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DELAY DIFFERENTIAL EQUATIONS WITH HOMOGENEOUS INTEGRAL CONDITIONS

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Abstract. In this article we prove the existence and uniqueness of a strong solution of a delay differential equation with homogenous integral conditions using the method of semidiscretization in time. As an application, we include an example that illustrates the main result.

1. Introduction

This article concerns the delay differential equation having homogeneous integral conditions,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \lambda \frac{\partial^3 u}{\partial x^2 \partial t} = F(x,t,u_t) \quad \text{on } (0,1) \times (0,T], \tag{1.1}$$

$$u(x,t) = \phi(x,t) \quad \text{on } (0,1) \times [-T,0], \tag{1.2}$$

with integral conditions

$$\int_0^1 u(x,t)\,dx = 0, \quad t \in [0,T], \tag{1.3}$$

$$\int_0^1 xu(x,t)\,dx = 0, \quad t \in [0,T], \tag{1.4}$$

where $0 < T < \infty$, the unknown function $u : [-T,T] \to L^2(0,1)$, the history $\phi : [-T,0] \to L^2(0,1)$ and the nonlinear map $F : (0,T] \times \mathcal{C}_0 \to B(0,1)$, are defined by $u(t)(x) = u(x,t)$, $\phi(t)(x) = \phi(x,t)$ and $F(t,u_t)(x) = F(x,t,u_t)$, respectively. Here $L^2(0,1)$ is the real Hilbert of all square integrable real valued functions on $(0,1)$ with the standard inner product, and $B(0,1)$ is the completion of $C_0(0,1)$, the space of all continuous functions on $(0,1)$ having compact support in $(0,1)$, with the inner product defined by

$$(u,v)_B = \int_0^1 \Im_x u(x) \Im_x v(x)\,dx,$$

where $\Im_x u(x) = \int_0^x u(\xi)d\xi$. We recall that, if $\|\cdot\|_B$ denote the corresponding norm; that is,

$$\|\psi\|_B = \sqrt{(\psi,\psi)_B},$$

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then it follows that
$$\|\psi\|_B^2 \le \frac{1}{2}\|\psi\|^2.$$
Also for $t \in (0,T]$, $\mathcal{C}_t := C([-T,t];B(0,1))$ is the Banach space of all continuous functions from $[-T,t]$ into $B(0,1)$ endowed with the supremum norm
$$\|\psi\|_t = \sup_{-T\le\theta\le t}\|\psi(\theta)\|_B.$$
For $\psi \in \mathcal{C}_T$, we denote $\psi_t \in \mathcal{C}_0$ given by $\psi_t(\theta) = \psi(t+\theta), \theta \in [-T,0]$.

For the consideration of integral conditions, we use the space V introduce by Merazga and Bouziani [11],
$$V = \big\{\phi \in L^2(0,1) : \int_0^1 \phi(x)\,dx = \int_0^1 x\phi(x)\,dx = 0\big\}.$$
Note that V is a Hilbert space with respect to the standard inner product.

Since 1930, various classical types of initial boundary value problems have been investigated by many authors using the method of semidiscretization, see for instance, [6, 13, 14, 15] and references therein.

In this paper our aim is to extend the application of the method of semidiscretization in time to delay differential equations with homogeneous integral conditions and to establish the existence and uniqueness of a strong solution for a delay differential equation with homogeneous integral conditions given by (1.1)-(1.4).

The method of semidiscretization in time is a constructive method and has a strong numerical aspect. For the application of the method of semidiscretization to integrodifferential equations with nonclassical boundary conditions, we refer the readers to [2, 3, 4, 8, 9] and references therein.

Dubey [5] established the existence and uniqueness of a strong solution for the following nonlinear differential equation in a reflexive Banach space with a nonlocal history condition using the method of semidiscretization in time
$$\begin{gathered} u'(t) + Au(t) = f(t,u(t),u_t), \quad t \in (0,T], \\ h(u_0) = \phi \quad \text{on} \quad [-\tau,0], \end{gathered}$$
where $0 < T < \infty$, $\phi \in \mathcal{C}_0 := C([-\tau,0];X), \tau > 0$, the nonlinear operator A is single-valued and m-accretive defined from the domain $D(A) \subset X$ into X, the nonlinear map f is defined from $[0,T] \times X \times \mathcal{C}_0 := C([-\tau,0];X)$ into X, the map h is defined from $\mathcal{C}_0$ and $\mathcal{C}_0$. Bahuguna, Abbas, and Dabas [1] applied the method of semidiscretization to a semilinear functional partial differential equation with an integral condition.

Our problem is motivated by the work of Lakoud and Belakroum [10] and Dubey [5]. Lakoud and Balakroum [10] established the existence and uniqueness of a weak solution for the integro-differential evolution with a memory term,
$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - \lambda\frac{\partial^3 v}{\partial x^2 \partial t} = g(x,t) + \int_0^t a(t-s)k(s,v(x,s))\,ds \quad \text{on } (0,1)\times(0,T],$$
subject to the initial conditions
$$v(x,0) = V_0(x),$$
and integral conditions
$$\int_0^1 v(x,t)\,dx = E(t),$$

$$\int_0^1 xv(x,t)\,dx = G(t),$$

where f, V_0, G, E are given functions and T, λ are positive constants.

The plan of the paper is as follows. In section 2, we state all the assumptions and preliminaries. In section 3, we state the main result. In section 4, we state and prove all the lemmas that are required to prove the main result and at the end of this section, we prove the main result. In the last section, we give an application of the main result.

Throughout the paper we denote a generic constant by C. This constant may have different values in the same discussion.

2. Preliminaries

We will use the following assumptions:

(H1) The nonlinear map $F : (0,T] \times \mathcal{C}_0 \to B(0,1)$ satisfies a local Lipschitz condition

$$\|F(t_1,\psi_1) - F(t_2,\psi_2)\|_B \le L_F(r)[|t_1-t_2| + \|\psi_1-\psi_2\|_0],$$

for all $t_1, t_2 \in (0,T]$ and $\psi_1, \psi_2 \in \mathcal{C}_0$ with $\|\psi_i - \phi(0)\|_0 \le r$, $i = 1,2$ and $L_F(r)$ is a nondecreasing function of $r > 0$.

(H2) The history function $\phi : [-T,0] \to L^2(0,1)$ is uniformly Lipschitz continuous with Lipschitz constant $K > 0$; i.e., $\|\phi(t) - \phi(s)\| \le K|t-s|$.

(H3) $\int_0^1 \phi(x,0)\,dx = 0$, $\int_0^1 x\phi(x,0)\,dx = 0$.

Lemma 2.1. *If $-A$ is the infinitesimal generator of a C_0-semigroup of contractions in a Banach space X then A is m-accretive; i.e.,*

$$(Au, J(u)) \ge 0, \quad \text{for } u \in D(A),$$

where J is the duality mapping and $R(I+\lambda A) = X$ for $\lambda > 0$, I is the identity operator on X and $R(\cdot)$ is the range of an operator.

The proof of the above lemma follows from Lumer Phillips theorem [12, Thm. 1.4.3].

3. Main result

Theorem 3.1. *Suppose that assumptions* (H1)–(H3) *are satisfied. Then problem* (1.1)-(1.4) *has a unique strong solution on the interval* $[-T,T]$.

4. Discretization and a priori estimates

To apply the method of semidiscretization we divide the interval $[0,T]$ into the subintervals of length $h_n = \frac{T}{n}$. We set $u_0^n = \phi(0)$ for all $n \in \mathbb{N}$ and define $\{u_j^n\}$ successively as the unique solution of the problem

$$\delta u_j^n - \frac{\partial^2 u_j^n}{\partial x^2} - \lambda \frac{\partial^2 \delta u_j^n}{\partial x^2} = F(t_j^n, \tilde{u}_{j-1}^n), \tag{4.1}$$

$$\int_0^1 u_j^n\,dx = 0, \tag{4.2}$$

$$\int_0^1 x u_j^n\,dx = 0, \tag{4.3}$$

where $\tilde{u}_0^n = \phi(t)$ for $t \in [-T, 0]$ and $2 \le j \le n$,

$$\tilde{u}_{j-1}^n(t) = \begin{cases} \phi(t_{j-1}^n + t), & \text{if } t \in [-T, -t_{j-1}^n] \\ u_{i-1}^n + (t_{j-1}^n + t - t_{i-1}^n)\delta u_i^n, & \text{if } t \in [-t_{j-i}^n, -t_{j-i-1}^n],\ 1 \le i \le j-1, \end{cases}$$

and

$$\delta u_j^n = \frac{u_j^n - u_{j-1}^n}{h_n}.$$

Let $w_j^n = u_j^n + \lambda \delta u_j^n$. This implies that $\delta u_j^n = \frac{1}{h_n+\lambda} w_j^n - \frac{1}{h_n+\lambda} u_{j-1}^n$. So (4.1)-(4.3) will reduce to

$$-\frac{\partial^2 w_j^n}{\partial x^2} + \frac{1}{h_n + \lambda} w_j^n = f_j^n, \tag{4.4}$$

$$\int_0^1 w_j^n \, dx = 0, \tag{4.5}$$

$$\int_0^1 x w_j^n \, dx = 0, \tag{4.6}$$

where

$$f_j^n = \frac{1}{h_n + \lambda} u_{j-1}^n + F(t_j^n, \tilde{u}_{j-1}^n).$$

Now we show the existence and uniqueness of functions w_j^n satisfying (4.4)-(4.6). For this consider $H = L^2(0,1)$, the Hilbert space of all real valued square integrable functions on the interval $(0,1)$. Let the linear operator A be defined by

$$D(A) := \{u \in H : u'' \in H, \int_0^1 u(x)\, dx = \int_0^1 x u(x)\, dx = 0\}, \quad Au = -u''.$$

Then we know that $-A$ is the infinitesimal generator of a C_0-semigroup $S(t), t \ge 0$ of contractions in H.

The existence of unique w_j^n satisfying equations (4.4)-(4.6) is a consequence of Lemma 2.1. As

$$u_j^n = \frac{h_n}{\lambda + h_n} w_j^n + \frac{\lambda}{\lambda + h_n} u_{j-1}^n,$$

there exist unique $u_j^n \in D(A)$ satisfying (4.1)–(4.3). Now we define

$$U^n(t) = \begin{cases} \phi(t), & \text{if } t \in [-T, 0] \\ u_{j-1}^n + (t - t_{j-1}^n) \frac{u_j^n - u_{j-1}^n}{h_n}, & \text{if } t \in (t_{j-1}^n, t_j^n]. \end{cases} \tag{4.7}$$

Lemma 4.1. *For $n \in \mathbb{N}$ and $j = 1, 2, \cdots n$,*

$$\|u_j^n - \phi(0)\| \le C,$$

where C is a generic constant independent of n, j, h_n.

Proof. Now for any $\psi \in V$, from (4.1) we have

$$(\delta u_j^n, \psi)_B - \Big(\frac{\partial^2 u_j^n}{\partial x^2}, \psi\Big)_B - \lambda \Big(\frac{\partial^2 \delta u_j^n}{\partial x^2}, \psi\Big)_B = (F_j^n, \psi)_B, \tag{4.8}$$

where $F_j^n = F(t_j^n, \tilde{u}_{j-1}^n)$. By the definition of the inner product $(,)_B$, we have

$$\Big(\frac{\partial^2 u_j^n}{\partial x^2}, \psi\Big)_B = -\int_0^1 u_j^n \psi \, dx = -(u_j^n, \psi). \tag{4.9}$$

So (4.8) reduces to

$$(\delta u_j^n, \psi)_B + (u_j^n, \psi) + \lambda(\delta u_j^n, \psi) = (F_j^n, \psi)_B. \tag{4.10}$$

Taking $j = 1, \psi = u_1^n - u_0^n$ in (4.10),

$$\begin{aligned}&(u_1^n - u_0^n, u_1^n - u_0^n)_B + h_n(u_1^n, u_1^n - u_0^n) + \lambda(u_1^n - u_0^n, u_1^n - u_0^n)\\ &= h_n(F_1^n, u_1^n - u_0^n)_B.\end{aligned}$$

Now using (4.9), we obtain

$$\begin{aligned}&(u_1^n - u_0^n, u_1^n - u_0^n)_B + h_n(u_1^n - u_0^n, u_1^n - u_0^n) + \lambda(u_1^n - u_0^n, u_1^n - u_0^n)\\ &= h_n\left(F_1^n + \frac{d^2u_0^n}{dx^2}, u_1^n - u_0^n\right)_B.\end{aligned}$$

Now, we obtain

$$\|u_1^n - u_0^n\|_B^2 + h_n\|u_1^n - u_0^n\|^2 + \lambda\|u_1^n - u_0^n\|^2 \le h_n\big[\|F_1^n\|_B + \|\frac{d^2u_0^n}{dx^2}\|_B\big]\|u_1^n - u_0^n\|_B.$$

By ignoring first two terms on the left hand side, we obtain

$$\lambda\|u_1^n - u_0^n\|^2 \le h_n\big[\|F_1^n\|_B + h_n\|\frac{d^2u_0^n}{dx^2}\|_B\big]\|u_1^n - u_0^n\|_B.$$

As $\|u_1^n - u_0^n\|_B \le \frac{1}{\sqrt{2}}\|u_1^n - u_0^n\|$, we have

$$\|u_1^n - u_0^n\| \le \frac{h_n}{\lambda\sqrt{2}}\big[\|F_1^n\|_B + \|\frac{d^2u_0^n}{dx^2}\|_B\big].$$

By using assumption (H1), and the inequality $h_n \le T$, we obtain

$$\|u_1^n - u_0^n\| \le \frac{T}{\lambda\sqrt{2}}\big[L_F(r)(T + r) + \|F(0, \phi(0))\|_B + \|\frac{d^2u_0^n}{dx^2}\|_B\big] \le C.$$

By putting $\psi = u_j^n - u_0^n$ in (4.10), we obtain

$$\begin{aligned}&(u_j^n - u_{j-1}^n, u_j^n - u_0^n)_B + h_n(u_j^n, u_j^n - u_0^n) + \lambda(u_j^n - u_{j-1}^n, u_j^n - u_0^n)\\ &= h_n(F_j^n, u_j^n - u_0^n)_B.\end{aligned}$$

Using (4.9), we obtain

$$\begin{aligned}&\big(u_j^n - u_0^n, u_j^n - u_0^n\big)_B + h_n\big(u_j^n - u_0^n, u_j^n - u_0^n\big) + \lambda\big(u_j^n - u_0^n, u_j^n - u_0^n\big)\\ &= h_n\big(F_j^n, u_j^n - u_0^n\big)_B + \big(u_{j-1}^n - u_0^n, u_j^n - u_0^n\big)_B + h_n\big(\frac{d^2u_0^n}{dx^2}, u_j^n - u_0^n\big)_B\\ &\quad + \lambda\big(u_{j-1}^n - u_0^n, u_j^n - u_0^n\big).\end{aligned}$$

By ignoring the first two terms on the left hand side, we obtain

$$\begin{aligned}\lambda\|u_j^n - u_0^n\|^2 &\le h_n\|F_j^n\|_B\|u_j^n - u_0^n\|_B + \|u_{j-1}^n - u_0^n\|_B\|u_j^n - u_0^n\|_B\\ &\quad + h_n\|\frac{d^2u_0^n}{dx^2}\|_B\|u_j^n - u_0^n\|_B + \lambda\|u_{j-1}^n - u_0^n\|\|u_j^n - u_0^n\|.\end{aligned}$$

As $\|u_j^n - u_0^n\|_B \le \frac{1}{\sqrt{2}}\|u_j^n - u_0^n\|$, we have

$$\|u_j^n - u_0^n\| \le \frac{1}{\lambda}\big[\frac{h_n}{\sqrt{2}}(\|F_j^n\|_B + \|\frac{d^2u_0^n}{dx^2}\|_B) + (\frac{1}{2} + \lambda)\|u_{j-1}^n - u_0^n\|\big]. \tag{4.11}$$

By assumption (H1), we have

$$\begin{aligned} \|F_j^n\|_B &= \|F(t_j^n, \tilde{u}_{j-1}^n)\|_B \\ &= \|F(t_j^n, \tilde{u}_{j-1}^n) - F(0,\phi(0))\|_B + \|F(0,\phi(0))\|_B \\ &\le L_F(r)[|t_j^n| + \|\tilde{u}_{j-1}^n - \phi(0)\|_0] + \|F(0,\phi(0))\|_B \\ &\le L_F(r)[T+r] + \|F(0,\phi(0))\|_B. \end{aligned} \tag{4.12}$$

Using (4.12) in (4.11), we obtain

$$\begin{gathered} \|u_j^n - u_0^n\| \le \frac{h_n}{\lambda\sqrt{2}}\big[L_F(r)(T+r) + \|F(0,\phi(0))\|_B + \|\frac{d^2u_0^n}{dx^2}\|_B\big] \\ + \Big(1+\frac{1}{2\lambda}\Big)\|u_{j-1}^n - u_0^n\| \\ \|u_j^n - u_0^n\| \le h_nK + \Big(1+\frac{1}{2\lambda}\Big)\|u_{j-1}^n - u_0^n\|, \end{gathered}$$

where $K = \frac{1}{\lambda\sqrt{2}}\big[L_F(r)(T+r) + \|F(0,\phi(0))\|_B + \|\frac{d^2u_0^n}{dx^2}\|_B\big]$. Repeating the above procedure, we obtain

$$\|u_j^n - u_0^n\| \le C.$$

This completes the proof. □

Lemma 4.2. *For $j = 1, 2, \cdots, n$,*

$$\|\frac{u_j^n - u_{j-1}^n}{h_n}\| \le C.$$

Proof. By putting $j=1$, $\psi = u_j^n - u_0^n$ in (4.10) and using (4.9), we obtain

$$\begin{aligned} &\Big(\frac{u_1^n - u_0^n}{h_n}, u_1^n - u_0^n\Big)_B + (u_1^n - u_0^n, u_1^n - u_0^n) + \lambda\Big(\frac{u_1^n - u_0^n}{h_n}, u_1^n - u_0^n\Big) \\ &= (F_1^n, u_1^n - u_0^n)_B + \Big(\frac{d^2u_0^n}{dx^2}, u_1^n - u_0^n\Big)_B. \end{aligned}$$

By ignoring the first two terms on the left hand sides, we obtain

$$\frac{\lambda}{h_n}\|u_1^n - u_0^n\|^2 \le \Big(F_1^n + \frac{d^2u_0^n}{dx^2}, u_1^n - u_0^n\Big)_B. \tag{4.13}$$

As $\|u_j^n - u_0^n\|_B \le \frac{1}{\sqrt{2}}\|u_j^n - u_0^n\|$, we have

$$\|\frac{u_1^n - u_0^n}{h_n}\| \le \frac{1}{\lambda\sqrt{2}}\big[\|F_1^n\|_B + \|\frac{d^2u_0^n}{dx^2}\|_B\big].$$

Using assumption (H1), we obtain

$$\|\frac{u_1^n - u_0^n}{h_n}\| \le \frac{1}{\lambda\sqrt{2}}\big[L_F(r)(T+r) + \|F(0,\phi(0))\|_B + \|\frac{d^2u_0^n}{dx^2}\|_B\big] \le C.$$

Subtracting (4.10) written for j, from the same identity for $j-1$ and then putting $\psi = u_j^n - u_{j-1}^n$, we obtain

$$\begin{aligned} &(\delta u_j^n, u_j^n - u_{j-1}^n)_B + (u_j^n - u_{j-1}^n, u_j^n - u_{j-1}^n) + \lambda(\delta u_j^n, u_j^n - u_{j-1}^n) \\ &= (F_j^n - F_{j-1}^n, u_j^n - u_{j-1}^n)_B + (\delta u_{j-1}^n, u_j^n - u_{j-1}^n)_B + \lambda\left(\delta u_{j-1}^n, u_j^n - u_{j-1}^n\right). \end{aligned}$$

Ignoring the first two terms on the left hand side,

$$\frac{\lambda}{h_n}\|u_j^n - u_{j-1}^n\|^2 \le \|F_j^n - F_{j-1}^n\|_B\|u_j^n - u_{j-1}^n\|_B + \|\frac{u_{j-1}^n - u_{j-2}^n}{h_n}\|_B\|u_j^n - u_{j-1}^n\|_B$$

$$+\lambda\|\frac{u_{j-1}^n-u_{j-2}^n}{h_n}\|\|u_j^n-u_{j-1}^n\|.$$

As $\|\psi\|_B \le \|\psi\|/\sqrt{2}$, we have

$$\|\frac{u_j^n-u_{j-1}^n}{h_n}\| \le \frac{1}{\lambda\sqrt{2}}\|F_j^n-F_{j-1}^n\|_B + (1+\frac{1}{2\lambda})\|\frac{u_{j-1}^n-u_{j-2}^n}{h_n}\|. \tag{4.14}$$

Now using assumption (H1),

$$\begin{aligned}\|F_j^n-F_{j-1}^n\|_B &= \|F(t_j^n,\tilde{u}_{j-1}^n)-F(t_{j-1}^n,\tilde{u}_{j-2}^n)\|_B\\ &\le L_F(r)[|t_j^n-t_{j-1}^n| + \|\tilde{u}_{j-1}^n-\tilde{u}_{j-2}^n\|_0]\\ &\le L_F(r)[T+2r].\end{aligned}$$

Using the above inequality in (4.14), we obtain

$$\|\frac{u_j^n-u_{j-1}^n}{h_n}\| \le \frac{1}{\lambda\sqrt{2}}L_F(r)[T+2r] + (1+\frac{1}{2\lambda})\|\frac{u_{j-1}^n-u_{j-2}^n}{h_n}\|. \tag{4.15}$$

Repeating the above procedure, we finally obtain

$$\|\frac{u_j^n-u_{j-1}^n}{h_n}\| \le C. \tag{4.16}$$

This completes the proof. □

Now we introduce a sequence of step functions $\{X^n(t)\}$ defined by

$$X^n(t) = \begin{cases}\phi(0), & \text{if } t=0\\ u_j^n, & \text{if } t\in(t_{j-i}^n,t_j^n].\end{cases} \tag{4.17}$$

Remark 4.3. From Lemma 4.2 it follows that the functions U^n are uniformly Lipschitz continuous on $[-T,T]$ and $U^n(t)-X^n(t)\to 0$, as $n\to\infty$ on $[0,T]$.

Let $F^n(t) = F(t_j^n,\tilde{u}_{j-1}^n)$. By assumption (H1) and remark 4.3, we see that $F^n(t)\to F(t,u_t)$. Using (4.7) and (4.17), in (4.1), we obtain

$$\frac{d^-}{dt}U^n(t) - \frac{\partial^2}{\partial x^2}X^n(t) - \lambda\frac{\partial^3}{\partial x^2\partial t}X^n(t) = F^n(t). \tag{4.18}$$

Integrating with respect to t, we obtain

$$-\int_0^t \big[\frac{\partial^2}{\partial x^2}X^n(s) + \lambda\frac{\partial^3}{\partial x^2\partial s}X^n(s)\big]\,ds = \phi(0) - U^n(t) + \int_0^t F^n(s)\,ds. \tag{4.19}$$

Lemma 4.4. *There exists $u\in C([-T,T];B(0,1))$ such that $U^n(t)\to u(t)$ uniformly on $[-T,T]$. Moreover $u(t)$ is Lipschitz continuous on $[-T,T]$.*

Proof. From (4.18), we have

$$\begin{aligned}&\Big(\frac{d^-}{dt}U^n(t)-\frac{d^-}{dt}U^k(t),U^n(t)-U^k(t)\Big)_B + (X^n(t)-X^k(t),U^n(t)-U^k(t))\\ &+\lambda\Big(\frac{\partial}{\partial t}X^n(t)-\frac{\partial}{\partial t}X^k(t),U^n(t)-U^k(t)\Big)\\ &=(F^n(t)-F^k(t),U^n(t)-U^k(t))_B.\end{aligned}$$

Now,

$$\begin{aligned}&\frac{1}{2}\frac{d^-}{dt}\|U^n(t)-U^k(t)\|_B^2 + \|X^n(t)-X^k(t)\|^2 + \frac{\lambda}{2}\frac{\partial}{\partial t}\|X^n(t)-X^k(t)\|^2\\ &=(X^n(t)-X^k(t),X^n(t)-X^k(t)-U^n(t)+U^k(t))\end{aligned}$$

$$
\begin{aligned}
&+\lambda\Big(\frac{\partial}{\partial t}(X^n(t)-X^k(t)),X^n(t)-X^k(t)-U^n(t)+U^k(t)\Big)\\
&+(F^n(t)-F^k(t),U^n(t)-U^k(t))_B.
\end{aligned}
$$

By ignoring the last two terms on the left hand side, we obtain

$$
\frac{1}{2}\frac{d^-}{dt}\|U^n(t)-U^k(t)\|_B^2\le \delta_{nk}(t)+\|F^n(t)-F^k(t)\|_B\|U^n(t)-U^k(t)\|_B.
$$

where

$$
\begin{aligned}
\delta_{nk}(t)&=\|X^n(t)-X^k(t)\|[\|X^n(t)-U^n(t)\|+\|X^k(t)-U^k(t)\|]\\
&\quad+\frac{\lambda}{2}\|\frac{\partial}{\partial t}(X^n(t)-X^k(t))\|\big[\|X^n(t)-U^n(t)\|+\|X^k(t)-U^k(t)\|\big].
\end{aligned}
$$

By Remark 4.3, it is clear that $\delta_{nk}(t)\to 0$ as $n,k\to\infty$ uniformly on the interval $[0,T]$. Now by assumption (H1), we have

$$
\begin{aligned}
\|F^n(t)-F^k(t)\|_B &= \|F(t_j^n,\tilde{u}_{j-1}^n)-F(t_l^k,\tilde{u}_{l-1}^k)\|_B \qquad (4.20)\\
&\le \delta'_{nk}(t)+L_F(r)\|U^n(t)-U^k(t)\|_B, \qquad (4.21)
\end{aligned}
$$

where

$$
\delta'_{nk}(t)=L_F(r)[|t_j^n-t_l^k|+\|U^n(t)-\tilde{u}_{j-1}^n\|_0+\|U^k(t)-\tilde{u}_{l-1}^k\|_0].
$$

Clearly $\delta'_{nk}(t)\to 0$ as $n,k\to\infty$ uniformly on $[0,T]$. This implies that for a.e. $t\in[0,T]$,

$$
\frac{1}{2}\frac{d^-}{dt}\|U^n(t)-U^k(t)\|_B^2\le \delta'_{nk}(t)+L_F(r)\|U^n(t)-U^k(t)\|_B^2, \qquad (4.22)
$$

Integrating the above inequality over $(0,t)$ with $0\le t\le T$, we obtain

$$
\|U^n(t)-U^k(t)\|_B^2\le 2\delta'_{nk}T+2L_F(r)\int_0^t\|U^n(s)-U^k(s)\|_B^2\,ds.
$$

Applying Gronwall's inequality, we obtain that $U^n\to u$ in $C([-\tau,T],B(0,1))$. As each U^n is uniformly Lipschitz continuous, and by assumption (H2), u is Lipschitz continuous. This completes the proof. □

Proof of Theorem 3.1. Taking limits as $n\to\infty$ in (4.19), we obtain

$$
-\int_0^t\big[\frac{\partial^2}{\partial x^2}u(t)+\lambda\frac{\partial^3}{\partial x^2\partial s}u(t)\big]\,ds=\phi(0)-u(t)+\int_0^t F(s,u_t)\,ds.
$$

This implies that

$$
\frac{\partial u(t)}{\partial t}-\frac{\partial^2 u(t)}{\partial x^2}-\lambda\frac{\partial^3 u(t)}{\partial x^2\partial t}=F(t,u_t),\quad \text{a.e. } t\in[0,T].
$$

Clearly $u(t)$ is differentiable with $u(t)\in V$ a.e. on $[0,T]$ and $u(t)=\phi(t)$, $t\in[-T,0]$. This implies that $u(t)(x)=u(x,t)$ is a strong solution of (1.1)–(1.4). Now we show the uniqueness of the strong solution. To do this, suppose that u_1,u_2 are two strong solutions of (1.1)-(1.4).

Let $u=u_1-u_2$, then for $\psi\in V$, we have

$$
\Big(\frac{\partial u}{\partial t},\psi\Big)_B+(u,\psi)+\lambda\Big(\frac{\partial u}{\partial t},\psi\Big)=\Big(F(t,(u_1)_t)-F(t,(u_2)_t),\psi\Big)_B.
$$

Putting $\psi=u$ and ignoring last two terms in the left hand side, we obtain

$$
\Big(\frac{\partial u}{\partial t},u\Big)_B\le\Big(F(t,(u_1)_t)-F(t,(u_2)_t),u\Big)_B.
$$

By using assumption (H1), we obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|u(t)\|_B^2 &\leq L_F(r)\|(u_1)_t-(u_2)_t\|_B\|u(t)\|_B\\ &\leq L_F(r)\sup_{-T\leq t+\theta\leq t}\|u_t(\theta)\|_B\sup_{-T\leq\theta\leq t}\|u(\theta)\|_B.\end{aligned}$$

Integrating between 0 and t, we obtain

$$\begin{gathered}\sup_{-T\leq\theta\leq t}\|u(\theta)\|_B^2\leq 2L_F(r)\int_0^t\|u\|_s^2\,ds\\ \|u\|_t^2\leq 2L_F(r)\int_0^t\|u\|_s^2\,ds.\end{gathered}$$

Applying Gronwall's inequality, we obtain $u=0$ on $[-T,T]$. Hence we obtain a unique strong solution of problem (1.1)-(1.4) on the interval $[-T,T]$. □

5. Application

Consider the partial differential equation

$$\begin{gathered}\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}-\lambda\frac{\partial^3 v}{\partial x^2\partial t}=g(x,t)+\int_0^t a(t-s)k(s,v(x,s))\,ds\quad\text{on } (0,1)\times(0,T],\\ v(x,t)=\phi(x,t)\quad\text{on}\quad(0,1)\times[-T,0],\end{gathered}\tag{5.1}$$

with the integral conditions

$$\int_0^1 v(x,t)\,dx=0,\tag{5.2}$$

$$\int_0^1 xv(x,t)\,dx=0.\tag{5.3}$$

In the above problem, we identify the unknown function $v:(0,T]\to B(0,1)$, by $v(t)(x)=v(x,t)$, $g:(0,T]\to B(0,1)$ by $g(t)(x)=g(x,t)$, $k:(0,T]\times\mathbb{R}\to B(0,1)$ by $k(t,v(x,t))=k(t,v(t))(x)$ and the history function $\phi:[-T,0]\to B(0,1)$ by $\phi(t)(x)=\phi(x,t)$. Also we take

$$V=\big\{\phi\in L^2(0,1):\int_0^1\phi(x)\,dx=\int_0^1 x\phi(x)\,dx=0\big\}.$$

Putting $t=s-\eta$ in the integral term, problem (5.1)-(5.3) reduces to

$$\begin{gathered}\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}-\lambda\frac{\partial^3 v}{\partial x^2\partial t}=g(t)+\int_{-t}^0 a(-\eta)k(t+\eta,v(t+\eta))d\eta\quad\text{on } (0,T],\\ v(t)=\phi(t),\quad t\in[-T,0].\end{gathered}\tag{5.4}$$

Now we consider the following assumptions:

(i) There exists $k_1>0$, such that for all $t,s\in(0,T]$
$$\|g(t)-g(s)\|_B\leq k_1|t-s|.$$

(ii) There exists $k_2>0$, such that for all $t,s\in(0,T]$ and $\psi_1,\psi_2\in\mathcal{C}_0$,
$$\|k(t,\psi_1(t))-k(s,\psi_2(s))\|_B\leq k_2[|t-s|+\|\psi_1-\psi_2\|_0].$$

(iii) Also there exist $M>0$, such that
$$\|a(t)\|_B\leq M,\quad t\in[-T,0].$$

Now we define $G : (0, T] \times \mathcal{C}_0 \to B(0, 1)$ by

$$G(t, \psi) = g(t) + \int_{-t}^{0} a(-\eta)k(t + \eta, \psi)d\eta.$$

Thus (5.4) reduces to

$$\begin{gathered} \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - \lambda \frac{\partial^3 v}{\partial x^2 \partial t} = G(t, v_t) \quad \text{on } (0, T], \\ v(t) = \phi(t), \quad t \in [-T, 0]. \end{gathered} \tag{5.5}$$

Now we show that G satisfies assumption (H1). For this take $t, s \in (0, T]$ and $\psi_1, \psi_2 \in \mathcal{C}_0$

$$\begin{aligned} &\|G(t, \psi_1) - G(s, \psi_2)\|_B \\ &\leq \|g(t) - g(s)\|_B + \Big\| \int_{-t}^{0} a(-\eta)k(t + \eta, \psi_1(\eta))d\eta - \int_{-s}^{0} a(-\eta)k(s + \eta, \psi_2(\eta))d\eta \Big\|_B. \end{aligned}$$

Using the given conditions on g, a and k, we obtain

$$\begin{aligned} \|G(t, \psi_1) - G(s, \psi_2)\|_B &\leq k_1|t - s| + Mk_2 \int_{-t}^{0} \{|t - s| + \|\psi_1 - \psi_2\|_0\} d\eta \\ &\quad + M \int_{-s}^{-t} \|k(s + \eta, \psi_2(\eta))\|_B d\eta. \end{aligned}$$

After some simplifications, we obtain

$$\begin{aligned} \|G(t, \psi_1) - G(s, \psi_2)\|_B &\leq (k_1 + Mk_2T + MK)|t - s| + Mk_2T\|\psi_1 - \psi_2\|_0 \\ &\leq L[|t - s| + \|\psi_1 - \psi_2\|_0], \end{aligned}$$

where

$$L = \max\{(k_1 + Mk_2T + MK), Mk_2T\}, \quad \|k(s + \eta, \psi_2(\eta))\|_B \leq K.$$

As G satisfies a Lipschitz like condition, we apply the result of Theorem 3.1 to ensure the existence and uniqueness of a strong solution of (5.1)-(5.3) .

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References

[1] D. Bahuguna, S. Abbas, J. Dabas; *Partial functional differential equation with an integral condition and applications to population dynamics,* Nonlinear Analysis 69, (2008), 2623–2635.
[2] D. Bahuguna, V. Raghavendra; *Rothe's method to parabolic integrodifferential equation via abstract integrodifferential equations,* Applicable Analysis, vol. 33, no., 3–4, 153–167, 1989.
[3] D. Bahuguna, A. K. Pani, V. Raghavendra, *Rothe's method to semilinear hyperbolic integrodifferential equations,* Journal of Applied Mathematics and Stochastic Analysis, vol. 3, no. 4, pp. 245–252, 1990.
[4] A. Bouziani, R. Mechri; *The Rothe's Method to a parabolic Integrodifferential Equation with a Nonclassical Boundary conditions,* Int. J. Stoch. Anal. 2010, Art. ID 519684, 1–16.
[5] Shruti A. Dubey; *The method of lines applied to nonlnear nonlocal functional differential equations,* J. Math. Anal. Appl. 376(2011) 275–281.
[6] J. Kacur; *Method of Rothe in Evolution Equations,* Teubner-Text zur Mathematik, Vol. 80, 1985.

[7] N. Kikuchi, J. Kacur; *Convergence of Rothe's method in Hӧlder spaces,* Appl. Math. 48 (2003), no. 5, 353-365.

[8] A. G. Lakoud, A. Chaoni; *Rothe's Method Applied to semilinear Hyperbolic Integrodifferetial equation with integral condition,* Int. J. open Problem Compt. Math. Vol. 4, No. 1, March 2011.

[9] A. G. Lakoud, M. S. Jasmati, A. Chaoni; *Rothe's method for an integrodifferential equation with integral conditions,* Nonlinear Analysis 72 (2010) 1552–1530.

[10] A. G. Lakoud, D. Belakroum; *Time-descretization schema for an integrodifferential Sobolev type equation with integral conditions,* Appied Math. Comp. Vol. 218, 9 (2012), 4695–4702.

[11] N. Merazga, A. Bouziani; *Rothe time-discretization method for a nonlocal problem arising in thermoelasticity,* Journal of Applied Mathematics and Stochastic Analysis 2005:1,(2005) 13–28.

[12] A. Pazy; *Semigroup of linear operators and application to partial differential equations,* Springer-Verlag, New York, 1983.

[13] K. Rektorys; *The Method of Discretization in time and Partial Differential Equations,* D. Reidel Publishing Company, 1982.

[14] K. Rektorys; *On application of direct variational metheods to the solution of parabolic boundary value problems of arbitrary order in the space variables,* Czech. Math. J., 21(96), 1971, 318–339.

[15] K. Rektorys; *Numerical and theoretical treating of evolution problems by the method of discretization in time,* Equadiff 6 (Brno, 1985), 71–84, Lecture Notes in Math., 1192, Springer, Berlin, 1986.

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