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OSCILLATION CRITERIA FOR SECOND-ORDER NONLINEAR PERTURBED DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the oscillation of solutions to the nonlinear second-order differential equation

$$\Big(r(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)\Big)' + P(t,x'(t))\psi(x(t)) + Q(t,x(t)) = 0.$$

We obtain sufficient conditions for the oscillation of all solutions to this equation.

1. Introduction

This article concerns the oscillation of solutions to the nonlinear second-order differential equation

$$\Big(r(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)\Big)' + P(t,x'(t))\psi(x(t)) + Q(t,x(t)) = 0, \quad t \ge t_0 \tag{1.1}$$

where $r \in C^1(I,\mathbb{R}^+)$, $P,Q \in C(I\times\mathbb{R},\mathbb{R})$, $\psi(x) \in C(\mathbb{R},\mathbb{R}^+)$, $I = [T_0,\infty) \subset \mathbb{R}$, $0 < \psi(x) < \gamma$ and α is a positive constant. Throughout this article, we assume the following conditions:

(E1) $Q \in C(I\times\mathbb{R},\mathbb{R})$ and there exist $f \in C^1(\mathbb{R},\mathbb{R})$ and a continuous function $q(t)$ such that

$$xf(x) > 0 \quad \text{and} \quad \frac{Q(t,x)}{f(x)} \ge q(t) \quad \text{for } x \ne 0.$$

(E2) $P \in C(I\times\mathbb{R},\mathbb{R})$ and there exists a continuous function $p(t)$ such that

$$\frac{P(t,x'(t))}{|x'(t)|^{\alpha-1}x'(t)} \ge p(t) \quad \text{for } x' \ne 0.$$

We restrict our attention to solutions satisfying $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_0$.

A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is said to be non-oscillatory. If all solutions of (1.1) are oscillatory, (1.1) is called oscillatory.

The oscillatory behavior of solutions of second-order ordinary differential equations, including the existence of oscillatory and non-oscillatory solutions, has been

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the subject of intensive investigations; see for example [1]–[13]. Some criteria involve the behavior of the integral of alternating coefficients. In this article, we give general integral criteria for the oscillation of (1.1), which contain some of the results in the references as particular cases.

2. Main results

Let $h(\cdot)$ and $K(\cdot,\cdot,\cdot) : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ be continunous functions such that for each fixed t, s, the function $K(t, s, .)$ is nondecreasing. Then there exists a solution to the integral equation

$$v(t) = h(t) + \int_{t_0}^{t} K(t, s, v(s))\, ds, \quad t \ge t_0. \tag{2.1}$$

Furthermore there exists a "minimal solution" v in the sense that any solution y of this equation satisfies $v(t) \le y(t)$ for all $t \ge t_0$. See [1, p. 322].

Lemma 2.1 (citew1)**.** *If v is the minimal solution of (2.1) and*

$$u(t) \ge h(t) + \int_{t_0}^{t} K(t, s, u(s))ds, \quad t \ge t_0,$$

then $u(t) \ge v(t)$ for all $t \ge t_0$.

Similarly for a maximal solution $w(t)$ of (2.1): if $u(t) \le h(t)+\int_{t_0}^{t} K(t, s, u(s))ds$, then $u(t) \le w(t)$ for all $t \ge t_0$.

Our main results reads as follows.

Theorem 2.2. *Assume (E1), $f'(x) \ge 0$, $p(t) \le 0$, $q(t) > 0$ and $\int_{t_0}^{\infty}(\frac{1}{r^{1/\alpha}(t)})dt = \infty$. Also assume that there exists a positive function $\rho(t)$ such that*

$$\int_{t_0}^{\infty} q(t)\rho(t)dt = \infty, \tag{2.2}$$

$$p(t)\rho(t) \ge r(t)\rho'(t). \tag{2.3}$$

Then every solution of (1.1) is oscillatory.

Proof. For the shake of contradiction, suppose that (1.1) has a non-oscillatory solution $x(t)$. Without loss of generality, suppose that it is an eventually positive solution (if it is an eventually negative solution, the proof is similar), that is, $x(t) > 0$ for all $t \ge t_0$. We consider the following three cases.

Case 1. Suppose that $x'(t)$ is oscillatory. Then there exists $t_1 \ge t_0$ such that $x'(t_1) = 0$. From (1.1), we have

$$\begin{aligned}
&\Big[r(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)\exp\Big(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\Big)\Big]' \\
&= [r(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)]'\exp\Big(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\Big) \\
&\quad + p(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)\exp\Big(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\Big) \\
&= (-P(t, x'(t))\psi(x(t)) - Q(t, x(t)))\exp\Big(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ p(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)\exp\Big(\int_{t_0}^t \frac{p(s)}{r(s)}ds\Big)\\
&\le (-p(t)|x'(t)|^{\alpha-1}x'(t)\psi(x(t)) - q(t)f(x(t)))\exp\Big(\int_{t_0}^t \frac{p(s)}{r(s)}ds\Big)\\
&\quad + p(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)\exp\Big(\int_{t_0}^t \frac{p(s)}{r(s)}ds\Big)\\
&= -q(t)f(x(t))\exp\Big(\int_{t_0}^t \frac{p(s)}{r(s)}ds\Big) < 0
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&r(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)\exp\Big(\int_{t_0}^t \frac{p(s)}{r(s)}ds\Big)\\
&< r(t_1)\psi(x(t_1))|x'(t_1)|^{\alpha-1}x'(t_1)\exp\Big(\int_{t_0}^{t_1} \frac{p(s)}{r(s)}ds\Big) = 0, \quad \forall t \ge t_1\,.
\end{aligned}
$$

it follows that $x'(t) < 0$ for all $t > t_1$, which contradicts to the assumption that $x'(t)$ is oscillatory.

Case 2. Assume that $x'(t) < 0$. From (1.1), we obtain

$$
\begin{aligned}
-[r(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)]' &= [r(t)\psi(x(t))(-x'(t))^{\alpha}]'\\
&= P(t,x'(t))\psi(x(t)) + Q(t,x(t))\\
&\ge p(t)|x'(t)|^{\alpha-1}x'(t)\psi(x(t)) + q(t)f(x(t))\\
&= -p(t)(-x'(t))^{\alpha}\psi(x(t)) + q(t)f(x(t)) \ge 0
\end{aligned}
$$

then there exists an $M > 0$ and a $t_1 \ge t_0$, such that

$$
r(t)\psi(x(t))(-x'(t))^{\alpha} \ge M, \quad \forall t \ge t_1. \tag{2.4}
$$

It follows that

$$
\begin{gathered}
\gamma(-x'(t))^{\alpha} \ge \frac{M}{r(t)},\\
x(t) \le -\int_{t_1}^{\infty} \big(\frac{M}{\gamma}\big)^{1/\alpha}\frac{1}{r^{1/\alpha}(t)}dt, \quad \forall t \ge t_1
\end{gathered}
$$

which implies $\lim_{t\to\infty} x(t) = -\infty$; this contradicts the assumption that $x(t) > 0$.

Case 3. Suppose that $x'(t) > 0$. Define $w(t) = \rho(t)r(t)\psi(x(t))(x'(t))^{\alpha}$. Differentiating $w(t)$ and using (1.1),

$$
w'(t) = [r(t)\psi(x(t))(x'(t))^{\alpha}]'\rho(t) + r(t)\psi(x(t))(x'(t))^{\alpha}\rho'(t), \quad \forall t \ge t_0. \tag{2.5}
$$

Then we obtain

$$
\begin{aligned}
\frac{w'(t)}{f(x(t))} &= -\frac{P(t,x'(t))\psi(x(t))\rho(t)}{f(x(t))} - \frac{Q(t,x(t))\rho(t)}{f(x(t))}\\
&\quad + \frac{\rho'(t)r(t)\psi(x(t))(x'(t))^{\alpha}}{f(x(t))}, \quad \forall t \ge t_0\,.
\end{aligned}
$$

Noticing that

$$
\Big(\frac{w(t)}{f(x(t))}\Big)' = \frac{w'(t)f(x(t)) - w(t)f'(x(t))x'(t)}{f^2(x(t))}
$$

$$
\begin{aligned}
&= -\frac{P(t,x'(t))\psi(x(t))\rho(t)}{f(x(t))} - \frac{Q(t,x(t))\rho(t)}{f(x(t))}\\
&\quad + \frac{\rho'(t)r(t)\psi(x(t))(x'(t))^\alpha}{f(x(t))} - \frac{w(t)f'(x(t))x'(t)}{f^2(x(t))}, \quad \forall t \ge t_0\,.
\end{aligned}
$$

Integrating the above from t_0 to t, we obtain

$$
\begin{aligned}
\frac{w(t)}{f(x(t))} = \frac{w(t_0)}{f(x(t_0))} - \int_{t_0}^{t}\Big[&\frac{P(s,x'(s))\psi(x(s))\rho(s)}{f(x(s))} + \frac{Q(s,x(s))\rho(s)}{f(x(s))}\\
&- \frac{\rho'(s)r(s)\psi(x(s))(x'(s))^\alpha}{f(x(s))} + \frac{w(s)f'(x(s))x'(s)}{f^2(x(s))}\Big]ds,
\end{aligned}
$$

$$
\begin{aligned}
\frac{w(t)}{f(x(t))} \le \frac{w(t_0)}{f(x(t_0))} - \int_{t_0}^{t}\Big[&q(s)\rho(s) + \frac{(\rho(s)p(s)-\rho'(s)r(s))(x'(s))^\alpha\psi(x(s))}{f(x(s))}\\
&+ \frac{w(s)f'(x(s))x'(s)}{f^2(x(s))}\Big]ds.
\end{aligned}
$$

Using (2.2), (2.3) and $x'(t) > 0$, we have

$$
0 \le \lim_{t\to\infty}\frac{w(t)}{f(x(t))} = -\infty,
$$

this is a contradiction. The proof is complete. □

Theorem 2.3. *Assume that $f'(x) \ge 0$ and $\psi(x(t)) \equiv 1$. Also assume that*

$$
\rho_0(t) = \exp\Big(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\Big), \tag{2.6}
$$

$$
\int_{t_0}^{\infty}\frac{dt}{(\rho_0(t)r(t))^{1/\alpha}} = \infty, \tag{2.7}
$$

and $\rho_0(t)$ satisfies (2.2)*. Then every solution of* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of (1.1). Without loss of generality, we assume that $x(t)$ is eventually positive. Let $w(t) = \rho_0(t)r(t)|x'(t)|^{\alpha-1}x'(t)$. Then

$$
w(t)x'(t) = \rho_0(t)r(t)|x'(t)|^{\alpha-1}(x'(t))^2 \ge 0 \quad \text{for } t \ge t_0
$$

and

$$
w'(t) = (r(t)|x'(t)|^{\alpha-1}x'(t))'\rho_0(t) + r(t)|x'(t)|^{\alpha-1}x'(t)\rho_0'(t) \quad \forall t \ge t_0\,. \tag{2.8}
$$

In view of (1.1) and (2.6), we obtain

$$
\begin{gathered}
w'(t) = (-P(t,x'(t)) - Q(t,x(t)))\rho_0(t) + |x'(t)|^{\alpha-1}x'(t)p(t)\rho_0(t),\\
w'(t) \le (-p(t)|x'(t)|^{\alpha-1}x'(t) - q(t)f(x(t)))\rho_0(t) + |x'(t)|^{\alpha-1}x'(t)p(t)\rho_0(t),\\
\frac{w'(t)}{f(x(t))} \le -q(t)\rho_0(t) \quad \forall t \ge t_0\,.
\end{gathered} \tag{2.9}
$$

Since

$$
\begin{aligned}
\Big(\frac{w(t)}{f(x(t))}\Big)' &= \frac{w'(t)f(x(t)) - w(t)f'(x(t))x'(t)}{f^2(x(t))}\\
&\le -q(t)\rho_0(t) - \frac{w(t)f'(x(t))x'(t)}{f^2(x(t))} \quad \forall t \ge t_0,
\end{aligned}
$$

integrating from t_0 to t, we have

$$-\frac{w(t)}{f(x(t))} \geq -\frac{w(t_0)}{f(x(t_0))} + \int_{t_0}^{t} q(s)\rho_0(s)ds + \int_{t_0}^{t} \frac{w(s)f'(x(s))x'(s)}{f^2(x(s))}ds, \quad \forall t \geq t_0.$$

By using (2.2), there exists a constant $m > 0$ and $t_1 \geq t_0$ such that

$$-\frac{w(t_0)}{f(x(t_0))} + \int_{t_0}^{t} q(s)\rho_0(s)ds + \int_{t_0}^{t_1} \frac{w(s)f'(x(s))x'(s)}{f^2(x(s))}ds \geq m \quad \forall t \geq t_0 \tag{2.10}$$

which means that

$$-\frac{w(t)}{f(x(t))} \geq m + \int_{t_1}^{t} \frac{w(s)f'(x(s))x'(s)}{f^2(x(s))}ds. \tag{2.11}$$

Because that $x(t)$ is positive, (2.11) implies $-w(t) > 0$, or equivalently $x'(t) < 0$.

Let

$$u(t) = -w(t) = -\rho_0(t)r(t)|x'(t)|^{\alpha-1}x'(t) = \rho_0(t)r(t)(-x'(t))^{\alpha}, \tag{2.12}$$

thus (2.11) can be written as

$$u(t) \geq mf(x(t)) + \int_{t_1}^{t} \frac{f(x(t))f'(x(s))(-x'(s))}{f^2(x(s))}u(s)ds. \tag{2.13}$$

Define

$$K(t,s,u) = \frac{f(x(t))f'(x(s))(-x'(s))}{f^2(x(s))}u. \tag{2.14}$$

Then, for any fixed t and s, $K(t,s,u)$ is nondecreasing in u. Let $v(t)$ be the minimal solution of the equation

$$v(t) = mf(x(t)) + \int_{t_1}^{t} \frac{f(x(t))f'(x(s))(-x'(s))}{f^2(x(s))}v(s)ds. \tag{2.15}$$

Applying Lemma 2.1, we obtain

$$u(t) \geq v(t) \quad \forall t \geq t_0. \tag{2.16}$$

Dividing both sides of (2.15) by $f(x(t))$ and deriving both sides of (2.15),

$$\Big(\frac{v(t)}{f(x(t))}\Big)' = \Big(m + \int_{t_1}^{t} \frac{f'(x(s))(-x'(s))}{f^2(x(s))}v(s)ds\Big)' = \frac{f'(x(t))(-x'(t))}{f^2(x(t))}v(t). \tag{2.17}$$

On the other hand

$$\Big(\frac{v(t)}{f(x(t))}\Big)' = \frac{v'(t)}{f(x(t))} + \frac{f'(x(t))(-x'(t))}{f^2(x(t))}v(t). \tag{2.18}$$

Combining (2.17) and (2.18), it follows that

$$v'(t) \equiv 0. \tag{2.19}$$

So $v(t) = v(t_1) = mf(x(t_1))$, $t \geq t_0$. From (2.16), we obtain

$$-x'(t) \geq (mf(x(t_1)))^{1/\alpha}\frac{1}{(\rho_0(t)r(t))^{1/\alpha}}, \quad \forall t \geq t_1. \tag{2.20}$$

Integrating both sides of this inequality above from t_1 to t, we have

$$-x(t) + x(t_1) \geq (mf(x(t_1)))^{1/\alpha}\int_{t_1}^{t} \frac{ds}{(\rho_0(s)r(s))^{1/\alpha}}.$$

Letting $t \to \infty$, and using (2.7), it follows that $\lim_{t\to\infty} x(t) \leq -\infty$, which contradicts to that $x(t)$ is eventually positive. The proof is complete. □

In what follows, we always assume that $H(t) \in C^2(\mathbb{R};\mathbb{R})$ and it satisfies the following two conditions:

(H1) $H(t) > 0$ for all $t \geq t_0$, $H(t)$ is a bounded;
(H2) $H'(t) = h(t)$ is a bounded.

Theorem 2.4. *Assume that $f'(x) \geq 0$, $\int_{t_0}^{\infty} \frac{dt}{(r(t))^{1/\alpha}} = \infty$, $\psi(x(t)) \equiv 1$, and*

$$p(t) \leq 0, \quad q(t) > 0, \tag{2.21}$$

or

$$p(t) \leq 0, \quad q(t) \leq 0, \quad \lim_{t\to\infty} \frac{p(t)}{q(t)} = M > 0. \tag{2.22}$$

Suppose further that there exists a function $H(t)$ that satisfies (H1), (H2)*, and such that*

$$\int_{t_0}^{\infty} H(t)\varphi(t)dt = \infty, \tag{2.23}$$

$$\limsup_{t\to\infty} v(t)r(t) < \infty, \tag{2.24}$$

where

$$\varphi(t) = v(t)(q(t) - p(t)h(t) - (r(t)h(t))'), \tag{2.25}$$

$$v(t) = \exp\Big(\int_{t_0}^{t} \big(\frac{p(s)}{r(s)} - \frac{h(s)}{H(s)}\big)ds\Big). \tag{2.26}$$

Then every solution of (1.1) *is oscillatory.*

Proof. Assume to the contrary that (1.1) has a non-oscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq t_0$. Define

$$u(t) = v(t)r(t)\Big(\frac{|x'(t)|^{\alpha-1}x'(t)}{f(x(t))} + h(t)\Big). \tag{2.27}$$

Differentiating, we obtain

$$\begin{aligned}
u'(t) = &\Big(\frac{p(t)}{r(t)} - \frac{h(t)}{H(t)}\Big)u(t) + v(t)\Big[-\frac{P(t,x'(t))}{f(x(t))} \\
&- \frac{Q(t,x(t))}{f(x(t))} - \frac{r(t)|x'(t)|^{\alpha-1}(x'(t))^2 f'(x(t))}{f^2(x(t))} + (r(t)h(t))'\Big],
\end{aligned}$$

$$\begin{aligned}
u'(t) &\leq \Big(\frac{p(t)}{r(t)} - \frac{h(t)}{H(t)}\Big)u(t) + v(t)\Big[-\frac{p(t)|x'(t)|^{\alpha-1}x'(t)}{f(x(t))} - q(t) \\
&\quad - \frac{r(t)|x'(t)|^{\alpha-1}(x'(t))^2 f'(x(t))}{f^2(x(t))} + (r(t)h(t))'\Big] \\
&\leq p(t)v(t)h(t) - \frac{h(t)}{H(t)}u(t) - q(t)v(t) + v(t)(r(t)h(t))' \\
&= -\frac{h(t)}{H(t)}u(t) - v(t)[q(t) - p(t)h(t) - (r(t)h(t))']
\end{aligned}$$

$$u'(t) \leq -\frac{h(t)}{H(t)}u(t) - \varphi(t).$$

Multiplying by $H(t)$, it follows that

$$\varphi(t)H(t) \leq -H(t)u'(t) - h(t)u(t). \tag{2.28}$$

We consider the following three cases.

Case 1. $u(t)$ is oscillatory. Then there exists a sequence $\{t_n\}$, $(n = 1, 2, \dots)$, $t_n \to \infty$ as $n \to \infty$ and such that $u(t_n) = 0$ $(n = 1, 2, \dots)$. Integrating both sides of (2.28) from t_0 to t_n, we obtain

$$\begin{aligned}\int_{t_0}^{t_n} H(t)\varphi(t)dt &\le -\int_{t_0}^{t_n} H(t)u'(t)dt - \int_{t_0}^{t_n} h(t)u(t)dt \\ &= -H(t)u(t)\mid_{t_0}^{t_n} - \int_{t_0}^{t_n} (-H'(t)u(t) + h(t)u(t))dt \\ &= H(t_0)u(t_0) - H(t_n)u(t_n) = H(t_0)u(t_0);\end{aligned}$$

that is,

$$\lim_{t_n\to\infty} \int_{t_0}^{t_n} H(t)\varphi(t)dt \le H(t_0)u(t_0),$$

which contradicts (2.23).

Case 2. $u(t)$ is eventually positive. Integrating both sides of (2.28) from t_0 to ∞, we obtain

$$\int_{t_0}^{\infty} H(t)\varphi(t)dt \le H(t_0)u(t_0) - \lim_{t\to\infty} H(t)u(t) \le H(t_0)u(t_0),$$

which also contradicts(2.23).

Case 3. $u(t)$ is eventually negative. If $\limsup_{t\to\infty} u(t) > -\infty$, then there exists a sequence $\{\bar{t}_n\}$, (n=1,2,...), that satisfies $\{\bar{t}_n\} \to \infty$ as $n \to \infty$ and such that $\lim_{\bar{t}_n\to\infty} u(\bar{t}_n) = \limsup_{t\to\infty} u(t) = M_1 > -\infty$. Because $H(t)$ is a bounded function, then there exists a $M_2 > 0$ such that $H(\bar{t}_n) \le M_2$, (n=1,2,...). According to (2.28), we obtain

$$\int_{t_0}^{\bar{t}_n} H(t)\varphi(t)dt \le H(t_0)u(t_0) - H(\bar{t}_n)u(\bar{t}_n) \le H(t_0)u(t_0) - M_2u(\bar{t}_n). \tag{2.29}$$

Using (2.23) and taking limit as $\bar{t}_n \to \infty$, it is easy to show that

$$\begin{aligned}\infty = \lim_{\bar{t}_n\to\infty} \int_{t_0}^{\bar{t}_n} H(t)\varphi(t)dt &\\ \le H(t_0)u(t_0) - \lim_{\bar{t}_n\to\infty} H(\bar{t}_n)u(\bar{t}_n) &\\ \le H(t_0)u(t_0) - M_1M_2 < \infty, &\end{aligned}$$

which is obviously a contradiction.

If $\limsup_{t\to\infty} u(t) = -\infty$, then $\lim_{t\to\infty} u(t) = -\infty$. From the definition of $h(t)$, combining (2.24) and (2.27), it follows that $x'(t) < 0$ and

$$\lim_{t\to\infty} (|x'(t)|^{\alpha-1}x'(t)/f(x(t))) = -\infty,$$

which implies that $\lim_{t\to\infty}((-x'(t))^{\alpha}/f(x(t))) = \infty$. Owing to $p(t) \le 0$, $q(t) \ge 0$, or $p(t) \le 0$, $q(t) \le 0$ and $\lim_{t\to\infty}(p(t)/q(t)) = M > 0$, using the similar method of the proof of Case 2 in Theorem 2.2, we will derive a contradiction. The proof is complete. □

Theorem 2.5. *Assume that* (2.24) *holds,* $f'(x) \geq 0$, $\int_{t_0}^{\infty} \frac{dt}{(r(t))^{1/\alpha}} = \infty$, *and* (2.21) *or* (2.22) *hold. Suppose further that there exists a function* $H(t)$ *that satisfies* (H1), (H2)*, and such that*

$$\int_{t_0}^{\infty} H(t)\bar{\varphi}(t)dt = \infty, \tag{2.30}$$

where

$$\bar{\varphi}(t) = v(t)(q(t) + p(t)h(t) + (r(t)h(t))'), \tag{2.31}$$

and $v(t)$ *is defined in* (2.26). *Then every solution of* (1.1) *is oscillatory when* $\psi(x(t)) \equiv 1$.

Proof. For the sake of contradiction, let (1.1) have a non-oscillatory solution. Without loss of generality, we may assume that (1.1) has an eventually positive $x(t) > 0$ for all $t \geq t_0$. Define

$$u(t) = v(t)r(t)\Big(\frac{|x'(t)|^{\alpha-1}x'(t)}{f(x(t))} - h(t)\Big).$$

The rest of proof is similar to Theorem 2.4 and is omitted. □

Theorem 2.6. *Assume* (2.24)*,* $p(t) \leq 0$, $q(t) > 0$, $f'(x) \geq 0$ *and* $\int_{t_0}^{\infty} \frac{dt}{(r(t))^{1/\alpha}} = \infty$. *Suppose further that there exists a function* $H(t)$ *that satisfies* (H1), (H2)*, and such that*

$$\int_{t_0}^{\infty} H(t)\phi(t)dt = \infty, \tag{2.32}$$

where

$$\phi(t) = v(t)(-p(t)h(t) - (r(t)h(t))'), \tag{2.33}$$

where $v(t)$ *is defined in* (2.26). *Then every solution of* (1.1) *is oscillatory when* $\psi(x(t)) \equiv 1$.

Proof. To the contrary, assume that (1.1) has a non-oscillatory solution $x(t)$. Without loss of generality, we may assume that (1.1) has an eventually positive $x(t) > 0$ for all $t \geq t_0$. Define

$$u(t) = v(t)r(t)\Big(\frac{|x'(t)|^{\alpha-1}x'(t)}{x(t)} + h(t)\Big). \tag{2.34}$$

We use (E1) and noting that $xf(x) \geq 0$ for $x \neq 0$, so $\frac{f(x)}{x} \geq 0$ for $x \neq 0$. Differentiating (2.34), we obtain

$$\begin{aligned}
u'(t) &= \Big(\frac{p(t)}{r(t)} - \frac{h(t)}{H(t)}\Big)u(t) + v(t)\Big[-\frac{P(t,x'(t))}{x(t)} - \frac{Q(t,x(t))}{x(t)} \\
&\quad - \frac{r(t)|x'(t)|^{\alpha-1}(x'(t))^2}{x^2(t)} + (r(t)h(t))'\Big] \\
&\leq \Big(\frac{p(t)}{r(t)} - \frac{h(t)}{H(t)}\Big)u(t) + v(t)\Big[-\frac{p(t)|x'(t)|^{\alpha-1}x'(t)}{x(t)} - \frac{q(t)f(x(t))}{x(t)} \\
&\quad - \frac{r(t)|x'(t)|^{\alpha-1}(x'(t))^2}{x^2(t)} + (r(t)h(t))'\Big] \\
&\leq p(t)v(t)h(t) - \frac{h(t)}{H(t)}u(t) + v(t)(r(t)h(t))' \\
&= -\frac{h(t)}{H(t)}u(t) - v(t)[-p(t)h(t) - (r(t)h(t))']
\end{aligned}$$

$$= -\frac{h(t)}{H(t)}u(t) - \phi(t).$$

Multiplying by $H(t)$, it follows that

$$H(t)\phi(t) \le -H(t)u'(t) - h(t)u(t).$$

The rest of the proof is similar to Theorem 2.4, and it is omitted. □

Theorem 2.7. *Assume* (2.24), $p(t) \le 0$, $q(t) > 0$, $f'(x) \ge 0$ *and* $\int_{t_0}^{\infty} \frac{dt}{(r(t))^{1/\alpha}} = \infty$. *Suppose further that there exists a function* $H(t)$ *satisfying* (H1), (H2), *and such that*

$$\int_{t_0}^{\infty} H(t)\overline{\phi}(t)dt = \infty, \tag{2.35}$$

where

$$\overline{\phi}(t) = v(t)(p(t)h(t) + (r(t)h(t))'), \tag{2.36}$$

where $v(t)$ *is defined in* (2.26). *Then every solution of* (1.1) *is oscillatory when* $\psi(x(t)) \equiv 1$.

Proof. For the sake of contradiction, assume that (1.1) has a non-oscillatory solution. Without loss of generality, we may assume that (1.1) has an eventually positive $x(t) > 0$ for all $t \ge t_0$. Define

$$u(t) = v(t)r(t)\Big(\frac{|x'(t)|^{\alpha-1}x'(t)}{x(t)} - h(t)\Big).$$

The rest of the proof is similar to Theorem 2.4, and it is omitted here. □

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