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EXPONENTIAL DICHOTOMY OF NONAUTONOMOUS PERIODIC SYSTEMS IN TERMS OF THE BOUNDEDNESS OF CERTAIN PERIODIC CAUCHY PROBLEMS

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Abstract. We prove that a family of q-periodic continuous matrix valued function $\{A(t)\}_{t\in\mathbb{R}}$ has an exponential dichotomy with a projector P if and only if $\int_0^t e^{i\mu s}U(t,s)Pds$ is bounded uniformly with respect to the parameter μ and the solution of the Cauchy operator Problem

$$\dot{Y}(t) = -Y(t)A(t) + e^{i\mu t}(I-P), \quad t \ge s$$
$$Y(s) = 0,$$

has a limit in $\mathcal{L}(\mathbb{C}^n)$ as s tends to $-\infty$ which is bounded uniformly with respect to the parameter μ. Here, $\{U(t,s) : t, s \in \mathbb{R}\}$ is the evolution family generated by $\{A(t)\}_{t\in\mathbb{R}}$, μ is a real number and q is a fixed positive number.

1. Introduction

Let $\mathbb{C}^n$ be the linear space of all complex vectors and $\mathcal{L}(\mathbb{C}^n)$ the Banach algebra of all linear $\mathbb{C}^n$-valued operators. The norm on $\mathbb{C}^n$ and on $\mathcal{L}(\mathbb{C}^n)$ is denoted by the same symbol, namely $\|\cdot\|$.

Consider the nonautonomous q-periodic system

$$x'(t) = A(t)x(t), \quad x(t) \in \mathbb{C}^n,\ t \in \mathbb{R}. \tag{1.1}$$

Here, $q > 0$ is a given real number and $A(t)$ is a q-periodic continuous matrix valued function i.e. $A(t+q) = A(t)$ for all $t \in \mathbb{R}$.

Exponential dichotomy is one of the fundamental asymptotic properties of solution of the linear differential system (1.1) on $\mathbb{C}^n$, [4]. In the autonomous case i.e. when $A(t) = A$ is a constant matrix, the exponential dichotomy is equivalent to the fact that the matrix A has no eigenvalues on the imaginary axis $i\mathbb{R}$. Nevertheless, when A depends on time variable, the study of the exponential dichotomy is more difficult. Important results on this topic are obtained in [3] and [5].

In this article, we aim to investigate the exponential dichotomy of the evolution family generated by the system (1.1). More exactly, we prove that this evolution family has an exponential dichotomy with respect to a projector P (see the next

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section for definitions) if and only if the solution of the operator Cauchy Problem

$$\begin{gathered} \dot{X}(t) = A(t)X(t) + e^{i\mu t}P, \quad X(t) \in \mathcal{L}(\mathbb{C}^n),\ t \in \mathbb{R} \\ X(0) = 0 \end{gathered} \tag{1.2}$$

is bounded, uniformly with respect to μ and the solution $V_\mu(\cdot)$ of the Cauchy Problem

$$\begin{gathered} \dot{Y}(t) = -Y(t)A(t) + e^{i\mu t}Q, \quad Y(t) \in \mathcal{L}(\mathbb{C}^n),\ t \geq s \\ Y(s) = 0 \end{gathered} \tag{1.3}$$

has a limit in $\mathcal{L}(\mathbb{C}^n)$ as $s \to -\infty$; i.e. $\int_{-\infty}^t e^{i\mu\tau} QU(\tau, t)d\tau$ exists in $\mathcal{L}(\mathbb{C}^n)$, and

$$\sup_{\mu\in\mathbb{R}} \sup_{t\in\mathbb{R}} \Big\| \int_{-\infty}^t e^{i\mu s} QU(s,t)ds \Big\| := M_2 < \infty.$$

Here, μ is a real number and Q denotes the projector $I - P$.

2. Definitions, notation and preliminary results

For $A \in \mathcal{L}(\mathbb{C}^n)$, we denote by $\sigma(A)$ spectrum of A; i.e., the set of all complex scalars $z \in \mathbb{C}$ for which the operator $zI - A$ is not invertible. I denotes the identity linear operator on $\mathbb{C}^n$. As it is well-known, the solution of the operator Cauchy Problem

$$\begin{gathered} \dot{X}(t) = A(t)X(t), \quad t \in \mathbb{R} \\ X(0) = I, \end{gathered}$$

denoted by $P(\cdot)$ is called the fundamental matrix associated to the family $\{A(t)\}_{t\in\mathbb{R}}$.

For every $t \in \mathbb{R}$, $P(t)$ is invertible and its inverse is the solution of the following operator Cauchy Problem

$$\begin{gathered} \dot{Y}(t) = -Y(t)A(t), \quad t \in \mathbb{R} \\ Y(0) = I. \end{gathered}$$

It is not difficult to verify that the family $\mathcal{U} := \{U(t,s) := P(t)P^{-1}(s),\ t, s \in \mathbb{R}\}$ satisfies the following properties:

- $U(t,t) = I$, for all $t \in \mathbb{R}$.
- $U(t,s)U(s,r) = U(t,r)$, for all $t, s, r \in \mathbb{R}$.
- The map
$$(t,s) \mapsto U(t,s) : \{(t,s) \in \mathbb{R}^2 :\ t \geq s\} \to \mathcal{L}(\mathbb{C}^n)$$
is continuous.
- $U(t+q, s+q) = U(t,s)$ for all $t, s \in \mathbb{R}$.
- $\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s)$ for all $t, s \in \mathbb{R}$.
- $\frac{\partial}{\partial s}U(t,s) = -U(t,s)A(s)$ for all $t, s \in \mathbb{R}$.
- There exist two real constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that
$$\|U(t,s)\| \leq Me^{\omega|t-s|} \quad \text{for all } t, s \in \mathbb{R}.$$

Definition 2.1. The evolution family $\mathcal{U}$ is said to have a uniform exponential dichotomy with respect to the projector P (i.e. $P \in \mathcal{L}(\mathbb{C}^n)$ and $P^2 = P$) if there exist positive constants N_1, N_2, ν_1 and ν_2 such that

1. $U(t,s)P = PU(t,s)$, for all $t, s \in \mathbb{R}$
2. $\|U(t,s)P\| \leq N_1 e^{-\nu_1(t-s)}$, for all $t \geq s \in \mathbb{R}$

3. $\|QU(s,t)\| \le N_2 e^{-\nu_2(t-s)}$, for all $t \ge s \in \mathbb{R}$.

Here, $Q := I - P$ and $U(s,t)$ is the inverse of $U(t,s)$.

It is clear that $Q^2 = Q$ and $PQ = QP = 0$.

Remark 2.2. For the special case when $P = I$ (and so $Q = 0$), we recognize the uniform exponential stability of the evolution family $\mathcal{U}$.

Example 2.3. Set two 1-periodic continuous functions $a(\cdot)$ and $b(\cdot)$; i.e., $a(t+1) = a(t)$ and $b(t+1) = b(t)$, for all $t \in \mathbb{R}$, and so, the 1-periodic continuous matrix valued map $t \mapsto A(t)$ given by

$$A(t) = \begin{pmatrix} a(t) & 0 \\ 0 & b(t) \end{pmatrix}.$$

The system $X(t) = A(t)X(t)$ leads to the evolution family $\mathcal{U} := \{P(t)P^{-1}(s),\ s,t \in \mathbb{R}\}$, where for each real t,

$$P(t) = \begin{pmatrix} e^{\int_0^t a(s)ds} & 0 \\ 0 & e^{\int_0^t b(s)ds} \end{pmatrix}.$$

We take the projectors $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. and $Q = I - P$. It is easy to verify that for all $t, s \in \mathbb{R}$, $PU(t,s) = U(t,s)P$.

Throughout this article, we assume that there exists a projector P such that $U(t,s)P = PU(t,s)$ for all $s,t \in \mathbb{R}$.

Before announcing the main result of this article, we recall some known results which will be used in its proof. State first the following proposition.

Proposition 2.4 ([1, 2])**.** *Let $\mathcal{U} := \{U(t,s) : s,t \in \mathbb{R}\}$ be a strongly continuous and q-periodic evolution family acting on the Banach space X. Then, the following assertions are equivalent:*

1. *The family $\mathcal{U}$ is uniformly exponentially stable.*
2. *There are two real constants $N \ge 1$ and $\nu > 0$ such that for all $t \ge 0$, we have $\|U(t,0)\| \le Ne^{-\nu t}$.*
3. *The spectral radius of $U(q,0)$,*
$$r(U(q,0)) := \sup\{|\lambda|;\ \lambda \in \sigma(U(q,0))\} = \lim_{n\to\infty} \|U(q,0)^n\|^{1/n},$$
is less than 1.
4. *For each $\mu \in \mathbb{R}$, we have*
$$\sup_{n\in\mathbb{N}} \Big\| \sum_{k=1}^{n} e^{i\mu k} U(q,0)^k \Big\|$$
is finite.

The following technical lemma will be an important ingredient of our proof.

Lemma 2.5 ([1])**.** *Consider the functions h_1 and h_2 defined from $[0,q]$ to $\mathbb{C}$, respectively, by*

$$h_1(t) = \begin{cases} t, & \text{if } t \in [0,q/2] \\ q-t, & \text{if } t \in [q/2,q], \end{cases}$$
$$h_2(t) = t(q-t) \quad \text{for all } t \in [0,q].$$

If we denote $H_j(\mu) := \int_0^q h_j(s)e^{-i\mu s}ds$, *for* $j = 1, 2$ *and* $\mathcal{A} := \{\frac{4k\pi}{q} : k \in \mathbb{Z} \setminus \{0\}\}$, *then*

- $H_1(\mu) = 0$ *if and only if* $\mu \in \mathcal{A}$,
- $H_2(\mu \neq 0$ *for all* $\mu \in \mathcal{A}$.

3. Main result and its proof

Theorem 3.1. *The following statements are equivalent:*

i. *The evolution family* $\mathcal{U}$ *has an exponential dichotomy with respect to the projector* P.
ii. *The following assertions hold:*
   1. $\sup_{\mu\in\mathbb{R}} \sup_{t\in\mathbb{R}} \big\| \int_0^t e^{i\mu s}U(t,s)Pds\big\| := M_1 < \infty$.
   2. *The solution of the equation* (1.3) *has a limit in* $\mathcal{L}(\mathbb{C}^n)$ *as* s *tends to* $-\infty$ *(i.e.* $\int_{-\infty}^t e^{i\mu s}QU(s,t)ds$ *exists) and*
$$\sup_{\mu\in\mathbb{R}} \sup_{t\in\mathbb{R}} \big\| \int_{-\infty}^t e^{i\mu s}QU(s,t)ds\big\| := M_2 < \infty,$$
   where M_1 *and* M_2 *are two absolutely positive constants.*

Proof. It is not difficult to show that the function $U_\mu(t) := \int_0^t e^{i\mu s}U(t,s)Pds$ is the solution of (1.2) and the function $V_\mu(t) := \int_s^t e^{i\mu s}QU(s,t)ds$ is the solution of (1.3).

• Let us first show that (i) implies (ii). If the evolution family has an exponential dichotomy with respect to the projector P, then in view of the Definition 2.1, we have $\|U(t,s)P\| \le N_1e^{-\nu_1(t-s)}$, for all $t \ge s \in \mathbb{R}$, for some positive constants N_1 and $\nu_1 > 0$. An easy calculation gives that for all $\mu, t \in \mathbb{R}$,
$$\big\| \int_0^t e^{i\mu s}U(t,s)Pds\big\| \le \frac{N_1}{\nu_1}.$$
Since $\|QU(s,t)\| \le N_2e^{-\nu_2(t-s)}$, for all $t \ge s \in \mathbb{R}$, it follows that the improper integral $\int_{-\infty}^t e^{i\mu s}QU(t,s)ds$ is well-defined which implies that the solution of (1.3),$V_\mu(\cdot)$, has a limit in $\mathcal{L}(\mathbb{C}^n)$ as s tends to $-\infty$ in $\mathcal{L}(\mathbb{C}^n)$, and, in addition,
$$\big\| \int_{-\infty}^t e^{i\mu s}QU(t,s)ds\big\| \le \frac{N_2}{\nu_2},$$

• Now, we show the converse. For $j = 1, 2$, we set the functions $f_j(t) := h_j(t)U(t,0)P$ defined on $[0,q]$ where h_j are those introduced in the Lemma 2.5. We extend them by periodicity on the whole real line.

If we put $t = (N+1)q$, where N is a positive integer number, for $j = 1, 2$, then denoting
$$L_k(f_j) := \int_{qk}^{q(k+1)} U((N+1)q,\tau)e^{-i\mu\tau}f_j(\tau)d\tau,$$
we obtain
$$\sup_{N\in\mathbb{Z}_+} \big\| \sum_{k=0}^{N} L_k(f_j)\big\| := M_1(\mu, f_j) < \infty.$$
Moreover, as $U(t+q, s+q) = U(t,s)$ for all $t, s \in \mathbb{R}$, it follows that $U(pq, kq) = U((p-k)q, 0) = U(q,0)^{p-k}$ for all $p, k \in \mathbb{Z}_+$ with $p \ge k$. Therefore, since P

commutes with $U(t,s)$, and so with every power of $U(t,s)$, we have

$$\begin{aligned}
L_k(f_j) &= \int_{qk}^{q(k+1)} U((N+1)q,(k+1)q)U((k+1)q,\tau)e^{-i\mu\tau}f_j(\tau)d\tau \\
&= \int_{qk}^{q(k+1)} U((N-k)q,0)U((k+1)q,\tau)e^{-i\mu\tau}f_j(\tau)d\tau \\
&= U(q,0)^{N-k}\int_{qk}^{q(k+1)} U((k+1)q,\tau)e^{-i\mu\tau}f_j(\tau)d\tau \\
&= U(q,0)^{N-k}e^{-i\mu kq}\int_0^q e^{-i\mu u}U(q,u)f_j(u)du \\
&= (U(q,0)P)^{N-k+1}e^{i\mu(N-k+1)q}e^{-i\mu(N+1)}H_j(\mu).
\end{aligned}$$

Thanks to the Lemma 2.5, we obtain

$$\begin{aligned}
(U(q,0)P))^{N-k+1}e^{i\mu(N-k+1)q} &= \frac{e^{i\mu(N+1)}}{H_1(\mu)}L_k(f_1), \quad \text{for all } \mu\notin\mathcal{A}, \\
(U(q,0)P))^{N-k+1}e^{i\mu(N-k+1)q} &= \frac{e^{i\mu(N+1)}}{H_1(\mu)}L_k(f_2), \quad \text{for all } \mu\in\mathcal{A}.
\end{aligned}$$

Therefore, we have

$$\begin{gathered}
\Big\|\sum_{k=0}^{N}(U(q,0)P))^{N-k+1}e^{i\mu(N-k+1)q}\Big\| \le \frac{1}{|H_1(\mu)|}M_1(\mu,f_1), \quad \text{if } \mu\notin\mathcal{A} \\
\Big\|\sum_{k=0}^{N}(U(q,0)P))^{N-k+1}e^{i\mu(N-k+1)q}\Big\| \le \frac{1}{|H_2(\mu)|}M_1(\mu,f_2), \text{if } \mu\in\mathcal{A}.
\end{gathered}$$

This implies that

$$\sup_{N\in\mathbb{Z}_+}\Big\|\sum_{k=0}^{N+1}(U(q,0)P))^{k}e^{i\mu kq}\Big\| < \infty.$$

Using Proposition 2.4, we deduce that the spectral radius $r(UP)<1$, where $U:=U(q,0)$, which implies that there exist two constants $N_1\ge 0$ and $\nu_1>0$ such that

$$\big\|U(t,s)P\big\| \le N_1e^{-\nu_1(t-s)} \quad \forall t\ge s\in\mathbb{R}\,.$$

From the second assumption, we have that for all $s\in\mathbb{R}$ large enough,

$$\Big\|\int_s^t e^{i\mu\tau}QU(\tau,t)d\tau\Big\| \le M_2+1.$$

Consider, for $j=1,2$, the functions g_j defined on $[0,q]$ by $g_j(\tau)=h_j(\tau)U(0,\tau)Q$, where the functions h_j are defined as in the Lemma 2.5 and we extend this functions by periodicity on the whole real line. Besides, by derivation, we can show easily that, for $j=1,2$, the function $\int_s^t e^{i\mu\tau}g_j(\tau)U(\tau,t)d\tau$ is the solution of the the differential equation $Z'(t)=-Z(t)A(t)+e^{i\mu t}g_j(t)$, $Z(s)=0$, $t\ge s$. We remark also that this functions are bounded. We proceed as in [1, Theorem 3.2].

If we put $t=(N+1)q$ and $s=mq$, for $N>m$ two integer numbers, then we have that

$$\Big\|\int_{mq}^{(N+1)q} e^{-i\mu\tau}g_j(\tau)U(\tau,(N+1)q)d\tau\Big\| := M_2(\mu,g_j) < \infty.$$

It follows that

$$\sup_{N\in\mathbb{Z}_+}\sum_{k=m}^{N}\|S_k(g_j)\| = M_2(\mu, g_j) < \infty;$$

where $S_k(g_j) = \int_{kq}^{(k+1)q} e^{-i\mu\tau} g_j(\tau)U(\tau,(N+1)q)d\tau$, for each $k = m,\dots,N$. For each $k = m, m+1,\dots$, similarly to the previous calculation, we have

$$\begin{aligned}
S_k(g_j) &= \int_{kq}^{(k+1)q} e^{-i\mu\tau} g_j(\tau)U(\tau,(k+1)q)U((k+1)q,(N+1)q)d\tau\\
&= \int_{kq}^{(k+1)q} e^{-i\mu\tau} g_j(\tau)U(\tau,(k+1)q)U(0,(N-k)q)d\tau\\
&= \int_{kq}^{(k+1)q} e^{-i\mu\tau} g_j(\tau)U(\tau,(k+1)q)d\tau U(0,q)^{N-k}\\
&= \int_{0}^{q} e^{-i\mu u} g_j(u)U(u,q)du e^{-i\mu kq}U(0,q)^{N-k}\\
&= \int_{0}^{q} e^{-i\mu u} h_j(u)du e^{-i\mu kq}(QU(0,q))^{N-k+1}\\
&= H_j(\mu)e^{-i\mu(N+1)q}e^{i\mu(N-k+1)}(QU(0,q))^{N-k+1}.
\end{aligned}$$

By using Lemma 2.5, we can write

$$\begin{aligned}
e^{i\mu(N-k+1)}(QU(0,q))^{N-k+1} &= \frac{1}{H_1(\mu)}e^{i\mu(N+1)q}S_k(g_1), \quad \text{if } \mu\notin\mathcal{A}\\
e^{i\mu(N-k+1)}(QU(0,q))^{N-k+1} &= \frac{1}{H_2(\mu)}e^{i\mu(N+1)q}S_k(g_2), \quad \text{if } \mu\in\mathcal{A}.
\end{aligned}$$

Then

$$\begin{aligned}
\|\sum_{k=0}^{N}(QU(0,q)))^{N-k+1}e^{i\mu(N-k+1)q}\| &\le \frac{1}{|H_1(\mu)|}M_2(\mu,g_1), \quad \text{if } \mu\notin\mathcal{A}\\
\|\sum_{k=0}^{N}(QU(0,q)))^{N-k+1}e^{i\mu(N-k+1)q}\| &\le \frac{1}{|H_2(\mu)|}M_2(\mu,g_2), \quad \text{if } \mu\in\mathcal{A}.
\end{aligned}$$

By Proposition 2.4, if we denote $V := U(0,q)$, we deduce that $r(QV) < 1$, and then there exist constants $N_2 \ge 0$, $\nu_2 > 0$ such that

$$\|QU(s,t)\| \le N_2 e^{-\nu_2(t-s)} \quad \forall t \ge s \in \mathbb{R},$$

which completes the proof. □

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