Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 12, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

AN EXTENSION PROBLEM RELATED TO THE SQUARE ROOT OF THE LAPLACIAN WITH NEUMANN BOUNDARY CONDITION

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Abstract. In this work we define the square root of the Laplacian operator with Neumann boundary condition via harmonic extension method. By using Fourier series and periodic even extension we define the non-local operator square root in three type of bounded domains such as an interval, square or a ball. Also, as application we study the existence of weak solutions for a class of nonlinear elliptic problems.

1. Introduction

The fractional powers of the Laplacian operator can be seen as infinitesimal generators of Levy stable diffusion processes. They arise in population dynamics, chemical reactions in liquids and other applications in mathematical physics, see for example [4].

From mathematical theory the fractional powers of the Laplacian can be defined using Fourier transform, formula of Riesz fractional derivative or else using harmonic extension techniques, see for example [8, 13, 21, 26]. The harmonic extension techniques have been frequently used and consist in considering an operator T given by

$$u \mapsto T(u)(x) = -v_z(x,0),$$

where $u : \mathbb{R}^n \to \mathbb{R}$ is a smooth bounded function and $v : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ is the unique solution of the problem

$$\begin{aligned} \Delta v(x,z) &= 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ v(x,0) &= u(x) \quad \text{on } \mathbb{R}^n. \end{aligned} \tag{1.1}$$

It is well known that the operator T that maps the Dirichlet condition u to the Neumann condition $-v_z(\cdot,0)$ is exactly the operator $(-\Delta)^{1/2}$, namely, the fractional power $s = 1/2$ of the Laplacian.

In [7] the authors generalized the above method using a similar extension problem with $s \in (0,1)$. Essentially, given a smooth bounded function $u : \mathbb{R}^n \to \mathbb{R}$, they

2000 *Mathematics Subject Classification.* 35J50, 35S05.
Key words and phrases. Harmonic extension; Neumann boundary condition;
square root of the Laplacian; nonlinear problem.
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Submitted June 17, 2013. Published January 8, 2014.

considered the extension problem

$$\begin{gathered}\Delta_x v(x,z) + \frac{a}{z} v_z(x,z) + v_{zz}(x,z) = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ v(x,0) = u(x) \quad \text{on } \mathbb{R}^n.\end{gathered}$$

where $a = 1 - 2s$ with $s \in (0,1)$, and showed that the following equality holds up to a multiplicative constant

$$(-\Delta)^s u(x) = -C_s \lim_{z \to 0^+} z^a v_z(x,z),$$

where $C_s = \frac{4^{s-\frac{1}{2}}\Gamma(s)}{\Gamma(1-s)}$.

Concerning a smooth bounded domain of $\mathbb{R}^n$ we can also define the fractional powers of the Laplacian. For example in [6] the authors studied the square root of the Laplacian operator with Dirichlet boundary condition. In this case the operator $(-\Delta)^{1/2}$ was defined using the harmonic extension problem

$$\begin{gathered}\Delta v = 0 \quad \text{in } \Omega \times (0,\infty), \\ v = 0 \quad \text{on } \partial\Omega \times [0,+\infty), \\ v = u \quad \text{on } \Omega \times \{0\},\end{gathered}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain.

In this sense there are some works defining the square root of the Laplacian with Neumann boundary condition in bounded domains, see e.g. [14] and [20]. The results in [14] were obtained by considering only the interval $(0,1)$. In [20] the study was done on a $C^{2,\alpha}$-bounded domain of $\mathbb{R}^n$ defining the operator from a Hilbert space onto its dual.

The main purpose in this paper is to define the square root of the Laplacian operator with Neumann boundary condition through of the harmonic extension method. Using Fourier series and periodic even extension we define the square root of the Laplacian in three types of bounded domains. Furthermore as an application we study the existence of nontrivial weak solution for a class of nonlinear elliptic problems.

In the following we consider Ω as being either the interval, square or ball, and X denotes the Hilbert space of the $L^2(\Omega)$-functions with null average.

Let $\{\varphi_j\}_{j\in\mathcal{I}}$ be an orthonormal basis in X formed by eigenfunctions associated to eigenvalues $\{\lambda_j\}_{j\in\mathcal{I}}$ of the Laplacian operator $-\Delta$ in Ω with homogenous Neumann boundary condition; that is,

$$\begin{gathered}-\Delta\varphi_j = \lambda_j\varphi_j \quad \text{in } \Omega, \\ \frac{\partial\varphi_j}{\partial n} = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

where $\mathcal{I}$ denotes the set $\mathbb{N}^*$ when the domain is an interval, $(\mathbb{N}\times\mathbb{N}) - \{(0,0)\}$ when the domain is a square or $(\mathbb{N}^*\times\mathbb{N}) - \{(1,0)\}$ when the domain is a ball. Then

$$-\Delta u = \sum_{j\in\mathcal{I}} \lambda_j \langle u, \varphi_j\rangle \varphi_j, \quad \forall u \in D(-\Delta).$$

We define the operator

$$\begin{gathered}A_{1/2} : D(A_{1/2}) \subset X \to X \\ u \mapsto -(\tilde{v}_z(\cdot,0))|_\Omega,\end{gathered} \tag{1.2}$$

with domain

$$D(A_{1/2}) = \big\{u \in H^s(\Omega) : \frac{\partial u}{\partial n}\big|_{\partial\Omega} = 0 \text{ and } \int_\Omega u(x)dx = 0\big\}, \tag{1.3}$$

where $\tilde{v}$ is the unique classical solution of the extension problem

$$\begin{gathered} \Delta\tilde{v}(x,z) = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ \tilde{v}(x,0) = \tilde{u}(x) \quad \text{in } \mathbb{R}^n, \\ \lim_{z\to\infty} \|\tilde{v}(\cdot,z)\|_{L^2(\Omega)} = 0 \\ \lim_{z\to\infty} \|\tilde{v}_z(\cdot,z)\|_{L^2(\Omega)} = 0 \\ \int_\Omega \tilde{v}(x,z)dx = 0 \quad \forall z \geq 0, \end{gathered} \tag{1.4}$$

with $s > 3/2$ if Ω is an interval ($n = 1$) and $s > 2$ if Ω is a square or ball ($n = 2$). The definition of the function $\tilde{u}$ is given in Section 3.

Now we define the operator

$$\begin{gathered} B_{1/2} : Y \subset X \to X \\ u \mapsto \sum_{j\in\mathcal{I}} \lambda_j^{1/2} < u, \varphi_j > \varphi_j, \end{gathered} \tag{1.5}$$

and we shall see that $B_{1/2}$ is an extension of the operator $A_{1/2}$ and coincides with the operator $(-\Delta)^{1/2}$ in Ω, where

$$Y = \big\{u \in X : \sum_{j\in\mathcal{I}} \lambda_j |\langle u, \varphi_j\rangle|^2 < \infty\big\} \tag{1.6}$$

and λ_j and φ_j are the eigenvalues and eigenfunctions of $-\Delta$ with Neumann boundary condition on Ω, respectively.

Using the above definition for the square root of the Laplacian we will show the existence of nontrivial weak solution to the problem

$$(-\Delta)^{1/2}u = u^p \quad \text{in } \Omega, \tag{1.7}$$

where $p = 2 + \frac{1}{r}$ and $r > 1$ odd if Ω is a square or $p = \overline{p} + \frac{1}{r}$ with $\overline{p}$ even and $r \geq 1$ odd if Ω is a interval.

This article is organized as follows. In Section 2 we fix the notation and we enunciate the main theorem. In Section 3 we define $A_{1/2}$ and $B_{1/2}$, and we show that $B_{1/2}$ coincides with the square root of the Laplacian with Neumann boundary condition. This section was divided into two parts. The first one we consider Ω as an interval or a square. Then we also consider the case where the domain can be a ball. In Section 4, we show the existence of nontrivial weak solutions to the nonlinear problem (1.7).

2. Notation and statement of main results

We denote the upper half-space in $\mathbb{R}^{n+1}$ by

$$\mathbb{R}^{n+1}_+ = \{(x,z) \in \mathbb{R}^{n+1} : z > 0\};$$

also denote

$$Q^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n : |x_j| \leq \pi \text{for } j = 1,\dots,n\}. \tag{2.1}$$

Here Ω represents the domains

$$\Omega_i = (0,\pi), \quad \Omega_q = (0,\pi)\times(0,\pi), \quad \Omega_b = B(0,\pi)\,.$$

The Hilbert space is

$$X = \Big\{u \in L^2(\Omega) : \int_\Omega u(x)dx = 0\Big\},$$

with the $L^2(\Omega)$-inner product.

Given a domain U in $\mathbb{R}^n$, we denote by $H^s(U)$ the Banach space

$$H^s(U) = \{f \in D'(U) : \|f\|_{H^s(U)} < \infty\},$$

with the norm

$$\|f\|_{H^s(U)} = \Big(\|f\|^2_{L^2(U)} + \sum_{|\alpha|=s} \|D^\alpha f\|^2_{L^2(U)}\Big)^{1/2} \quad \text{for } s \in \mathbb{Z}$$

or

$$\|f\|_{H^s(U)} = \Big(\|f\|^2_{L^2(U)} + \sum_{|\alpha|=[s]} \int_{U\times U} \frac{|D^\alpha f(x) - D^\alpha f(y)|^2}{|x-y|^{n+2\{s\}}}\,dx\,dy\Big)^{1/2}$$

for $s > 0$, non-integer. Note that $s = [s] + \{s\}$ with $[s]$ is the integer part and $\{s\} \in (0,1)$; see e.g. [25, pp. 316, 322-324].

The set of periodic smooth functions is denoted by

$$C^\infty_{\rm per}(\mathbb{R}^n) = \big\{u \in C^\infty(\mathbb{R}^n) : u(x+2k\pi) = u(x),\ \forall k \in \mathbb{Z}^n, x \in \mathbb{R}^n\big\},$$

and

$$C^\infty_{\rm per}(\overline{\mathbb{R}^{n+1}_+}) = \big\{u \in C^\infty(\overline{\mathbb{R}^{n+1}_+}) : u(x+2k\pi, z) = u(x,z),\ \forall k \in \mathbb{Z}^n, (x,z) \in \overline{\mathbb{R}^{n+1}_+}\big\},$$

see e.g. [17, chapter 2].

Let $s \in \mathbb{R}$. We consider the periodic Sobolev spaces $H^s_{\rm per}(\mathbb{R}^n) = \overline{C^\infty_{\rm per}(\mathbb{R}^n)}$ equipped with the norm

$$\|u\|_{H^s_{\rm per}(\mathbb{R}^n)} = \Big(\sum_{k\in\mathbb{Z}^n} (1+|k|^2)^s |\widehat{u}(k)|^2\Big)^{1/2},$$

where $\widehat{u}(k)$ are the Fourier coefficients of u and the spaces $H^s_{\rm per}(\mathbb{R}^{n+1}_+) = \overline{C^\infty_{\rm per}(\overline{\mathbb{R}^{n+1}_+})}$ equipped with the norm

$$\|\tilde u\|_{H^s_{\rm per}(\mathbb{R}^{n+1}_+)} = \Big(\int_0^\infty \sum_{j=0}^{s} \|D^j_z \tilde u(\cdot,z)\|^2_{H^{s-j}_{\rm per}(\mathbb{R}^n)} dz\Big)^{1/2},$$

see e.g. [17, chapter 2]. Our main result is the following.

Theorem 2.1. *Under above conditions, the operator $B_{1/2}$ defined in* (1.5) *and* (1.6) *is well defined. Moreover, $B_{1/2}$ is an extension of the operator $A_{1/2}$ and coincides with the operator $(-\Delta)^{1/2}$ in Ω; that is,*

$$\langle u, B_{1/2}u\rangle \geq 0, \quad \forall u \in D(B_{1/2}),$$
$$B_{1/2}\circ B_{1/2}u = -\Delta u, \quad \forall u \in D(-\Delta).$$

The proof of Theorem 2.1 will be given in the next section. As an application we study the existence of a nontrivial weak solution of the nonlinear elliptic problem (1.7) on Ω_i and Ω_q. In fact, since $(-\Delta)^{1/2}$ is a non-local operator, then from harmonic extension method, problem (1.7) is equivalent to the problem

$$\begin{gathered}
\Delta\tilde{v}(x,z)=0 \quad \text{in } \mathbb{R}^{n+1}_+,\\
\tilde{v}_z(x,0)=-\tilde{u}^p(x) \quad \text{in } \mathbb{R}^n,\\
\lim_{z\to\infty}\|\tilde{v}(\cdot,z)\|_{L^2(\Omega)}=0\\
\lim_{z\to\infty}\|\tilde{v}_z(\cdot,z)\|_{L^2(\Omega)}=0\\
\int_\Omega \tilde{v}(x,z)=0 \quad \forall z\ge 0,
\end{gathered} \tag{2.2}$$

where $\tilde{v}$ is even and periodic with respect to x, $\tilde{u}$ is an even and periodic extension of u.

3. Proof of the main result

In this section we prove Theorem 2.1. First we need to verify the existence and uniqueness of a classical solution to the problem (1.4). The proof of this result will be given in the Theorems 3.1 and 3.7. These theorems are particular cases of [23, Theorem 1.1], considering that in our case the function $\tilde{u}$ is more regular. Here we use convergence properties of series, see e.g. [10, 15, 24].

3.1. **Operator in Ω_i and Ω_q.** Let $\{\lambda_j\}_{j\in\mathcal{I}}$ and $\{\varphi_j\}_{j\in\mathcal{I}}$ be the eigenvalues and corresponding eigenfunctions of $-\Delta$ with Neumann boundary condition on Ω_i or Ω_q, then

$$\lambda_j=j^2 \quad\text{and}\quad \varphi_j(x)=\sqrt{\frac{2}{\pi}}\cos(jx), \quad \forall j\in\mathcal{I}$$

when the domain is Ω_i, and

$$\lambda_{lk}=l^2+k^2 \quad\text{and}\quad \varphi_{lk}(x)=\beta_{lk}\cos(lx)\cos(ky), \quad \forall j=(l,k)\in\mathcal{I}$$

with

$$\beta_{lk}=\begin{cases}\sqrt{2/\pi} & \text{if } k=0 \text{ or } l=0,\\ 2/\pi & \text{if } l,k\ge 1,\end{cases}$$

when the domain is Ω_q.

Theorem 3.1. *Let $u\in D(A_{1/2})$ and $\tilde{u}$ its $2\pi-$periodic even extension as in* [2]*. Then the function $\tilde{v}:\mathbb{R}^{n+1}_+\to\mathbb{R}$ given by*

$$\tilde{v}(x,z)=\sum_{j\in\mathcal{I}} e^{-\sqrt{\lambda_j}z}\langle\tilde{u},\varphi_j\rangle\varphi_j(x)$$

is the unique classical solution of (1.4) *where the convergence is uniform with respect to x, λ_j and φ_j are the eigenvalues and eigenfunctions of $-\Delta$ with Neumann boundary condition on Ω, respectively.*

Proof. It is well known that the φ_j are even and 2π-periodic, then $\tilde{v}$ is even and 2π-periodic with respect to x.

Consider the inequality

$$e^{-2\sqrt{\lambda_j}z}\le\frac{K}{\lambda_j^2}, \quad \forall j\in\mathcal{I},\ \forall z>0,$$

where K is a constant that depends on z. Thus,

$$\left|e^{-\sqrt{\lambda_j}z}\langle \tilde{u},\varphi_j\rangle\varphi_j(x)\right| \le C|\langle u,\varphi_j\rangle|^2 + \frac{K}{\lambda_j^2},$$

for every $(x,z)\in\mathbb{R}^{n+1}_+$ and $j\in\mathcal{I}$, where C is a constant. Therefore by [16] and Weierstrass criterion it follows that $\tilde{v}\in C_{\rm per}(\mathbb{R}^{n+1}_+)$.

We have that $\tilde{v}\in C^2_{\rm per}(\mathbb{R}^{n+1}_+)$ by [16] and

$$\lambda_j^m e^{-2\sqrt{\lambda_j}z} \le \frac{K}{\lambda_j^2}, \quad \forall j\in\mathcal{I},\ \forall m\in\mathbb{Z}^*_+,\ \forall z>0,$$

where K is a constant that depends on z.

Using the convergence properties we obtain

$$\Delta\tilde{v}(x,z)=0, \quad \forall (x,z)\in\mathbb{R}^{n+1}_+.$$

Let $u\in D(A_{1/2})\subset X$, then $u=\sum_{j\in\mathcal{I}}\langle u,\varphi_j\rangle\varphi_j$ and $\tilde{u}=\sum_{j\in\mathcal{I}}\langle u,\varphi_j\rangle\varphi_j$ in $L^2(\Omega)$ and $L^2_{\rm per}(\mathbb{R}^n)$, respectively. We easily verify that

$$\lim_{m\to\infty}\|\tilde{v}-\psi_m\|_{H^1_{\rm per}(\mathbb{R}^{n+1}_+)}=0,$$

where

$$\psi_m(x,z)=\sum_{j\in\mathcal{I}}^{m} e^{-\sqrt{\lambda_j}z}\langle\tilde{u},\varphi_j\rangle\varphi_j(x),$$

then $\tilde{v}\in H^1_{\rm per}(\mathbb{R}^{n+1}_+)$ and by Trace theorem from [17] follows that

$$\tilde{v}(\cdot,0)=\sum_{j\in\mathcal{I}}\langle u,\varphi_j\rangle\varphi_j$$

in $L^2_{\rm per}(\mathbb{R}^n)$. Therefore, $\tilde{v}(\cdot,0)=\tilde{u}(\cdot)$ almost everywhere in $\mathbb{R}^n$. Moreover,

$$\|\tilde{v}(\cdot,z)\|^2_{L^2(\Omega)} \le \Big(\sum_{j\in\mathcal{I}}|\langle u,\varphi_j\rangle|^2\Big)e^{-2z}\to 0 \quad \text{as } z\to\infty.$$

We have that

$$e^{-\sqrt{\lambda_j}z}\langle\frac{4}{\lambda_j z^2},\ \forall z\rangle 0, \quad j\in\mathcal{I},$$

then

$$\begin{aligned}\|\tilde{v}_z(\cdot,z)\|^2_{L^2(\Omega)} &= \sum_{j\in\mathcal{I}}\lambda_j|\langle u,\varphi_j\rangle|^2 e^{-2\sqrt{\lambda_j}z}\\ &\le 4\Big(\sum_{j\in\mathcal{I}}|\langle u,\varphi_j\rangle|^2\Big)\frac{e^{-z}}{z^2}\to 0 \quad \text{as } z\to\infty.\end{aligned}$$

From the uniform convergence properties we have

$$\int_\Omega\tilde{v}(x,z)dx=\sum_{j\in\mathcal{I}}\langle u,\varphi_j\rangle e^{-\sqrt{\lambda_j}z}\Big(\int_\Omega\varphi_j(x)dx\Big)=0, \quad \forall z>0.$$

The same holds for $z=0$, since $\tilde{u}$ has null average and

$$\tilde{u}(\cdot)=\tilde{v}(\cdot,0)$$

almost everywhere in $\mathbb{R}^n$. Therefore, $\tilde{v}$ is a classical solution of (1.4).

Consider $\tilde{v}_1$ and $\tilde{v}_2$ classical solutions to (1.4). Let $\mathcal{H}$ be the Hilbert space of functions $w\in H^1_{\rm per}(\mathbb{R}^{n+1}_+)$ satisfying

(1) w is almost everywhere even with respect to x,
(2) $w(\cdot, 0) = 0$ almost everywhere in $\mathbb{R}^n$,
(3) $\lim_{z\to\infty} \|w(\cdot, z)\|_{L^2(\Omega)} = \lim_{z\to\infty} \|w_z(\cdot, z)\|_{L^2(\Omega)} = 0$,
(4) $\int_\Omega w(x, z)dx = 0$, for any $z \geq 0$,

with the inner product

$$(\tilde{\psi}, \tilde{\varphi})_{\mathcal{H}} = \int_0^\infty \int_\Omega \nabla\tilde{\psi}(x, z)\nabla\tilde{\varphi}(x, z)\, dx\, dz, \quad \forall\tilde{\psi}, \tilde{\varphi} \in \mathcal{H}.$$

Applying the Riesz representation theorem it follows that $\tilde{v}_1 = \tilde{v}_2$. Note that we proved the uniqueness of the weak solution for extension problem (1.4). □

Through the existence and uniqueness of classical solution of the harmonic extension problem (1.4), we have the following lemma.

Lemma 3.2. *The operator $A_{1/2}$ defined in* (1.3) *and* (1.2) *is well defined and*

$$A_{1/2}u = \sum_{j\in\mathcal{I}} \lambda_j^{1/2}\langle u, \varphi_j\rangle\varphi_j \quad \text{in } L^2(\Omega),$$

where φ_j and λ_j are the eigenfunctions and the eigenvalues of the $-\Delta$ with Neumann boundary condition on Ω, respectively.

Proof. We know that $A_{1/2}u \in X$, because

$$\tilde{v}_z(\cdot, 0) = -\sum_{j\in\mathcal{I}} \lambda_j^{1/2}\langle u, \varphi_j\rangle\varphi_j \quad \text{in } L^2(\Omega).$$

Then by the uniqueness of solution of problem (1.4) it follows that $A_{1/2}$ is well defined and

$$A_{1/2}u = \sum_{j\in\mathcal{I}} \lambda_j^{1/2}\langle u, \varphi_j\rangle\varphi_j \text{in } L^2(\Omega).$$

□

Finally, we will conclude this section by proving Theorem 2.1.

Theorem 3.3. *The operator $B_{1/2}$ defined in* (1.5) *and* (1.6) *is well defined. Moreover, $B_{1/2}$ is an extension of the operator $A_{1/2}$ and coincides with the operator $(-\Delta)^{1/2}$ in Ω; that is,*

$$\langle u, B_{1/2}u\rangle \geq 0, \quad \forall u \in D(B_{1/2}),$$
$$B_{1/2} \circ B_{1/2}u = -\Delta u, \quad \forall u \in D(-\Delta).$$

Proof. Let $u \in Y$. Then $\sum_{j\in\mathcal{I}} \lambda_j^{1/2}\langle u, \varphi_j\rangle\varphi_j$ converges in $L^2(\Omega)$. Considering the sequence of partial sums

$$s_m(x) = \sum_{j\in\mathcal{I}}^{m} \lambda_j^{1/2}\langle u, \varphi_j\rangle\varphi_j(x),$$

it follows that the convergence

$$\Big|\int_\Omega B_{1/2}u(x)dx\Big| \leq C\|B_{1/2}u - s_m\| \to 0 \quad \text{as } m \to \infty$$

implies $\int_\Omega B_{1/2}u(x)dx = 0$; then $B_{1/2}u \in X$ and the operator is well defined.

Let $u \in D(A_{1/2})$, then by Lemma 3.2,

$$\sum_{j\in\mathcal{I}} |\langle A_{1/2}u, \varphi_j\rangle|^2 = \sum_{j\in\mathcal{I}} \lambda_j |\langle u, \varphi_j\rangle|^2 < \infty,$$

and $D(A_{1/2}) \subset Y$. Then

$$\begin{aligned}
&\|A_{1/2}u - B_{1/2}u\|_{L^2(\Omega)} \\
&\leq C\Big\|A_{1/2}u - \sum_{j\in\mathcal{I}}^{m} \lambda_j^{1/2}\langle u, \varphi_j\rangle\varphi_j\Big\| + C\Big\|B_{1/2}u - \sum_{j\in\mathcal{I}}^{m} \lambda_j^{1/2}\langle u, \varphi_j\rangle\varphi_j\Big\| \to 0
\end{aligned}$$

as $m \to \infty$. Therefore, $A_{1/2}u = B_{1/2}u$ almost everywhere in Ω for any $u \in D(A_{1/2})$ and $B_{1/2}$ is an extension of the operator $A_{1/2}$.

Let $u \in D(-\Delta) \subset X$. As $\lambda_j \geq 1$ for any $j \in \mathcal{I}$, we have

$$\sum_{j\in\mathcal{I}} \lambda_j |\langle u, \varphi_j\rangle|^2 \leq \sum_{j\in\mathcal{I}} \lambda_j^2 |\langle u, \varphi_j\rangle|^2 < \infty.$$

Then $D(-\Delta) \subset D(B_{1/2})$ and

$$B_{1/2}u = \sum_{j\in\mathcal{I}} \lambda_j^{1/2}\langle u, \varphi_j\rangle\varphi_j.$$

As $B_{1/2}u \in X$ and

$$\sum_{j\in\mathcal{I}} \lambda_j |\langle B_{1/2}u, \varphi_j\rangle|^2 = \sum_{j\in\mathcal{I}} \lambda_j^2 |\langle u, \varphi_j\rangle|^2 < \infty,$$

then $B_{1/2}u \in D(B_{1/2})$.

By the orthonormality of the eigenfunctions it follows that

$$B_{1/2} \circ B_{1/2}u = \sum_{j\in\mathcal{I}} \lambda_j^{1/2}\langle B_{1/2}u, \varphi_j\rangle\varphi_j = \sum_{j\in\mathcal{I}} \lambda_j \langle u, \varphi_k\rangle\varphi_j = -\Delta u, \quad \forall u \in D(-\Delta).$$

Also note that

$$\langle u, B_{1/2}u\rangle = \sum_{j\in\mathcal{I}} \lambda_j^{1/2} |\langle u, \varphi_j\rangle|^2 \geq 0, \quad \forall u \in D(B_{1/2})$$

□

3.2. **Operator in Ω_b.** In this section we shall use the eigenvalues and corresponding eigenfunctions of $-\Delta$ with Neumann boundary condition on Ω_b. The eigenfunctions are given by the bessel and cosine, sine functions, see e.g. [12, page 108]. We shall use also the properties of Bessel functions, see e.g. [1, 5, 19, 22, 27]. We have that

$$(x, y) = (\alpha\cos\theta, \alpha\sin\theta), \quad \forall (x, y) \in \Omega_b,$$

where $\alpha \in (-\pi, \pi)$ and $\theta \in \mathbb{R}$. Thus consider $u \in D(A_{1/2})$ and define the function $\overline{u}$ such that

$$\overline{u}(x, y) = \begin{cases} U(\alpha, \theta) & \text{if } -\pi \leq \alpha \leq \pi \\ U(-\alpha - 2\pi, \theta) & \text{if } -3\pi \leq \alpha \leq -\pi, \end{cases}$$

where $U(\alpha, \theta) = u(\alpha\cos\theta, \alpha\sin\theta)$ for any $\alpha \in (-3\pi, \pi)$ and $\theta \in \mathbb{R}$.

Consider $\tilde{u} : \mathbb{R}^2 \to \mathbb{R}$ the 4π-periodic radial extension of $\overline{u}$ such that

$$\begin{aligned}\tilde{u}(x,y) &= \tilde{U}(\alpha,\theta)\\ &= \begin{cases} U(\alpha - 4k\pi, \theta) & \text{if } (4k-1)\pi \le \alpha \le (4k+1)\pi \\ U(-\alpha + 2(2k-1)\pi, \theta) & \text{if } (4k-3)\pi \le \alpha \le (4k-1)\pi, \end{cases}\end{aligned} \tag{3.1}$$

with $k \in \mathbb{Z}$.

Lemma 3.4 ([2, Lemma 9])**.** *The function defined in* (3.1) *satisfies the following properties*

(1) $\tilde{U}(\alpha + 4k\pi, \theta) = \tilde{U}(\alpha, \theta)$*, for all* $\alpha, \theta \in \mathbb{R}$*, and all* $k \in \mathbb{Z}$*.*
(2) $\tilde{U}(-\alpha - 2\pi, \theta) = \tilde{U}(\alpha, \theta)$*, for all* $\alpha, \theta \in \mathbb{R}$*.*
(3) $\tilde{u} \in C(\mathbb{R}^2)$*.*

Proof. The proof of (1) and (2) follows from the definition of $\tilde{U}$ in 3.1 and (3) follows from the fact that $D(A^{1/2})$ is embedded in a $C^{1,\alpha}$ space. $\square$

Similarly to the previous section, we first verify the existence and uniqueness of classical solution of problem (1.4). Consider two auxiliary results whose statements are in [2].

Proposition 3.5 ([2, Prop. 10])**.** *Consider the function* $V : [0,\pi) \times \mathbb{R} \times (0,\infty) \to \mathbb{R}$ *with*

$$V(r,\theta,z) = \sum_{(j,k)\in\mathcal{I}} e^{-\frac{\mu_{jk}}{\pi}z} J_k\big(\frac{\mu_{jk}}{\pi} r\big)\big[a_{jk}\cos(k\theta) + b_{jk}\sin(k\theta)\big],$$

where J_k *are the Bessel functions of order* k*,* μ_{jk} *are positive zeros from* J_k' *and*

$$\begin{aligned} a_{jk} &= \frac{2\mu_{jk}^2}{\pi^3(\mu_{jk}^2 - k^2)J_k^2(\mu_{jk})} \int_0^{2\pi}\int_0^{\pi} rU(r,\theta)\cos(k\theta) J_k\Big(\frac{\mu_{jk}}{\pi} r\Big)\, dr\, d\theta, \\ b_{jk} &= \frac{2\mu_{jk}^2}{\pi^3(\mu_{jk}^2 - k^2)J_k^2(\mu_{jk})} \int_0^{2\pi}\int_0^{\pi} rU(r,\theta)\sin(k\theta) J_k\Big(\frac{\mu_{jk}}{\pi} r\Big)\, dr\, d\theta. \end{aligned} \tag{3.2}$$

Then $V \in C^2([0,\pi) \times \mathbb{R} \times (0,\infty))$*.*

The proof of the above proposition follows from the properties of Bessel functions.

Theorem 3.6 ([2, Theorem 10])**.** *Let* $u \in D(A_{1/2})$ *and* $v : \Omega_b \times (0,\infty) \to \mathbb{R}$ *given by*

$$\begin{aligned} v(x,y,z) &= V(r,\theta,z) \\ &= \sum_{(j,k)\in\mathcal{I}} e^{-\frac{\mu_{jk}}{\pi}z} J_k\big(\frac{\mu_{jk}}{\pi} r\big)\big[a_{jk}\cos(k\theta) + b_{jk}\sin(k\theta)\big], \end{aligned}$$

for every $(x,y,z) \in (\Omega_b \backslash \{(0,0)\}) \times (0,\infty)$*, where* a_{jk} *and* b_{jk} *are given by* (3.2) *and for every* $z > 0$ *we have:*

(1)
$$v(0,0,z) = \sum_{j=2}^{\infty} a_{j0} e^{-\frac{\mu_{j0}}{\pi}z},$$

(2)

$$\begin{aligned}
\frac{\partial v}{\partial x}(0,0,z) &= \frac{1}{2\pi}\sum_{j=1}^{\infty} a_{j1}\mu_{j1}e^{-\frac{\mu_{j1}}{\pi}z},\\
\frac{\partial v}{\partial y}(0,0,z) &= \frac{1}{2\pi}\sum_{j=1}^{\infty} b_{j1}\mu_{j1}e^{-\frac{\mu_{j1}}{\pi}z},\\
\frac{\partial v}{\partial z}(0,0,z) &= -\frac{1}{\pi}\sum_{j=2}^{\infty} a_{j0}\mu_{j0}e^{-\frac{\mu_{j0}}{\pi}z},
\end{aligned}$$

(3)

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2}(0,0,z) &= \frac{1}{4\pi^2}\sum_{j=1}^{\infty} a_{j2}\mu_{j2}^2 e^{-\frac{\mu_{j2}}{\pi}z} - \frac{1}{2\pi^2}\sum_{j=2}^{\infty} a_{j0}\mu_{j0}^2 e^{-\frac{\mu_{j0}}{\pi}z},\\
\frac{\partial^2 v}{\partial y^2}(0,0,z) &= -\frac{1}{4\pi^2}\sum_{j=1}^{\infty} a_{j2}\mu_{j2}^2 e^{-\frac{\mu_{j2}}{\pi}z} - \frac{1}{2\pi^2}\sum_{j=2}^{\infty} a_{j0}\mu_{j0}^2 e^{-\frac{\mu_{j0}}{\pi}z},\\
\frac{\partial^2 v}{\partial z^2}(0,0,z) &= \frac{1}{\pi^2}\sum_{j=2}^{\infty} a_{j0}\mu_{j0}^2 e^{-\frac{\mu_{j0}}{\pi}z},\\
\frac{\partial^2 v}{\partial x\partial y}(0,0,z) &= \frac{1}{4\pi^2}\sum_{j=1}^{\infty} b_{j2}\mu_{j2}^2 e^{-\frac{\mu_{j2}}{\pi}z},\\
\frac{\partial^2 v}{\partial x\partial z}(0,0,z) &= -\frac{1}{2\pi^2}\sum_{j=1}^{\infty} a_{j1}\mu_{j1}^2 e^{-\frac{\mu_{j1}}{\pi}z},\\
\frac{\partial^2 v}{\partial y\partial z}(0,0,z) &= -\frac{1}{2\pi^2}\sum_{j=1}^{\infty} b_{j1}\mu_{j1}^2 e^{-\frac{\mu_{j1}}{\pi}z}.
\end{aligned}$$

Then v is the unique function that satisfies the following conditions:

(1) $v \in C^2(\Omega_b \times (0,\infty))$.
(2) $\Delta v(x,y,z) = 0$ *in* $\Omega_b \times (0,\infty)$.
(3) $v(\cdot,\cdot,z) = u(\cdot,\cdot)$ *almost everywhere on* Ω_b.
(4) $\frac{\partial v}{\partial n} = 0$ *on* $\partial\Omega_b \times [0,\infty)$.
(5) $\lim_{z\to\infty} \|v(\cdot,\cdot,z)\|_{L^2(\Omega_b)} = \lim_{z\to\infty} \|v_z(\cdot,\cdot,z)\|_{L^2(\Omega_b)} = 0$.
(6) $\int_{\Omega_b} v(x,z) \quad dx = 0$ *for any* $z \geq 0$.

Proof. The proof follows by considering the eigenfunction decomposition of the Neumann Laplacian in the ball and the properties of Bessel functions. Let us show that v is continuous at $(0,0,z)$ for any $z > 0$ the others cases are analogous.

Let $z > 0$ and $(x_m, y_m, z_m) \to (0,0,z)$ as $m \to \infty$. Then (r_m, θ_m, z_m) is a associated sequence with (x_m, y_m, z_m) where $r_m \to 0$ and $z_m \to z$ as$m \to \infty$. Thus

$$\begin{aligned}
|v(x_m,y_m,z_m) - v(0,0,z)| &\leq \sum_{j=2}^{\infty} \big|J_0\big(\frac{\mu_{j0}}{\pi} r_m\big) e^{-\frac{\mu_{j0}}{\pi}z_m} - e^{-\frac{\mu_{j0}}{\pi}z}\big||a_{j0}|\\
&\quad + \sum_{j,k\geq 1}^{\infty} \big|J_k\big(\frac{\mu_{jk}}{\pi} r_m\big)\big| e^{-\frac{\mu_{jk}}{\pi}z_m}[|a_{jk}| + |b_{jk}|].
\end{aligned}$$

Note that

$$\lim_{m\to\infty} J_0\big(\frac{\mu_{j0}}{\pi} r_m\big) = 1 \quad \text{and} \quad \lim_{m\to\infty} J_k\big(\frac{\mu_{jk}}{\pi} r_m\big) = 0,$$

then

$$\lim_{m\to\infty} v(x_m, y_m, z_m) = v(0,0,z).$$

□

Theorem 3.7. *Let $\tilde{v} : \mathbb{R}^3_+ \to \mathbb{R}$ the 4π-periodic radial extension of the function v of Theorem 3.6, namely,*

$$\begin{aligned}\tilde{v}(x,y,z) &= \tilde{V}(\alpha,\theta,z)\\ &= \begin{cases}\widehat{V}(\alpha - 4k\pi, \theta, z) & \text{if } (4k-1)\pi \le \alpha \le (4k+1)\pi,\\ \widehat{V}(-\alpha + 2(2k-1)\pi, \theta, z) & \text{if } (4k-3)\pi \le \alpha \le (4k-1)\pi,\end{cases}\end{aligned}$$

where $\widehat{V}(\alpha,\theta,z) = v(\alpha\cos\theta, \alpha\sin\theta, z)$ for all $\alpha,\theta \in \mathbb{R}$, and all $z > 0$. Then, $\tilde{v}$ is the unique classical solution of (1.4).

Proof. We have that $v \in C^2(\Omega_b \times (0,\infty))$ by the previous theorem. Then $\tilde{v} \in C(\mathbb{R}^3_+)$ by Lemma 3.4. Because of the periodicity the derivatives of $\tilde{v}$ are continuous in $\mathbb{R}^3_+$, except possibly at the points

$$(x,y,z) = (m\pi\cos\theta, m\pi\sin\theta, z) \text{ with } m \in \mathbb{Z}^*.$$

Using the periodicity, symmetry and chain rule, we will verify the continuity of the functions

$$\frac{\partial\tilde{V}}{\partial\alpha}, \quad \frac{\partial\tilde{V}}{\partial\theta}, \quad \frac{\partial^2\tilde{V}}{\partial\theta^2}, \quad \frac{\partial^2\tilde{V}}{\partial\alpha^2}, \quad \frac{\partial^2\tilde{V}}{\partial\theta\partial\alpha}, \quad \frac{\partial^2\tilde{V}}{\partial\alpha\partial z}, \quad \frac{\partial^2\tilde{V}}{\partial\theta\partial z}$$

at points $(\pm\pi, \theta, z)$.

As $v \in C^2(\overline{\Omega_b} \times (0,\infty))$, then V and $\widehat{V}$ are C^2 at points (π,θ,z) for any $\theta \in \mathbb{R}$, $z > 0$. Thus,

$$\begin{aligned}\lim_{h\to 0^+} \frac{\tilde{V}(\pi,\theta+h,z) - \tilde{V}(\pi,\theta,z)}{h} &= \frac{\partial\widehat{V}}{\partial\theta}(\pi,\theta,z),\\ \lim_{h\to 0^-} \frac{\tilde{V}(\pi,\theta+h,z) - \tilde{V}(\pi,\theta,z)}{h} &= \frac{\partial\widehat{V}}{\partial\theta}(\pi,\theta,z),\end{aligned}$$

Then there exists $\frac{\partial\tilde{V}}{\partial\theta}(\pi,\theta,z)$ such that

$$\frac{\partial\tilde{V}}{\partial\theta}(\pi,\theta,z) = -\sum_{(j,k)\ge 1} kJ_k(\mu_{jk}) e^{-\frac{\mu_{jk}}{\pi}z}(-a_{jk}\sin(k\theta) + b_{jk}\cos(k\theta)).$$

Therefore, $\frac{\partial\tilde{V}}{\partial\theta}$ is continuous in (π,θ,z) and $(-\pi,\theta,z)$ for any $\theta \in \mathbb{R}$, $z > 0$. Similarly

$$\frac{\partial\tilde{V}}{\partial\alpha}, \quad \frac{\partial^2\tilde{V}}{\partial\theta\partial\alpha}, \quad \frac{\partial^2\tilde{V}}{\partial\theta^2}, \quad \frac{\partial^2\tilde{V}}{\partial\alpha^2}, \quad \frac{\partial^2\tilde{V}}{\partial\alpha\partial z}, \quad \frac{\partial^2\tilde{V}}{\partial\theta\partial z}$$

are continuous in $(\pm\pi,\theta,z)$ for any $\theta \in \mathbb{R}$, $z > 0$. It is easy to verify the smoothness of the derivatives of $\tilde{V}$ with respect the variable z; then $\tilde{v} \in C^2(\mathbb{R}^3_+)$. Note that

$$\begin{aligned}\tilde{V}(\alpha + 4k\pi, \theta, 0) &= \lim_{z\to 0^+} \tilde{V}(\alpha + 4k\pi, \theta, z)\\ &= \lim_{z\to 0^+} \tilde{V}(\alpha,\theta,z)\\ &= \tilde{V}(\alpha,\theta,0), \quad \forall k \in \mathbb{Z},\ \forall \alpha,\theta \in \mathbb{R}.\end{aligned}$$

By the extension properties,

$$\tilde{V}(\alpha+4k\pi,\theta,z)=\tilde{V}(\alpha,\theta,z),\quad \forall k\in\mathbb{Z},\ \forall\alpha,\theta\in\mathbb{R},\ \forall z>0.$$

Analogously we have

$$\tilde{V}(-\alpha-2\pi,\theta,z)=\tilde{V}(\alpha,\theta,z),\quad \forall k\in\mathbb{Z},\ \forall\alpha,\theta\in\mathbb{R},\ \forall z\geq 0.$$

As $\tilde{v}=v$ on $\Omega_b\times(0,\infty)$ and

$$v(\cdot,0)=u(\cdot)\quad \text{almost everywhere on } \Omega_b,$$

it follows that $\tilde{v}(\cdot,0)=\tilde{u}(\cdot)$ almost everywhere on $\mathbb{R}^2$.

We have that $\Delta v=0$ in $\Omega_b\times(0,\infty)$; then

$$\Delta\tilde{v}(x,y,z)=\frac{1}{\pi^2}\sum_{(j,k)\in\mathcal{I}}T_{jk}(\mu_{jk})e^{-\frac{\mu_{jk}}{\pi}}z[a_{jk}\cos(k\theta)+b_{jk}\sin(k\theta)]=0$$

for all $\theta\in\mathbb{R},\ z>0$, where $T_{jk}(\mu_{jk})=\mu_{jk}^2J_k''(\mu_{jk})+\mu_{jk}J_k'(\mu_{jk})+(\mu_{jk}^2-k^2)J_k(\mu_{jk})$. Moreover,

$$\Delta\tilde{v}(x,y,z)=\Delta\tilde{v}(\pi\cos(\theta+\pi),\pi\sin(\theta+\pi),z)=0,\quad \forall\theta\in\mathbb{R},z>0.$$

We have

$$\begin{gathered}\int_{\Omega_b}\tilde{v}(x,y,z)\,dx\,dy=\int_{\Omega_b}v(x,y,z)\,dx\,dy=0,\quad \forall z\geq 0,\\ \lim_{z\to\infty}\|\tilde{v}(\cdot,\cdot,z)\|_{L^2(\Omega_b)}=\lim_{z\to\infty}\|v(\cdot,\cdot,z)\|_{L^2(\Omega_b)}=0,\\ \lim_{z\to\infty}\|\tilde{v}_z(\cdot,\cdot,z)\|_{L^2(\Omega_b)}=\lim_{z\to\infty}\|v(\cdot,\cdot,z)\|_{L^2(\Omega_b)}=0.\end{gathered}$$

Therefore, $\tilde{v}$ is classical solution of the problem (1.4). The uniqueness follows similarly to the previous section. $\square$

Lemma 3.8. *The operator $A_{1/2}$ defined in* (1.3) *and* (1.2) *is well defined and*

$$A_{1/2}u=\sum_{j\in\mathcal{I}}\lambda_j^{1/2}\langle u,\varphi_j\rangle\varphi_j,$$

where $(\varphi_j)_{j\in\mathcal{I}}$ and $(\lambda_j)_{j\in\mathcal{I}}$ are the eigenfunctions and the eigenvalues of the $-\Delta$ with Neumann boundary condition on Ω_b, respectively.

The proof of the above lemma is analogous to Lemma 3.2 and is omitted. We conclude this section by proving the Theorem 2.1.

Theorem 3.9. *The operator $B_{1/2}$ defined in* (1.5) *and* (1.6) *is well defined. Moreover, $B_{1/2}$ is an extension of the operator $A_{1/2}$ and coincides with the operator $(-\Delta)^{1/2}$ in Ω_b, that is,*

$$\begin{gathered}\langle u,B_{1/2}u\rangle\geq 0,\quad \forall u\in D(B_{1/2}),\\ B_{1/2}\circ B_{1/2}u=-\Delta u,\quad \forall u\in D(-\Delta).\end{gathered}$$

Proof. Let $u\in Y$. Then the series

$$\sum_{j\in\mathcal{I}}\lambda_j^{1/2}<u,\varphi_j>\varphi_j,$$

converges in $L^2(\Omega_b)$ and thus $B_{1/2}u\in L^2(\Omega_b)$.

Consider the partial sum

$$s_m(r,\theta) = \sum_{j\in\mathcal{I}}^{m} \lambda_j^{1/2}\langle U,\varphi_j\rangle\varphi_j(r,\theta),$$

it follows by Holder's inequality which

$$\begin{aligned}\Big|\int_{\Omega_b} B_{1/2}u(x,y)\,dx\,dy\Big| &\le \int_0^{2\pi}\int_0^{\pi} r\big|B_{1/2}U(r,\theta) - s_m(r,\theta)\big|\,dr\,d\theta \\ &\le M\|B_{1/2}U - s_m\|_{L^2((0,\pi)\times(0,2\pi);r)} \to 0 \quad \text{as } m\to\infty;\end{aligned}$$

then $B_{1/2}u \in X$. Thus the operator $B_{1/2}$ is well defined by uniqueness of the Fourier-Bessel series.

The proof that $B_{1/2}$ is an extension of the operator $A_{1/2}$ and coincides with $(-\Delta)^{1/2}$ in Ω is analogous to cases of the domains Ω_i e Ω_q. $\square$

4. Application

In this section we study the existence of nontrivial weak solution of the nonlocal problem (1.7), namely, the existence of a nontrivial function u with $u = (\tilde{v}(\cdot,0))|_\Omega$ where $\tilde{v}$ is almost everywhere even with respect to x, $\tilde{v} \in H^1_{\rm per}(\mathbb{R}^{n+1}_+)$,

$$\begin{gathered}\int_\Omega \tilde{v}(x,z)dx = 0, \quad \forall z\ge 0,\\ \lim_{T\to\infty}\|\tilde{v}(\cdot,T)\|_{L^2(\Omega)} = \lim_{T\to\infty}\|\tilde{v}_z(\cdot,T)\|_{L^2(\Omega)} = 0,\\ \int_0^\infty\int_\Omega \nabla\tilde{v}(x,z)\nabla\tilde{\varphi}(x,z)\,dx\,dz = \int_\Omega u^p(x)\tilde{\varphi}(x,0)\,dx,\end{gathered}$$

for every $\tilde{\varphi} \in H^1_{\rm per}(\mathbb{R}^{n+1}_+)$ satisfying the same conditions as $\tilde{v}$.

Consider for the nontrivial weak solution in Ω_i with the condition

$$\tilde{v}(x+\pi,0) = -\tilde{v}(x,0) \tag{4.1}$$

almost everywhere for $x\in\mathbb{R}$, and in Ω_q with the condition

$$\tilde{v}(x+\pi,y,0) = -\tilde{v}(x,y,0), \tag{4.2}$$

almost everywhere for $(x,y)\in\mathbb{R}^2$. Note that the condition in Ω_q could be with respect to y.

Our goal is to apply the Lagrange multiplier theorem in linear topological spaces from [3] and thus obtain the existence of nontrivial weak solution of nonlinear problem (1.7).

Lemma 4.1. *Consider the set H of functions $\tilde{v}$, even almost everywhere in $\mathbb{R}^{n+1}$ with respect to x, which satisfy* (4.1) *or* (4.2) *according with the domain Ω,*

$$\begin{gathered}\int_\Omega \tilde{v}(x,z)dx = 0, \quad \forall z\ge 0,\\ \lim_{T\to\infty}\|\tilde{v}(\cdot,T)\|_{L^2(\Omega)} = \lim_{T\to\infty}\|\tilde{v}_z(\cdot,T)\|_{L^2(\Omega)} = 0.\end{gathered}$$

Then there are nontrivial functions in $(H,\|\cdot\|_H)$ which is a Hilbert space with the norm

$$\|\tilde{v}\|_H = \Big(\int_0^\infty\int_\Omega |\nabla\tilde{v}(x,z)|^2\,dx\,dz\Big)^{1/2}.$$

Proof. Consider the functions

$$\tilde v_1(x) = e^{-z}\cos(x), \quad \tilde v_2(x) = e^{-z}\cos(x_1)\cos(x_2),$$

with $x = (x_1, x_2) \in \mathbb{R}^n$. Note that $\tilde v_1, \tilde v_2 \in H$ according with the domain Ω. Follows from [2, Lemma 12] that $(H, \|\cdot\|_H)$ is a Hilbert space. $\square$

For any $a > 0$ consider the functional $I : H \to \mathbb{R}$ given by

$$I(\tilde v) = \frac{1}{2}\int_0^\infty \int_\Omega |\nabla \tilde v|^2 \,dx\,dz,$$

and consider the set

$$H_a = \big\{\tilde v \in H : \int_\Omega (\tilde v(x,0))^{p+1} dx = a\big\}.$$

Proposition 4.2. *There is $\tilde v \in H_a$ with $I(\tilde v) = \min_{\tilde w \in H_a} I(\tilde w)$.*

Proof. Let $m = \inf\{I(\tilde v) : \tilde v \in H_a\}$. We have that $\{I(\tilde v) : \tilde v \in H_a\} \neq \emptyset$, and by the definition of infimum,

$$\lim_{j\to\infty} I(\tilde v_j) = m,$$

where $\{\tilde v_j\}_{j\in\mathbb{N}} \subset H_a$.

Since $\{I(\tilde v_j)\}_{j\in\mathbb{N}} \subset \mathbb{R}$, there exists $M > 0$ such that

$$\|\tilde v_j\|_H^2 = \int_0^\infty \int_\Omega |\nabla \tilde v_j|^2 \,dx\,dz = 2I(\tilde v_j) \le 2M\,.$$

Then $\{\tilde v_j\}_{j\in\mathbb{N}}$ is bounded in H. Thus, using compact immersion (see [25, Theorem 4.10.1]) there is a subsequence $\{\tilde v_{jk}\}_{k\in\mathbb{N}}$ and $\tilde v \in H$ such that

$$\tilde v_{jk} \rightharpoonup \tilde v \quad \text{in } H,$$
$$\tilde v_{jk}(\cdot,0) \to \tilde v(\cdot,0) \quad \text{in } L^{p+1}(Q^n),$$

with Q^n defined in (2.1).

By the properties of convex functions, we have

$$\begin{aligned}(p+1)\int_\Omega (\tilde v_{jk}(x,0))^p(\tilde v(x,0) - \tilde v_{jk}(x,0))dx &\le \int_\Omega (\tilde v(x,0))^{p+1}dx - a\\ &\le (p+1)\int_\Omega (\tilde v(x,0))^p(\tilde v(x,0) - \tilde v_{jk}(x,0))dx.\end{aligned}$$

Note that

$$\begin{gathered}\Big|\int_\Omega (\tilde v_{jk}(x,0))^p(\tilde v(x,0) - \tilde v_{jk}(x,0))dx\Big|\\ \le \|\tilde v_{jk}(\cdot,0)\|_{L^{\frac{p+1}{p}}}\|\tilde v(\cdot,0) - \tilde v_{jk}(\cdot,0)|\|_{L^{p+1}} \to 0,\\ \Big|\int_\Omega (\tilde v(x,0))^p(\tilde v(x,0) - \tilde v_{jk}(x,0))dx\Big|\\ \le \|\tilde v(\cdot,0)\|_{L^{\frac{p+1}{p}}}\|\tilde v(\cdot,0) - \tilde v_{jk}(\cdot,0)|\|_{L^{p+1}} \to 0,\end{gathered}$$

thus

$$\int_\Omega (\tilde v(x,0))^{p+1}dx = a.$$

Then $\tilde v \in H_a$ and $I(\tilde v) \ge m$. Using the properties of convex function, we obtain that

$$I(\tilde v_{jk}) \ge -I(\tilde v) + \int_0^\infty \int_\Omega \nabla\tilde v \cdot \nabla \tilde v_{jk}\,dx\,dz. \tag{4.3}$$

Thus, making $k \to \infty$ in (4.3), it follows that $m \geq I(\tilde{v})$ and therefore $I(\tilde{v}) = \min_{\tilde{w} \in H_a} I(\tilde{w})$. □

Theorem 4.3. *The nonlinear problem* (1.7) *admits a nontrivial weak solution in* Ω.

Proof. Consider the functions $g : H \to \mathbb{R}$ given by

$$\tilde{w} \mapsto \frac{1}{p+1} \int_\Omega (\tilde{w}(x,0))^{p+1} dx - \frac{a}{p+1},$$

and $f : H \to \mathbb{R}$ given by

$$\tilde{w} \mapsto \frac{1}{2} \int_0^\infty \int_\Omega |\nabla \tilde{w}|^2 \, dx \, dz.$$

Thus,

$$|f(\tilde{v} + \tilde{\varphi}) - f(\tilde{v}) - \langle Df(\tilde{v}), \tilde{\varphi} \rangle| = \frac{1}{2} \|\tilde{\varphi}\|_H^2,$$

where

$$\langle Df(\tilde{v}), \tilde{\varphi} \rangle = \int_0^\infty \int_\Omega \nabla \tilde{v} \nabla \tilde{\varphi} \, dx \, dz, \quad \forall \tilde{\varphi} \in H$$

and therefore, f is strongly H-differentiable at $\tilde{v}$.

Consider $Dg(\tilde{v}) : H \to \mathbb{R}$ such that

$$\langle Dg(\tilde{v}), \tilde{\varphi} \rangle = \int_\Omega (\tilde{v}(x,0))^p \tilde{\varphi}(x,0) dx, \quad \forall \tilde{\varphi} \in H.$$

Then

$$\begin{aligned} &\Big| \frac{g(\tilde{v} + t\tilde{\varphi}) - g(\tilde{v})}{t} - \langle Dg(\tilde{v}), \tilde{\varphi} \rangle \Big| \\ &= \frac{1}{|t|} \Big| \frac{1}{p+1} \int_{\Omega_q} (\tilde{v}(x,0) + t\tilde{\varphi}(x,0))^{p+1} dx - \frac{1}{p+1} \int_{\Omega_q} (\tilde{v}(x,0))^{p+1} dx \\ &\quad - t \int_{\Omega_q} (\tilde{v}(x,0))^p \tilde{\varphi}(x,0) dx \Big|. \end{aligned} \tag{4.4}$$

By Taylor's formula in [18], we have

$$\begin{aligned} &\frac{1}{p+1} \int_{\Omega_q} (\tilde{v}(x,0) + t\tilde{\varphi}(x,0))^{p+1} dx \\ &= \frac{1}{p+1} \int_{\Omega_q} (\tilde{v}(x,0))^{p+1} dx + t \int_{\Omega_q} (\tilde{v}(x,0)^p) \tilde{\varphi}(x,0) dx \\ &\quad + \frac{pt^2}{2!} \int_{\Omega_q} (\tilde{v}(x,0))^{p-1} (\tilde{\varphi}(x,y,0))^2 dx \\ &\quad + \frac{p(p-1)}{3!} \int_{\Omega_q} (\tilde{v}(x,0))^{p-2} (\tilde{\varphi}(x,y,0))^3 dx + \frac{1}{p+1} \int_{\Omega_q} r_3(t\tilde{\varphi}(x,0)) dx, \end{aligned}$$

where $\lim_{t\to 0} \frac{r_3(t\tilde{\varphi}(x,0))}{(t\tilde{\varphi}(x,0))^3} = 0$. Thus in (4.4) we have

$$\begin{aligned} \Big| \frac{g(\tilde{v} + t\tilde{\varphi}) - g(\tilde{v})}{t} - \langle Dg(\tilde{v}), \tilde{\varphi} \rangle \Big| &\leq \frac{p|t|}{2!} \int_{\Omega_q} |\tilde{v}(x,0)|^{p-1} |\tilde{\varphi}(x,0)|^2 dx \\ &\quad + \frac{p(p-1)|t|^2}{3!} \int_{\Omega_q} |\tilde{v}(x,0)|^{p-2} |\tilde{\varphi}(x,0)|^3 dx \end{aligned}$$

$$+\frac{1}{p+1}\int_{\Omega_q}|r_3(t\tilde{\varphi}(x,0))|dx.$$

Then, by Holder's inequality and the continuous immersions in [25, Theorem 4.6.1], it follows that

$$H^{1/2}_{\rm per}(\mathbb{R}^n)\hookrightarrow L^4(Q^n),\quad H^{1/2}_{\rm per}(\mathbb{R}^n)\hookrightarrow L^{2(p-1)}(Q^n),\quad H^{1/2}_{\rm per}(\mathbb{R}^n)\hookrightarrow L^{\frac{6(p-1)}{p}}(Q^n).$$

Then

$$\begin{aligned}
&\Big|\frac{g(\tilde{v}+t\tilde{\varphi})-g(\tilde{v})}{t}-\langle Dg(\tilde{v}),\tilde{\varphi}\rangle\Big|\\
&\leq\frac{p|t|}{2!}\|\tilde{v}(\cdot,0)\|^{p-1}_{L^{2(p-1)}(Q^n)}\|\tilde{\varphi}(\cdot,0)\|^2_{L^4(Q^n)}\\
&\quad+\frac{p(p-1)|t|^2}{3!}\|\tilde{v}(\cdot,0)\|^{p-2}_{L^{2(p-1)}(Q^n)}\|\tilde{\varphi}(\cdot,0)\|^{1/3}_{L^{\frac{6(p-1)}{p}}(Q^n)}+\frac{1}{p+1}\int_\Omega|r_3(t\tilde{\varphi}(x,0))|dx,
\end{aligned}$$

where $\lim_{t\to 0}\frac{r_3(t\tilde{\varphi}(x,0))}{(t\tilde{\varphi}(x,0))^3}=0$. Thus for any $\epsilon>0$, exists $\delta>0$ such that $|t|<\delta$ implies that

$$\begin{aligned}
&\Big|\frac{g(\tilde{v}+t\tilde{\varphi})-g(\tilde{v})}{t}-\langle Dg(\tilde{v}),\tilde{\varphi}\rangle\Big|\\
&\leq\frac{p\delta}{2!}\|\tilde{v}(\cdot,0)\|^{p-1}_{L^{2(p-1)}(Q^n)}\|\tilde{\varphi}(\cdot,0)\|^2_{L^4(Q^n)}\\
&\quad+\frac{p(p-1)\delta^2}{3!}\|\tilde{v}(\cdot,0)\|^{p-2}_{L^{2(p-1)}(Q^n)}\|\tilde{\varphi}(\cdot,0)\|^{1/3}_{L^{\frac{6(p-1)}{p}}(Q^n)}+\frac{\epsilon\delta^3}{p+1}\|\tilde{\varphi}(\cdot,0)\|^3_{L^3(Q^n)},
\end{aligned}$$

Moreover, there is $\overline{\delta}>0$ such that $P(\overline{\delta})<\epsilon$, where

$$\begin{aligned}
P(\overline{\delta})&=\frac{p\overline{\delta}}{2!}\|\tilde{v}(\cdot,0)\|^{p-1}_{L^{2(p-1)}(Q^n)}\|\tilde{\varphi}(\cdot,0)\|^2_{L^4(Q^n)}\\
&\quad+\frac{p(p-1)\overline{\delta}^2}{3!}\|\tilde{v}(\cdot,0)\|^{p-2}_{L^{2(p-1)}(Q^n)}\|\tilde{\varphi}(\cdot,0)\|^{1/3}_{L^{\frac{6(p-1)}{p}}(Q^n)}\\
&\quad+\frac{\epsilon\overline{\delta}^3}{p+1}\|\tilde{v}(\cdot,0)\|_{L^{2(p-1)(Q^n)}};
\end{aligned}$$

then g is $H-$differentiable at $\tilde{v}$. By Taylor's formula in [18] we have

$$\begin{aligned}
&|g(\tilde{w}+t\tilde{\varphi})-g(\tilde{w})|\\
&\leq|t|\int_\Omega|\tilde{w}(x,0)|^p|\tilde{\varphi}(x,0)|dx+\frac{p|t|^2}{2!}\int_\Omega|\tilde{w}(x,0)|^{p-1}|\tilde{\varphi}(x,0)|^2dx\\
&\quad+\frac{p(p-1)|t|^3}{3!}\int_\Omega|\tilde{w}(x,0)|^{p-2}|\tilde{\varphi}(x,0)|^3dx+\frac{1}{p+1}\int_\Omega|r_3(t\tilde{\varphi}(x,0))|dx,
\end{aligned}$$

for every $(\tilde{w},\tilde{\varphi})\in H\times H$.

Using the immersions in [25, Theorem 4.6.1]

$$H^{1/2}_{\rm per}(\mathbb{R}^n)\hookrightarrow L^{\frac{4p}{3}}(Q^n),\quad H^{1/2}_{\rm per}(\mathbb{R}^n)\hookrightarrow L^4(Q^n)$$

we conclude that $(\tilde{w}(\cdot,0))^p\in L^{4/3}(Q^n)$ and $\tilde{\varphi}(\cdot,0)\in L^4(Q^n)$. Then, by Holder's inequality it follows that g is H-continuous on H.

Suppose that $Dg(\tilde{v})=0$, then

$$0=\langle Dg(\tilde{v}),\tilde{v}\rangle=\int_\Omega(\tilde{v}(x,0))^{p+1}dx=a,$$

which is an absurd, then $Dg(\tilde{v}) \neq 0$. Therefore by Lagrange multiplier theorem there is $\lambda \in \mathbb{R}$ such that

$$\int_0^\infty \int_\Omega \nabla\tilde{v}\nabla\tilde{\varphi}\,dx\,dz = \lambda \int_\Omega (\tilde{v}(x,0))^p \tilde{\varphi}(x,0)dx, \quad \forall \tilde{\varphi} \in H$$

Taking $\tilde{w} = \lambda^{-\frac{1}{1-p}}\tilde{v} \in H$ we have

$$\int_0^\infty \int_\Omega \nabla\tilde{w}\nabla\tilde{\varphi}\,dx\,dz = \int_\Omega (\tilde{w}(x,0))^p \tilde{\varphi}(x,0)dx, \quad \forall \tilde{\varphi} \in H$$

Thus the nonlinear problem (1.7) admits a weak solution. Note that the solution $\tilde{w}$ is nontrivial. If $\tilde{w} = 0$ then $\lambda = 0$ and

$$\int_0^\infty \int_\Omega |\nabla\tilde{v}|^2\,dx\,dz = 0$$

which implies $\tilde{v} = 0$ almost everywhere, which is an absurd. Therefore, the nonlinear problem (1.7) admits nontrivial weak solution. □

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