^2(J_2, w_2)$ with their usual inner products

$$(f,g)_r = \int_{Jr} f\overline{g}w_r,\ r = 1,2 \tag{4.15}$$

is well known [33] and it is a routine exercise to show that if S_r is a self-adjoint operator in H_r, $r = 1, 2$ then the direct sum of S_1 and S_2 is a self-adjoint operator in the direct sum space $H_u = L^2(J_1, w_1) \dotplus L^2(J_2, w_2)$ where each of H_1 and H_2 is endowed with the usual inner product (4.15). Everitt and Zettl [18] showed that there are many self-adjoint operators in H_u which are not generated as direct sums in this way. These 'new' self-adjoint operators involve interactions between the the intervals J_1 and J_2. In [18] all these interactions are characterized in terms of boundary conditions at the endpoints. Mukhtarov and Yakubov [9] observed that the theory in [18] can be significantly extended by using different multiples of the inner usual inner products ((4.15):

$$(f,g)_r = h_r \int_{Jr} f\overline{g}w_r,\ h_r > 0,\ r = 1,2. \tag{4.16}$$

Wang, Sun and Zettl [16] exploited this observation to characterize this enlarged set of self-adjoint operators in terms of boundary conditions at the endpoints. Recently Wang and Zettl in [17] further enlarged this set by removing the positivity restriction on h_r.This requires a different proof since (4.16) is not an inner product if h_r is negative. In Section 5 we give some examples to illustrate these interactions between the two intervals which generate self-adjoint extensions including those found in [9] and [17] and relate these to the comments we made in the Introduction about transmission and interface conditions.

5. Examples

In this section we give examples to illustrate that the construction of the two-interval Green's function applies to problems with transmission and interface conditions as mentioned in the Introduction.

These examples are taken from [17]. They are for the special case when the right endpoint of J_1 is the same as the left endpoint of J_2, i.e. $a_2 = b_1$. In order to avoid unnecessary subscripts and to make the notation more consistent with the literature on transmission and interface conditions we let

$$J_1 = (a, b), \quad J_2 = (c, d), \quad b = c \tag{5.1}$$

and use $c^+ = b$ for the right endpoint of J_1 and $c^- = c$ for the left endpoint of J_2. Also we let $A = A_1$, $B = B_1$, $C = A_2$, $D = B_2$ in (3.4).

Using this notation we make the following simple but key observation.

Remark 5.1. To apply the above construction of the Green's function to problems with transmission and interface conditions a simple but important observation is that when $b = c$ the direct sum of the Hilbert spaces from the two intervals can be identified with the Hilbert space of the 'outer' interval:

$$L^2((a,b),w_1) \dotplus L^2((c,d),w_2) = L^2((a,d),w) \tag{5.2}$$

where w_1 is the restriction of w to J_1 and w_2 is the restriction of w to J_2. In each example below the given boundary conditions generate a self-adjoint operator in the Hilbert space $L^2((a,d),w)$.

The first example has separated boundary conditions: these are generally called 'transmission conditions' in the literature.

Example 5.2 (Transmission Conditions)**.** Separated boundary conditions:

$$\begin{aligned}
A_1y(a) + A_2y^{[1]}(a) = 0, &\quad A_1, A_2 \in \mathbb{R},\ (A_1, A_2) \neq (0,0);\\
B_1y(b) + B_2y^{[1]}(b) = 0, &\quad B_1, B_2 \in \mathbb{R},\ (B_1, B_2) \neq (0,0);\\
C_1y(c) + C_2y^{[1]}(c) = 0, &\quad C_1, C_2 \in \mathbb{R},\ (C_1, C_2) \neq (0,0);\\
D_1y(d) + D_2y^{[1]}(d) = 0, &\quad D_1, D_2 \in \mathbb{R},\ (D_1, D_2) \neq (0,0).
\end{aligned} \tag{5.3}$$

Let

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ B_1 & B_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ C_1 & C_2 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ D_1 & D_2 \end{bmatrix}.$$

In this case the 4×8 matrix (A, B, C, D) has full rank and

$$0 = AEA^* = BEB^* = CEC^* = DED^*. \tag{5.4}$$

Considering $(a, c] \cup [c, d)$ as one interval (a, d) the next example has transmission conditions at the outer endpoint a, d and interface conditions at c. This example is chosen to highlight the (discontinuous) interface conditions at an interior point c. The roles of the endpoints a, c^+, c^-, d can be interchanged in this example (but care must be taken regarding the signs of the matrices A, B, C, D, see [17]).

Example 5.3. Let $h, k \in \mathbb{R}$, $h \neq 0 \neq k$. Separated boundary conditions at a and at d and coupled jump conditions at c.

$$\begin{aligned}
A_1y(a) + A_2(py')(a) = 0, &\quad A_1, A_2 \in \mathbb{R},\ (A_1, A_2) \neq (0,0);\\
D_1y(d) + D_2(py')(d) = 0, &\quad D_1, D_2 \in \mathbb{R},\ (D_1, D_2) \neq (0,0).
\end{aligned}$$

and

$$\begin{gathered}
Y(c) = e^{i\gamma}KY(b), \quad Y = \begin{bmatrix} y \\ y^{[1]} \end{bmatrix}, \quad K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ 1 \leq i,j \leq 2,\\
\det K \neq 0,\ -\pi < \gamma \leq \pi\,.
\end{gathered} \tag{5.5}$$

Let A, D be as in Example 5.2, then $\operatorname{rank}(A, D) = 2$ and $k\,AEA^* - h\,DED^* = 0$ for any h, k since $0 = AEA^* = DED^*$. Let

$$C = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad B = e^{i\gamma} \begin{bmatrix} 0 & 0 \\ k_{11} & k_{12} \\ k_{21} & k_{22} \\ 0 & 0 \end{bmatrix}, \quad -\pi < \gamma \leq \pi. \tag{5.6}$$

Then a straightforward computation shows that

$$h\,CEC^* = k\,BEB^*$$

is equivalent to

$$h\,E = k\,(\det K)\,E$$

which is equivalent to

$$h = k\det K. \tag{5.7}$$

Since (5.7) holds for any $h, k \in \mathbb{R}$, $h \neq 0 \neq k$, it follows from [17, Theorem 2] that the boundary conditions of this example are self-adjoint for any $K \in M_2(\mathbb{R})$ with $\det(K) \neq 0$.

The next remark highlights a remarkable comparison with the well known classical one-interval self adjoint boundary conditions, see [33].

Remark 5.4. It is well known that in the one-interval theory $\det K = 1$ is required for self-adjointness of the boundary conditions. We find it remarkable that the one-interval condition $\det K = 1$ extends to $det(K) \neq 0$ in the two-interval theory. And that this generalization follows from two simple observations: (i) The Mukhtarov-Yakubov [9] observation that for $h > 0$ and $k > 0$ using inner product multiples produces an interaction between the two intervals yielding $det(K) > 0$ and (ii) the Wang-Zettl observation that the boundary value problem is invariant under muliplication by -1 and this yields the further extension $det(K) \neq 0$. Note that the parameters h, k play no role in Example 5.2.

The next example illustrates the situation when there are two sets of coupled i.e. 'jump' boundary conditions, in one case the jumps are between the outer endpoints a, d and the other between the inner 'endpoints, $b = c^+$ and $c = c^-$.

Example 5.5. Two pairs of coupled conditions, with $-\pi < \gamma_1, \gamma_2 \leq \pi$,

$$\begin{gathered} Y(d) = e^{i\gamma_1} G Y(a), \quad G = (g_{ij}),\ g_{ij} \in \mathbb{R},\ i,j = 1,2,\ \det G \neq 0, \\ Y(c) = e^{i\gamma_2} K Y(b), \quad K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ i,j = 1,2,\ \det K \neq 0, \\ Y = \begin{bmatrix} y \\ y^{[1]} \end{bmatrix}. \end{gathered} \tag{5.8}$$

Proceeding as in the previous example we obtain the equivalence of the conditions for self-adjointness:

$$\begin{gathered} k\,GEG^* = h\,E \quad \text{and} \quad k\,KEK^* = hE; \\ k \det G = h \quad \text{and} \quad k \det K = h; \end{gathered}$$

i.e.,

$$\det G = \det K = \frac{h}{k}.$$

This shows that (5.8) are self-adjoint boundary conditions for any h, k positive or negative.

More examples can be found in [17] where singular analogues of the regular self-adjoint boundary conditions are also found. We plan to construct the Green's function for singular self-adjoint problems in a subsequent paper.

6. The Neuberger Construction

The remark below is written by J. W. Neuberger and published here with his permission. We believe it is of interest not only because we refer to 'a construction of Neuberger' in the Introduction but also for pedagogical reasons.

Remark 6.1 (J. W. Neuberger)**.** In the spring of 1958, I taught my first graduate course. It was an introduction to functional analysis by means of Sturm-Liouville problems. As was, and still is, my custom, I didn't lecture, but rather I broke up material for the class into a sequence of problems. The night before I was concerned with finding problems which gave a good introduction to Green's functions to the class. The standard 'recipe' with its prescribed discontinuity, seemed contrived. I managed to come up with the algebraic method mentioned at the start of this paper. Problems for some simple examples quickly led to the general case, again algebraically. To me this remains an example of how 'teaching' and 'research' can impact one another, particularly in a non lecture situation. If I had been lecturing, I would have given the standard approach, the only one I knew the day before. The algebraic approach to Green's functions might have never seen the light of day and some nice mathematics would have been missed.

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References

[1] Z. Akdogan, M. Dimirci, O. S. Mukhtarov; *Green function of discontinuous boundary value problem with transmission conditions*, Math. Meth. Appl. Sci. (2007), 1719-1738.

[2] J. J. Ao, J. Sun, M. Z. Zhang; *The finite spectrum of Sturm-Liouville problems with transmission conditions*, Appl. Math. Comp. 218 (2011), 1166-1173.

[3] E. A. Coddington, N. Levinson; *Theory of Ordinary Differential Equations*, McGraw-Hill, 1955.

[4] Sonja Currie, Bruce A. Watson; *Green's functions and regularized traces of Sturm-Liouville operators on graphs*, Proceedings of the Edinburgh Mathematical Society (2008) 51, 315-335.

[5] O. Demir, Z.Akdoğan; *Green function of one discontinuous Sturm-Liouville problem*, Bulletin of Pure and Applied Sciences Volume 29 E Issue 2 (2010),435 -452.

[6] M. Kadakal, O. S. Muktarov; *Sturm-Liouville problems with discontinuities at two points*, Comput. Math. Appl. 54 (2007), 1367-1370.

[7] L. Kong, Q. Kong, M. K. Kwong, J. S. W. Wong; *Linear Sturm-Liouville problems with multi-point boundary conditions*, pre-print.

[8] Q. Kong, Q. R. Wang; *Using time scales to study multi-interval Sturm-Liouville problems with interface conditions*, pre-print.

[9] O. S. Mukhtarov, S. Yakubov; *Problems for differential equations with transmission conditions*, Applicable Analysis 81 (2002), 1033-1064.

[10] O. S. Mukhtarov, M. Kadakal, F. S. Muhtarov; *Eigenvalues and normalized eigenfunctions of discontinuous Sturm-Liouville problems with transmission conditions*, Report on Math. Physics 54 (2004), 41-58.

[11] O. Sh. Mukhtarov, Mahir Kadakal, F. S. Muhtarov; *On discontinuous Sturm-Liouville problems with transmission conditions*, J. Math. Kyoto Univ. (JMKYAZ)44-4 (2004), 779-798.

[12] J. W. Neuberger; *Concerning Boundary Value Problems*, Pacific J. Math. 10 (1960) 1385-1392.

[13] Jorge M. Ramirez; *Green's functions for Sturm-Liouville problems on directed tree graphs*, arXiv:1108.0621.

[14] C.-T. Shieh, V. A. Yurko; *Inverse nodal and inverse spectral problems for discontinuous boundary value problems*, J. Math. Anal. Appl. 347 (2008), 266-272.

[15] M. M. Tharwat; *Discontinuous Sturm-Liouville problems and associated sampling theories*, Hindawi Publishing Corporation Abstract and Applied Analysis,Volume 2011, Article ID 610232, 30 pages, doi:10.1155/2011/610232.
[16] A. P. Wang, J. Sun, A. Zettl; *Two-Interval Sturm-Liouville Operators in Modified Hilbert Spaces*, Journal of Mathematical Analysis and Applications, vol. 328 (2007), 390-399.
[17] A. P. Wang, A. Zettl; *What is a Self-Adjoint Sturm Liouville Problem?*, pre-print.
[18] W. N. Everitt, A. Zettl; *Sturm-Liouville Differential Operators in Direct Sum Spaces*, Rocky Mountain J. of Mathematics, (1986), 497-516.
[19] Q. Kong, H. Wu, A. Zettl; *Sturm-Liouville Problems With Finite Spectrum*, J. Math. Anal. Appl. (2001), 1-15.
[20] Q. Kong, H. Volkmer, A. Zettl; *Matrix Representations of Sturm-Liouville Problems*, Results in Mathematics 54 (2009), 103-116.
[21] W. S. Loud; *Self-Adjoint multi-point boundary value problems*, Pacific J. Math. 24 (1968), 303-317.
[22] M. A. Naimark; *Linear Differential Operators*, English transl., Ungar, New York, 1968.
[23] Yu. V. Pokornyi, A. V. Borovskikh; *Differential Equations on Networks (Geometric Graphs)*, Journal of Mathematical Sciences, 119 (2004)
[24] Yu. V. Pokornyi, V. L. Pryadiev; *Sturm-Liouville Theory on Spacial Networks*, Journal of Mathematical Sciences, 119 (2004).
[25] J. Sun, A. Wang; *Sturm-Liouville operators with interface conditions*, The progress of research for Math, Mech, Phys. and High Tech. Science Press, Beijing 12 (2008), 513-516.
[26] J. Sun, A. P. Wang, A. Zettl; *Two-Interval Sturm-Liouville Operators in Direct Sum Spaces with Inner Product Multiples*, Results in Mathematics, 50 (2007), 155-168.
[27] E. Tunc, O. S. Mukhtarov; *Fundamental Solutions and Eigenvalues of one Boundary Value Problem with Transmission Conditions*, Appl. Math. Comp. 157 (2004), 347-355.
[28] A. Wang, J. Sun, X. Hao, S. Yao; *Completeness of Eigenfunctions of Sturm-Liouville Problems with Transmission Conditions*, Methods and Applications of Analysis 16 (2009), 299-312.
[29] A. P. Wang, J. Sun, A. Zettl; *Two-Interval Sturm-Liouville Operators in Modified Hilbert Spaces*, Journal of Mathematical Analysis and Applications, vol. 328 (2007), 390-399.
[30] J. Weidmann; *Spectral theory of ordinary differential operators*, Lecture Notes in Mathematics 1258, Springer-Verlag, Berlin, 1987.
[31] A. Zettl; *The lack of self-adjointness in three point boundary value problems*, Proc. Amer. Math. Soc. 17 (1966), 368-371.
[32] A. Zettl; *Adjoint and self-adjoint problems with interface conditions*, SIAM J. Applied Math. 16 (1968), 851-859.
[33] A. Zettl; *Sturm-Liouville Theory*, American Mathematical Society, Mathematical Surveys and Monographs, v. 121, 2005.

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