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GLOBAL EXISTENCE AND BLOWUP FOR FREE BOUNDARY PROBLEMS OF COUPLED REACTION-DIFFUSION SYSTEMS

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Abstract. This article concerns a free boundary problem for a reaction-diffusion system modeling the cooperative interaction of two diffusion biological species in one space dimension. First we show the existence and uniqueness of a local classical solution, then we study the asymptotic behavior of the free boundary problem. Our results show that the free boundary problem admits a global solution if the inter-specific competitions are strong, while, if the inter-specific competitions are weak, there exist the blowup solution and a global fast solution.

1. Introduction

We consider the free boundary problem

$$\begin{gathered}
u_t - d_1 u_{xx} = u(a_1 - b_1 u^r + v^p), \quad t > 0,\ 0 < x < h(t), \\
v_t - d_2 v_{xx} = v(a_2 - b_2 v^s + u^q), \quad t > 0,\ 0 < x < h(t), \\
u = v = 0, \quad t > 0,\ x = 0, \\
u = v = 0, \quad h'(t) = -\mu(u_x + \rho v_x), \quad t > 0,\ x = h(t), \\
h(0) = h_0, \quad 0 < h_0 < \infty, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad 0 \le x \le h_0,
\end{gathered} \tag{1.1}$$

where $a_i \geq 0, p, q, r, s, b_i, d_i$ $(i = 1, 2)$ and μ are positive constants, $x = h(t)$ is the free boundary to be determined together with $u(t, x)$ and $v(t, x)$. System (1.1) is usually referred as the cooperative system. It provides a simple model to describe, for instance, the cooperative interaction of two diffusing biological species. u and v represent the densities of two species, a_1 and a_2 are their growth rates. Here, it is assumed that each species finds its subsistence from the activity of the other one (represented by the reaction terms v^p and u^q), and disappears by a destruction mechanism, corresponding for instance to overcrowding or the action of a predator (represented by the absorption terms $b_1 u^r$ and $b_2 v^s$). For more background for the system, we can refer to [10, 12] and references therein.

As we know, the free boundary problems have been used to describe different types of mathematical models. For the study of free boundary problems for some biological models, we refer to, for instance [3, 4, 5, 8, 9, 15] and references cited

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therein. Let us recall some work about the blow-up results to the reaction-diffusion equations or systems with free boundaries. In [17], Zhang and Lin investigated the behavior of the positive solution $u(t,x)$ to a parabolic model with double fronts free boundaries:

$$\begin{gathered} u_t - du_{xx} = u^p, \quad t > 0,\ g(t) < x < h(t), \\ u(t,g(t)) = 0, \quad g'(t) = \mu u_x(t, g(t), \quad t > 0, \\ u(t,h(t)) = 0, \quad h'(t) = \mu u_x(t, h(t), \quad t > 0, \\ g(0) = -h_0, \quad h(0) = h_0, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad -h_0 \le x \le h_0. \end{gathered} \tag{1.2}$$

The result showed that when $p > 1$ blowup occurs if the initial datum is large enough and that the solution is global and fast, which decays uniformly at an exponential rate if the initial datum is small, while there is a global and slow solution provided that the initial value is suitably large. In [13], Ling et al. studied the global existence and blow-up for a parabolic equation with a nonlocal source and absorption

$$u_t - du_{xx} = \int_{g(t)}^{h(t)} u^p(t,x)dx - ku^q, \quad t > 0,\ g(t) < x < h(t),$$

with the same initial and boundary conditions as in (1.2). As far as the coupled system is concerned, Kim et al. [10] considered the mutualistic model

$$\begin{gathered} u_t - d_1 u_{xx} = u(a_1 - b_1 u + c_1 v), \quad t > 0,\ 0 < x < h(t), \\ v_t - d_2 v_{xx} = v(a_2 - b_2 v + c_2 u), \quad t > 0,\ 0 < x < \infty, \\ u(t,x) = 0, \quad t > 0,\ h(t) < x < \infty, \\ u = 0, \quad h'(t) = -\mu u_x, \quad t > 0,\ x = h(t), \\ u_x(t,0) = v_x(t,0) = 0, \quad t > 0, \\ h(0) = b, \quad 0 < b < \infty, \\ u(0,x) = u_0(x) \ge 0, \quad 0 \le x \le b, \\ v(0,x) = v_0(x) \ge 0, \quad 0 \le x \le \infty. \end{gathered}$$

They showed the existence and uniqueness of a classical local solution and the asymptotic behavior of the solution. And they showed that the free boundary problem admits a global slow solution if the inter-specific competitions are strong, while if the inter-specific competitions are weak there exist the blowup solution and global fast solution.

As we know, sometimes both species have a tendency to emigrate from the boundaries to obtain their new habitat; i.e., they will move outward along the unknown curves (free boundaries) as time increases. It is assumed that the movement speeds of free boundaries are proportional to the sum of gradient of these two species, i.e.

$$h'(t) = -\mu(u_x + \rho v_x),$$

which is the well-known Stefan type condition and whose ecological background can be found in [1].

In this article, our interests in studying the long time behavior of the solution of (1.1) is motivated by previous discussion. Differently from above, we put zero Dirichlet boundary conditions at the fixed boundary. This condition means that the habitat is restricted by a hostile environment from the left and the species cannot survive on the fixed boundary. We will show that if $pq < rs$, the solution of (1.1) is global while if $pq > rs$, there exist a blowup solution and a global fast solution of (1.1). To this end, we assume that the initial functions $u_0(x)$ and $v_0(x)$ satisfy

$$\begin{gathered} u_0, v_0 \in C^2([0, h_0]), \quad u_0(0) = v_0(0) = u_0(h_0) = v_0(h_0) = 0, \\ u_0(x), v_0(x) > 0 \quad \text{in } (0, h_0). \end{gathered} \tag{1.3}$$

Now let us recall some blowup results of the corresponding problem on a fixed domain under Dirichlet boundary condition with nonnegative initial data:

$$\begin{gathered} u_t = d_1\Delta u + u(a_1 - b_1u^r + v^p), \quad t > 0,\ x \in \Omega, \\ v_t = d_2\Delta v + v(a_2 + u^q - b_2v^s), \quad t > 0,\ x \in \Omega, \\ u = v = 0, \quad t > 0,\ x \in \partial\Omega, \end{gathered} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. By constructing blowup sub-solution or bounded super-solutions, Li and Wang [12] obtained the optimal conditions on the exponent of reaction and absorption terms for the existence or nonexistence of global solutions. The main results in [12] are stated as follows.

Proposition 1.1 ([12, Theorem 1]). *If $pq < rs$, then all solutions of (1.4) are global and uniformly bounded.*

Proposition 1.2 ([12, Theorem 3]). *Suppose that $pq > rs$. If $b_1^q b_2^{r_0} < 1$ for some $r_0 > 0$ satisfying $pq = r_0 s$, or $b_1^{s_0} b_2^p < 1$ for some $s_0 > 0$ satisfying $pq = rs_0$, then all solutions of (1.4) blows up in finite time with suitable initial data.*

The rest of the paper is organized as follows. In the next section, local existence and uniqueness of the free boundary problem are obtained by using the contraction mapping theorem. In Section 3 a priori estimates will be derived and the global existence will be given for the case $pq < rs$. Section 4 deals with the global existence and nonexistence of a classical positive solution for the case $pq > rs$.

2. Existence and uniqueness

In this section, we first prove the existence and uniqueness of a local solution using the contraction mapping theorem.

Theorem 2.1. *For any given $(u_0(x), v_0(x))$ satisfying (1.3) and any $\alpha \in (0, 1)$, there is a $T > 0$ such that problem (1.1) admits a unique solution*

$$(u, v, h) \in (C^{\frac{1+\alpha}{2}, 1+\alpha}(\bar{D}_T))^2 \times C^{\frac{1+\alpha}{2}}([0, T]).$$

Moreover,

$$\|u, v\|_{C^{\frac{1+\alpha}{2}, 1+\alpha}(\bar{D}_T)} + \|h\|_{C^{\frac{1+\alpha}{2}}([0,T])} \le C, \tag{2.1}$$

where $D_T = (0, T] \times (0, h(t))$, C and T are positive constants only depending on $h_0, \alpha, \|u_0, v_0\|_{C^2([0,h_0])}$. Here and in the following,

$$\|u, v\|_X := \|u\|_X + \|v\|_X.$$

Proof. As in [2, 6], we first straighten the free boundaries. Let $\zeta(y)$ be a function in $C^3(\mathbb{R})$ satisfying

$$\zeta(y) = \begin{cases} 1 & \text{if } |y - h_0| < h_0/4, \\ 0 & \text{if } |y - h_0| > h_0/2, \end{cases}$$

$$\zeta'(y) < \frac{6}{h_0}, \quad \forall y.$$

Consider the transformation

$$(t, y) \mapsto (t, x), \quad \text{where } x = y + \zeta(y)(h(t) - h_0),\ 0 \le y < \infty.$$

As long as $|h(t) - h_0| \le h_0/8$, the above transformation is a diffeomorphism from $[0, \infty)$ onto $[0, \infty)$. Moreover, it changes the free boundary $x = h(t)$ to the line $y = h_0$. If we set

$$\begin{aligned} u(t, x) &= u(t, y + \zeta(y)(h(t) - h_0)) = w(t, y), \\ v(t, x) &= v(t, y + \zeta(y)(h(t) - h_0)) = z(t, y), \end{aligned}$$

then the free boundary problem (1.1) becomes

$$\begin{gathered} w_t - Ad_1 w_{yy} - (Bd_1 + h'C) w_y = w\left(a_1 - b_1 w^r + z^p\right), \quad t > 0,\ 0 < y < h_0, \\ z_t - Ad_2 z_{yy} - (Bd_2 + h'C) z_y = z\left(a_2 + w^q - b_2 z^s\right), \quad t > 0,\ 0 < y < h_0, \\ w(t, 0) = z(t, 0) = w(t, h_0) = z(t, h_0) = 0, \quad t > 0, \\ w(0, y) = u_0(y), \quad z(0, y) = v_0(y), \quad 0 \le y \le h_0, \end{gathered} \tag{2.2}$$

where

$$\begin{aligned} A &:= A(h(t), y) = \frac{1}{(1 + \zeta'(y)(h(t) - h_0))^2}, \\ B &:= B(h(t), y) = -\frac{\zeta''(y)(h(t) - h_0))}{(1 + \zeta'(y)(h(t) - h_0))^3}, \\ C &:= C(h(t), y) = \frac{\zeta(y)}{1 + \zeta'(y)(h(t) - h_0)}. \end{aligned}$$

Denote $h_1 = -\mu(u_0'(h_0) + \rho v_0'(h_0))$, and for $0 < T \le \frac{h_0}{8(1+h_1)}$, define $\Delta_T = [0, T] \times [0, h_0]$,

$$\begin{aligned} \mathcal{D}_{1T} = \big\{ & w \in C^{\frac{\alpha}{2}, \alpha}(\Delta_T) : w(t, y) \ge 0, w(0, y) = u_0(y), w(t, h_0) = 0, \\ & \|w - u_0\|_{C^{\frac{\alpha}{2}, \alpha}(\Delta_T)} \le 1 \big\}, \\ \mathcal{D}_{2T} = \{ & h \in C^1([0, T]) : h(0) = h_0,\ h'(0) = h_1,\ \|h' - h_1\|_{C([0,T])} \le 1 \}. \end{aligned}$$

It is easily seen that the set $\mathcal{D} = \mathcal{D}_{1T} \times \mathcal{D}_{2T}$ is a closed convex set in $C^{\frac{\alpha}{2}, \alpha}(\Delta_T) \times C^1([0, T])$.

Next, we shall prove the existence and uniqueness result by using the contraction mapping theorem. First, we observe that due to our choice of T, for any given $(w, h) \in \mathcal{D}$, we have

$$|h(t) - h_0| \le T(1 + h_1) \le \frac{h_0}{8}.$$

Therefore the transformation $(t, y) \to (t, x)$ introduced at the beginning of the proof is well defined. Applying standard L^p theory and then the Sobolev imbedding theorem, we find that for any $(w, h) \in \mathcal{D}$, the initial boundary value problem

$$\begin{gathered} z_t - Ad_2 z_{yy} - (Bd_2 + h'C) z_y = z\left(a_2 + w^q - b_2 z^s\right), \quad t > 0,\ 0 < y < h_0, \\ z(t, 0) = z(t, h_0) = 0, \quad t > 0, \\ z(0, y) = v_0(y), \quad 0 \le y \le h_0, \end{gathered} \tag{2.3}$$

admits a unique solution (see [11]) $z \in C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)$, and

$$\|z\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)} \le C_1.$$

Moreover, the initial boundary value problem

$$\begin{gathered}\overline{w}_t - Ad_1\overline{w}_{yy} - (Bd_1 + h'C)\overline{w}_y = w\left(a_1 - b_1 w^r + z^p\right), \quad t > 0,\ 0 < y < h_0,\\ \overline{w}(t,0) = \overline{w}(t,h_0) = 0, \quad t > 0,\\ \overline{w}(0,y) = u_0(y), \quad 0 \le y \le h_0,\end{gathered} \tag{2.4}$$

admits a unique solution $\overline{w} \in C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)$, and

$$\|\overline{w}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)} \le C_2, \tag{2.5}$$

where C_1, C_2 are two constants depending on h_0, α, u_0, v_0.

Defining

$$\overline{h}(t) = h_0 - \int_0^t \mu(\overline{w}_y(\tau, h_0) + \rho z_y(\tau, h_0))d\tau,$$

we have

$$\overline{h}'(t) = -\mu(\overline{w}_y(t,h_0) + \rho z_y(t,h_0)), \quad \overline{h}(0) = h_0, \quad \overline{h}'(0) = h_1, \tag{2.6}$$

and hence $\overline{h}' \in C^{\alpha/2}([0,t])$ with

$$\|\overline{h}'\|_{C^{\alpha/2}([0,t])} \le C_3 := \mu(C_2 + \rho C_1). \tag{2.7}$$

We now define $\mathcal{F} : \mathcal{D} \to C^{\frac{\alpha}{2},\alpha}(\Delta_T) \times C^1([0,T])$ by

$$\mathcal{F}(w,h) = (\overline{w}, \overline{h}).$$

Clearly $(w,h) \in \mathcal{D}$ is a fixed point of $\mathcal{F}$ if and only if it solves (2.2).

By (2.7)) and (2.5), we have

$$\|\overline{h}' - h_1\|_{C([0,T])} \le \|\overline{h}'\|_{C^{\alpha/2}([0,T])T^{\alpha/2}} \le C_3 T^{\alpha/2},$$

$$\begin{aligned}&\|\overline{w} - u_0\|_{C^{\frac{\alpha}{2},\alpha}(\Delta_T)}\\ &\le \|\overline{w}\|_{C^{\frac{1+\alpha}{2},0}(\Delta_T)} T^{\frac{1+\alpha}{2}} + \|\overline{w}\|_{C^{\frac{1+\alpha}{2},0}(\Delta_T)} T^{\frac{1}{2}} + h_0^{1-\alpha}\|\overline{w}_y\|_{C^{\alpha/2,0}(\Delta_T)} T^{\alpha/2}\|\\ &\le C_2\left(T^{\frac{1+\alpha}{2}} + T^{\frac{1}{2}} + h_0^{1-\alpha}T^{\alpha/2}\right).\end{aligned}$$

Therefore, if we take $T \le \min\{1, C_3^{-2/\alpha}, [(2 + h_0^{1-\alpha})C_1]^{-2/\alpha}\}$, then $\mathcal{F}$ maps $\mathcal{D}$ into itself.

Next we prove that $\mathcal{F}$ is a contraction mapping on $\mathcal{D}$ for $T > 0$ sufficiently small. Let $(w_i, h_i) \in \mathcal{D}(i = 1, 2)$ and denote $(\overline{w}_i, \overline{h}_i) = \mathcal{F}(w_i, h_i)$. Then it follows from (2.5) and (2.7) that

$$\|\overline{w}_i\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)} \le C_2, \quad \|\overline{h}_i'\|_{C^{\alpha/2}([0,t])} \le C_3. \tag{2.8}$$

Setting $U = \overline{w}_1 - \overline{w}_2$, $V = z_1 - z_2$, we find that $V(t,y)$ and $U(t,y)$ satisfy

$$\begin{aligned}&V_t - A(h_2,y)d_2V_{yy} - (B(h_2,y)d_2 + h_2'C(h_2,y))V_y\\ &= [A(h_1,y) - A(h_2,y)]d_2 z_{1,yy} + [B(h_1,y) - B(h_2,y)]d_2 z_{1,y}\\ &\quad + [h_1'C(h_1,y) - h_2'C(h_2,y)]z_{1,y} + (a_2 - b_2\Phi_2(t,y) + w_1^q)(z_1 - z_2)\\ &\quad + z_2\Psi_2(t,y)(w_1 - w_2), \quad t > 0,\ 0 < y < h_0,\\ &\qquad V(t,0) = V(t,h_0) = 0, \quad t > 0,\end{aligned}$$

$$V(0,y)=0, \quad 0\le y\le h_0,$$

and

$$\begin{aligned}
&U_t - A(h_2,y)d_1U_{yy} - (B(h_2,y)+h_2'C(h_2,y))U_y\\
&= [A(h_1,y)-A(h_2,y)]d_1w_{1,yy} + [B(h_1,y)-B(h_2,y)]d_1w_{1,y}\\
&\quad + [h_1'C(h_1,y)-h_2'C(h_2,y)]w_{1,y} + (a_1 - b_1\Phi_1(t,y)+z_1^p)(w_1-w_2)\\
&\quad + w_2\Psi_1(t,y)(z_1-z_2), \quad t>0,\ 0<y<h_0,\\
&\qquad U(t,0)=U(t,h_0)=0, \quad t>0,\\
&\qquad U(0,y)=0, \quad 0\le y\le h_0,
\end{aligned}$$

where

$$\begin{gathered}
\Phi_1(t,y)=\int_0^1 (r+1)(\theta w_1+(1-\theta)w_2)^r d\theta,\\
\Phi_2(t,y)=\int_0^1 (s+1)(\theta z_1+(1-\theta)z_2)^s d\theta,\\
\Psi_1(t,y)=\int_0^1 p(\theta z_1+(1-\theta)z_2)^{p-1} d\theta,\\
\Psi_2(t,y)=\int_0^1 q(\theta w_1+(1-\theta)w_2)^{q-1} d\theta.
\end{gathered}$$

Using standard partial differential equation theory [11], the L^p estimates for parabolic equations and Sobolev's imbedding theorem, we obtain

$$\|z_1-z_2\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)} \le C_4(\|w_1-w_2\|_{C(\Delta_T)} + \|h_1-h_2\|_{C^1([0,T])}), \tag{2.9}$$

$$\begin{aligned}
&\|\overline{w}_1-\overline{w}_2\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)}\\
&\le C_4(\|w_1-w_2\|_{C(\Delta_T)} + \|h_1-h_2\|_{C^1([0,T])} + \|z_1-z_2\|_{C(\Delta_T)})\\
&\le C_5(\|w_1-w_2\|_{C(\Delta_T)} + \|h_1-h_2\|_{C^1([0,T])}).
\end{aligned}\tag{2.10}$$

using (2.6), we have

$$\|\overline{h}_1'-\overline{h}_2'\|_{C^{\alpha/2}([0,T])} \le \mu(\|\overline{w}_1-\overline{w}_2\|_{C^{\alpha/2,0}(\Delta_T)} + \rho\|z_1-z_2\|_{C^{\frac{\alpha}{2},0}(\Delta_T)}). \tag{2.11}$$

Combing (2.9)-(2.11), assuming $T\le 1$, and applying mean value theorem, we obtain

$$\begin{aligned}
&\|\overline{w}_1-\overline{w}_2\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)} + \|\overline{h}_1'-\overline{h}_2'\|_{C^{\alpha/2}([0,T])}\\
&\le C_6(\|w_1-w_2\|_{C(\Delta_T)} + \|h_1-h_2\|_{C^1([0,T])}),
\end{aligned}$$

which implies

$$\begin{aligned}
\|\overline{w}_1-\overline{w}_2\|_{C^{\alpha/2,\alpha}(\Delta_T)} &\le \|w_1-w_2\|_{C^{\frac{1+\alpha}{2},0}(\Delta_T)}T^{\frac{1+\alpha}{2}} + \|w_1-w_2\|_{C^{\frac{1+\alpha}{2},0}(\Delta_T)}T^{1/2}\\
&\quad + h_0^{1-\alpha}\|\overline{w}_{1y}-\overline{w}_{2y}\|_{C^{\alpha/2,0}(\Delta_T)}T^{\frac{\alpha}{2}}\\
&\le (2+h_0^{1-\alpha})T^{\frac{\alpha}{2}}\|\overline{w}_1-\overline{w}_2\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)}.
\end{aligned}$$

Hence, for

$$T:=\min\big\{1,(4+2h_0^{1-\alpha})^{-2/\alpha}, C_3^{-2/\alpha}, [(2+h_0^{1-\alpha})C_1]^{-2/\alpha}, \frac{h_0}{8(1+h_1)}\big\},$$

we have

$$\begin{aligned}
&\|\overline{w}_1-\overline{w}_2\|_{C^{\alpha/2,\alpha}(\Delta_T)}+\|h_1-h_2\|_{C^1([0,T])}\\
&\le (2+h_0^{1-\alpha})T^{\frac{\alpha}{2}}\|\overline{w}_1-\overline{w}_2\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)}+2T^{\alpha/2}\|h_1'-h_2'\|_{C^{\alpha/2}([0,T])}\\
&\le (2+h_0^{1-\alpha})T^{\frac{\alpha}{2}}C_6(\|w_1-w_2\|_{C(\Delta_T)}+\|h_1-h_2\|_{C^1([0,T])})\\
&\le \frac{1}{2}(\|\overline{w}_1-\overline{w}_2\|_{C^{\alpha/2,\alpha}(\Delta_T)}+\|h_1-h_2\|_{C^1([0,T])}).
\end{aligned}$$

This shows that for this T, $\mathcal{F}$ is a contraction mapping on $\mathcal{D}$. It now follows from the contraction mapping theorem that $\mathcal{F}$ has a unique fixed point (w,h) in $\mathcal{D}$. Moreover, by the Schauder estimates, we have additional regularity for (w,z,h) as a solution of (2.2), namely, $h\in C^{1+\frac{\alpha}{2}}([0,T])$ and $w,z\in C^{\frac{1+\alpha}{2},1+\alpha}((0,T]\times[0,h_0])$, and (2.5), (2.7) hold. In other words, $(w(t,y),z(t,y),h(t))$ is a unique local classical solution of the problem (2.2). Hence, (u,v,h) is a unique classical solution of (1.1). □

Theorem 2.2. *The free boundary for the problem* (1.1) *is strictly monotone increasing; i.e., for any solution in* $(0,T]$, *we have* $h'(t)>0$ *for* $0<t\le T$.

Proof. Firstly, as $u>0$ for $0<x<h(t)$ and $u=0$ at $x=h(t)$, we see that $u_x(t,h(t))\le 0$ and so $h'(t)\ge 0$. Since we only know $h\in C^{1+\frac{\alpha}{2}}([0,\infty))$, it can not be guaranteed that the domain $(0,\infty)\times[0,h(t)]$ has an interior sphere property at the right boundary $x=h(t)$. hence, the Hopf lemma cannot be used directly to get $h'(t)>0$. To solve this, we use a transformation to straighten the free boundary $x=h(t)$. Define $y=x/h(t)$ and $w(t,y)=u(t,x),z(t,y)=v(t,x)$. A series of detailed calculation asserts that

$$\begin{gathered}
w_t-d_1\zeta(t)w_{yy}-\xi(t,y)w_y=w(a_1-b_1w^r+z^p),\quad t>0,\ 0<y<1,\\
z_t-d_2\zeta(t)z_{yy}-\xi(t,y)z_y=z(a_2-b_2z^s+w^q),\quad t>0,\ 0<y<1,\\
w(t,0)=w(t,1)=0,\quad t>0,\\
w(0,y)=u_0(h_0y),\quad z(0,y)=v_0(h_0y),\quad 0\le y\le 1,
\end{gathered}$$

where $\zeta(t)=h^{-2}(t)$, $\xi(t,y)=yh'(t)/h(t)$. This is an initial and boundary value problem with fixed boundary. Since $w>0,z>0$ for $t>0$ and $0<y<1$, by the Hopf lemma, we have $w(y,1)<0$, $z(y,1)<0$ for $t>0$. This combines with the relation $u_x=h^{-1}(t)w_y$ and $v_x=h^{-1}(t)z_y$ to derive that $u_x(t,h(t))<0$ and $v_x(t,h(t))<0$ and so $h'(t)>0$ for $t>0$. □

It is observed that there exists a T such that the solution exists in the time interval $[0,T]$. The maximal existing time of the solution $T_{\max}$ depends on a prior estimate with respect to $\|u\|_{L^\infty}$, $\|v\|_{L^\infty}$ and $h'(t)$. Next we show that if $\|u\|_{L^\infty}<\infty$ or $\|v\|_{L^\infty}<\infty$, the solution can be extended. Therefore we first give the following lemma.

Lemma 2.3. *Let* (u,v,h) *be a solution to problem* (1.1) *defined for* $t\in(0,T_0)$ *for some* $T_0\in(0,+\infty]$. *If* $M_1:=\|u\|_{L^\infty([0,T]\times[0,h(t)])}<\infty$, *then there exist constants* M_2 *and* M_3 *independent of* T_0 *such that*

$$0<v(t,x)\le M_2(M_1),\quad 0<h'(t)\le M_3(M_1)$$

for $0<t<T_0,\ 0\le x<h(t)$.

Proof. By (1.1), we obtain

$$v_t - d_2 v_{xx} \le v(a_2 + M_1^q - b_2 v^s), \quad 0 < t < T_0, \quad 0 \le x < h(t).$$

It follows from the comparison principle that $v(t,x) \le \overline{v}(t)$ for $t \in (0,T_0)$ and $x \in [0,h(t)]$, where $\overline{v}(t)$ is a unique solution of the problem

$$\frac{d\overline{v}}{dt} = \overline{v}(a_2 + M_1^q - b_2\overline{v}^s), \quad t > 0; \quad \overline{v}(0) = \|v_0\|_\infty.$$

It is obvious that $\bar{v}$ is globally bounded. Thus we have

$$v(t,x) \le M_2 := \sup_{t\ge 0} \overline{v}(t).$$

Moreover, by Theorem 2.2, we have $h'(t) > 0$ for $t \in (0,T_0)$. It remains to show that $h'(t) \le M_2$ for all $t \in (0,T_0)$ with some M_2 independent of T_0. To this end, we define

$$\Omega_M := \{(t,x) : 0 < t < T_0,\ h(t) - 1/M < x < h(t)\}$$

and construct an auxiliary function

$$\omega(t,x) := M_1[2M(h(t) - x) - M^2(h(t) - x)^2].$$

We will choose M so that $\omega(t,x) \ge u(t,x)$ holds over Ω_M.

Direct calculations show that, for $(t,x) \in \Omega_M$,

$$w_t = 2M_1 M h'(t)(1 - M(h(t) - x)) \ge 0, \quad -w_{xx} = 2M_1M^2,$$
$$u(a_1 - b_1u^r + v^p) \le M_1(a_1 + M_2^p).$$

It follows that

$$\omega_t - d_1\omega_{xx} \ge M_1(a_1 + M_2^p) \ge u(a_1 - b_1u^r + v^p)$$

if $M^2 \ge \frac{a_1+M_2^p}{2d_1}$. On the other hand,

$$\omega(t,h(t) - 1/M) = M_1 \ge u(t,h(t) - 1/M), \quad \omega(t,h(t)) = 0 = u(t,h(t)).$$

Thus, if we can choose M such that

$$u_0(x) \le \omega(0,x) \quad \text{for } x \in [h_0 - 1/M, h_0], \tag{2.12}$$

then we can apply the maximum principle to $\omega - u$ over Ω_M to deduce that $u(t,x) \le \omega(t,x)$ for $(t,x) \in \Omega_M$. It would then follow that

$$u_x(t,h(t)) \ge \omega_x(t,h(t)) = -2MM_1.$$

With the same method, we can deduce

$$v_x(t,h(t)) \ge \omega_x(t,h(t)) = -2MM_2,$$

if $M^2 \ge \frac{a_2+M_1^q}{2d_2}$. Hence, if $M^2 \ge \max\{\frac{a_1+M_2^p}{2d_1}, \frac{a_2+M_1^q}{2d_2}\}$, we have

$$h'(t) = -\mu(u_x(t,h(t)) + \rho v_x(t,h(t))) \le M_3 := 2M\mu(M_1 + \rho M_2).$$

To complete the proof, we need only find some M such that (2.12) holds. By direct calculation, we obtain

$$u_0(x) = \int_{h_0}^x u_0'(y)dy \le \|u_0'\|_{C([0,h_0])}(h_0 - x) \quad \text{on } [h_0 - 1/M, h_0],$$
$$\omega(0,x) = M_1[2M(h_0 - x) - M^2(h_0 - x)^2] \ge M_1M(h_0 - x).$$

Therefore, upon choosing $M \geq \frac{\|u_0'\|_{C([0,h_0])}}{M_1}$, (2.12) follows. To conclude, we choose

$$M := \max\Big\{\frac{a_1 + M_2^p}{2d_1}, \frac{a_2 + M_1^q}{2d_2}, \frac{\|u_0'\|_{C([0,h_0])}}{M_1}, \frac{\|v_0'\|_{C([0,h_0])}}{M_2}\Big\},$$

thus the proof is complete. □

With the same method as in proof of [6, Theorem 2.3], we can get the existence and uniqueness of a global solution for (1.1).

Theorem 2.4. *The solution of problem* (1.1) *exists and is unique, and it can be extended to* $[0, T_{\max})$ *where* $T_{\max} \leq \infty$. *Moreover, if* $T_{\max} < \infty$*, we have* $\limsup_{t\to T_{\max}} \|u, v\|_{L^\infty([0,h(t)]\times[0,t])} = \infty$.

Proof. It follows from the uniqueness that there is a $T_{\max}$ such that $[0, T_{\max})$ is the maximal time interval in which the solution exists. In order to prove the present theorem, it suffices to show that, when $T_{\max} < \infty$,

$$\limsup_{t\to T_{\max}} \|u, v\|_{L^\infty([0,t]\times[0,h(t)])} = \infty.$$

In what follows we use the contradiction argument. Assume that $T_{\max} < \infty$ and $\|u\|_{L^\infty([0,T_{\max})\times[0,h(t)])} < \infty$. Since $v \leq M_2(M)$ in $[0, h(t)] \times [0, T_{\max})$ and $0 < h'(t) \leq M_3$ in $[0, T_{\max})$ by Lemma 2.3, using a bootstrap argument and the Schauder's estimate yields a priori bound of $\|u(t, x), v(t, x)\|_{C^{1+\alpha}([0,h(t)]}$ for all $t \in [0, T_{\max})$. Let the bound be M_4. It follows from the proof of Theorem 2.1 that there exists a $\tau > 0$ depending only on M_1, M_2, M_3 and M_4 such that the solution of the problem (1.1) with the initial time $T_{\max} - \tau/2$ can be extended uniquely to the time $T_{\max} - \tau/2 + \tau$ that contradicts the assumption. Thus the proof is completed. □

3. Global solution for the case $pq < rs$

We first give a comparison principle, whose proof is standard and we omit it (see [16]).

Lemma 3.1. *Let* $a_i(x, t), b_i(x, t), c_i(x, t)$, $(i = 1, 2)$, *be continuous functions in* $\Omega \times (0, T)$. *Assume that* $a_i(x, t), c_i(x, t) \geq 0$ *in* $\Omega \times (0, T)$ *and* $b_i(x, t), c_i(x, t)$ *are bounded on* $\bar\Omega \times [0, T_0]$ *for any* $T_0 < T$. *If functions* u_i *belong to* $C^{2,1}(\Omega \times (0, T)) \cap C(\bar\Omega \times [0, T])$, $i = 1, 2$, *and satisfy*

$$\begin{gathered} u_{1t} \leq (\geq) a_1\Delta u_1 + b_1u_1 + c_1u_2, \quad 0 < t < T,\ x \in \Omega, \\ u_{2t} \leq (\geq) a_2\Delta u_2 + b_2u_2 + c_2u_1, 0 < t < T, \quad x \in \Omega, \\ u_1(0, x) \leq (\geq) 0, \quad u_2(0, x) \leq (\geq) 0, \quad x \in \Omega, \\ u_1(t, x) \leq (\geq) 0, \quad u_2(t, x) \leq (\geq) 0, \quad 0 < t < T,\ x \in \partial\Omega, \end{gathered} \tag{3.1}$$

then

$$(u_1, u_2) \leq (\geq) 0, \quad \forall (x, t) \in \bar\Omega \times [0, T),\ i = 1, 2.$$

In order to get the global existence, we aim to construct a constant supersolution of (1.1). Now, we state a simple fact without proof.

Lemma 3.2 ([12, Lemma 1])**.** *If* $p, q, r, s > 0$ *and* $pq < rs$*, then for any positive constants* A, B*, there exist two positive constants* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *such that* $A\mathcal{M}_1^r \geq \mathcal{M}_2^p$ *and* $B\mathcal{M}_2^s \geq \mathcal{M}_1^q$. *In addition, if* $(\mathcal{M}_1, \mathcal{M}_2)$ *is a solution to this inequalities,*

$(k\mathcal{M}_1, \ell\mathcal{M}_2)$ is a solution to such inequalities for every $k, \ell \geq 1$ satisfying $k^{q/s} \leq \ell \leq k^{r/p}$.

Lemma 3.3. *If $pq < rs$, the solution of the free boundary problem* (1.1) *satisfies*

$$0 < u(t,x) < \mathcal{M}_1, \quad 0 < v(x,t) \leq \mathcal{M}_2 \quad \text{for } 0 \leq t \leq T,\ 0 \leq x < h(t),$$

where $\mathcal{M}_i$ is independent of T for $i = 1, 2$.

Proof. Fix $C = \max\{\max_{\overline{\Omega}} u_0(x), \max_{\overline{\Omega}} v_0(x)\}$. We seek a pair of constant $\mathcal{M}_1$, $\mathcal{M}_2$ such that $\mathcal{M}_1, \mathcal{M}_2 \geq C$, and

$$b_1\mathcal{M}_1^r > a_1 + \mathcal{M}_2^p, \quad b_2\mathcal{M}_2^s \geq a_2 + \mathcal{M}_1^q. \tag{3.2}$$

To begin with, we choose $(\mathcal{M}_1, \mathcal{M}_2)$ such that $a_1 \leq \mathcal{M}_2^p$ and $a_2 \leq \mathcal{M}_1^q$. Therefore, (3.2) holds provided that

$$b_1\mathcal{M}_1^r > 2\mathcal{M}_2^p, \quad b_2\mathcal{M}_2^s \geq 2\mathcal{M}_1^q.$$

Since $pq < rs$, the existence of suitable $\mathcal{M}_1$ and $\mathcal{M}_2$ is guaranteed by Lemma 3.2.

Next we prove that for any $l > h_0$, $(u(t,x), v(t,x)) \leq (\mathcal{M}_1, \mathcal{M}_2) := (\overline{u}, \overline{v})$. From the above process, we have $(\overline{u}, \overline{v})$ satisfies

$$\begin{gathered}
\overline{u}_t - d_1\overline{u}_{xx} \geq \overline{u}(a_1 - b_1\overline{u}^r + \overline{v}^p), \quad 0 < t \leq T,\ 0 < x < l, \\
\overline{v}_t - d_2\overline{v}_{xx} \geq \overline{v}(a_2 + \overline{u}^q - b_2\overline{v}^s), \quad 0 < t \leq T,\ 0 < x < l, \\
\overline{u} \geq 0, \quad \overline{v} \geq 0, \quad 0 < t \leq T,\ x = 0, l, \\
\overline{u}(0,x) \geq u_0(x), \quad \overline{v}(0,x) \geq v_0(x) \quad 0 \leq x \leq l.
\end{gathered}$$

Set $w = \bar{u} - u,\ z = \bar{v} - v$, then we have

$$\begin{gathered}
w_t - d_1 w_{xx} \geq (a_1 - b_1\Phi_3(t,x) + v^p)w + \bar{u}\Psi_3(t,x)z, \quad 0 < t \leq T,\ 0 < x < l, \\
z_t - d_2 z_{xx} \geq (a_2 - b_2\Phi_4(t,x) + u^q)w + \bar{v}\Psi_4(t,x)z, \quad 0 < t \leq T,\ 0 < x < l, \\
w \geq 0,\ z \geq 0, \quad 0 < t \leq T,\ x = 0, l, \\
w(0,x) \geq 0, \quad z(0,x) \geq 0, \quad 0 \leq x \leq l,
\end{gathered}$$

where

$$\Phi_3(t,x) = \int_0^1 (r+1)(\theta\bar{u} + (1-\theta)u)^r d\theta, \quad \Phi_4(t,x) = \int_0^1 (s+1)(\theta\bar{v} + (1-\theta)v)^s d\theta,$$

$$\Psi_3(t,x) = \int_0^1 p(\theta\bar{v} + (1-\theta)v)^{p-1} d\theta, \quad \Psi_4(t,x) = \int_0^1 q(\theta\bar{u} + (1-\theta)u)^{q-1} d\theta.$$

Using Lemma 3.1 in $[0,T] \times [0,l]$ shows that $u \leq \overline{u}$ and $v \leq \overline{v}$. Now for any fixed $(t_0, x_0) \in [0,T] \times [0, h(t)]$, let l be sufficiently large so that $(t_0, x_0) \in [0,T] \times [0,l]$, and it follows from the above proof that

$$\begin{gathered}
u(t_0, x_0) \leq \overline{u}(t_0, x_0) = \mathcal{M}_1, \\
v(t_0, x_0) \leq \overline{v}(t_0, x_0) = \mathcal{M}_2,
\end{gathered}$$

which gives the desired estimates. □

Combining Theorem 2.4 with Lemma 3.3 yields the existence of a global solution.

Theorem 3.4. *If $pq < rs$, the free boundary problem* (1.1) *admits a unique global classical solution.*

4. Global and nonglobal solutions for the case $pq > rs$

In this section, we consider the asymptotic behavior of the solution for the case $pq > rs$. First we give the blowup result.

Theorem 4.1. *Assume that $pq > rs$. If $b_1^q b_2^{r_0} < 1$ for some $r_0 > 0$ satisfying $pq = r_0 s$, or $b_1^{s_0} b_2^p < 1$ for some $s_0 > 0$ satisfying $pq = rs_0$, then all solutions of* (1.1) *blow up in finite time with suitable initial data.*

Proof. To show this, it suffices to compare the free boundary problem with the corresponding problem in the fixed domain:

$$\begin{gathered} u_t - d_1 u_{xx} = u(a_1 - b_1 u^r + v^p), \quad t > 0,\ 0 < x < h_0, \\ v_t - d_2 v_{xx} = v(a_2 + u^q - b_2 v^s), \quad t > 0,\ 0 < x < h_0, \\ u(t,0) = v(t,0) = 0, \quad t > 0, \\ u(t,h_0) = v(t,h_0) = 0, \quad t > 0, \\ u(0,x) = u_0(x) \geq 0,\ v(0,x) = v_0(x) \geq 0, \quad 0 \leq x \leq h_0 \end{gathered} \tag{4.1}$$

It follows from Proposition 1.2 that the solution blows up if $b_1^q b_2^{r_0} < 1$ for some $r_0 > 0$ satisfying $pq = r_0 s$, or $b_1^{s_0} b_2^p < 1$ for some $s_0 > 0$ satisfying $pq = rs_0$. We conclude the result by using comparison principle for the fixed boundary. □

Now we present a comparison principle for u, v and the free boundary $x = h(t)$ which can be used to estimate the solution $(u(t,x), v(t,x))$ and the free boundary $x = h(t)$.

Lemma 4.2. *Suppose that $T \in (0,\infty)$, $\overline{h} \in C^1([0,T])$, $\overline{u}, \overline{v} \in C(\overline{D}^*_{1,T}) \cap C^{1,2}(D^*_{1,T})$ with $D^*_{1,T} = (0,T] \times (0, \overline{h}(t))$, and*

$$\begin{gathered} \overline{u}_t - d_1 \overline{u}_{xx} \geq \overline{u}(a_1 - b_1 \overline{u}^r + \overline{v}^p), \quad t > 0,\ 0 < x < \overline{h}(t), \\ \overline{v}_t - d_2 \overline{v}_{xx} \geq \overline{v}(a_2 + \overline{u}^q - b_2 \overline{v}^s), \quad t > 0,\ 0 < x < \overline{h}(t), \\ \bar{u}, \quad \bar{v} \geq 0, \quad t > 0,\ x = 0, \\ \bar{u} = \bar{v} = 0, \quad \bar{h}'(t) \geq -\mu(\bar{u}_x + \rho \bar{v}_x), \quad t > 0, \quad x = \bar{h}(t), \\ \bar{u}(0,x) \geq u_0(x), \bar{v}(0,x) \geq v_0(x), \quad 0 \leq x \leq h_0. \end{gathered}$$

If $h(0) \leq \overline{h}(0)$,

$$\begin{gathered} (\bar{u}(0,x), \bar{v}(0,x)) \geq (0,0) \quad \text{on } [0, \bar{h}(0)], \\ (u_0(x), v_0(x)) \leq (\overline{u}(0,x), \overline{v}(0,x)) \quad \text{on } [0, h_0], \end{gathered}$$

then the solution (u,v,h) of the free boundary problem (1.1) *satisfies $h(t) \leq \overline{h}(t)$ in $(0,T]$, $(u(t,x), v(t,x)) \leq (\overline{u}(t,x), \overline{v}(t,x))$ in $[0,T] \times (0, h(t))$.*

Proof. We first assume that $\overline{h}(0) > h(0)$. Then $\overline{h}(t) > h(t)$ for small $t > 0$. We can derive that $\overline{h}(t) > h(t)$ for all $t \geq 0$. If this is not true, there exists $t^* > 0$ such that $\overline{h}(t^*) = h(t^*)$ and $\overline{h}(t) > h(t)$ for all $t \in (0, t^*)$. Thus, $\overline{h}'(t^*) < h'(t^*)$. Recall that $(u_0(x), v_0(x)) \leq (\overline{u}(0,x), \overline{v}(0,x))$ on $[0, h_0]$, $u(t^*, h(t^*)) = 0 = \overline{u}(t^*, \overline{h}(t^*))$ and $v(t^*, h(t^*)) = 0 = \overline{v}(t^*, \overline{h}(t^*))$. As the proof of Lemma 3.3, and applying Lemma 3.1 for the fixed boundary, we can obtain that $(u(t,x), v(t,x)) \leq (\overline{u}(t,x), \overline{v}(t,x))$ in $(0,t^*) \times (0, h(t^*))$ and

$$\frac{\partial}{\partial x}(u - \overline{u})\big|_{(t^*, h(t^*))} \geq 0, \quad \frac{\partial}{\partial x}(v - \overline{v})\big|_{(t^*, h(t^*))} \geq 0,$$

which shows that

$$\begin{aligned}h'(t^*) &= -\mu\Big(\frac{\partial u}{\partial x}(t^*, h(t^*)) + \rho\frac{\partial v}{\partial x}(t^*, h(t^*))\Big)\\ &\le -\mu\Big(\frac{\partial \overline{u}}{\partial x}(t^*, \overline{h}(t^*)) + \rho\frac{\partial \overline{v}}{\partial x}(t^*, \overline{h}(t^*))\Big)\\ &\le \overline{h}'(t^*).\end{aligned}$$

This leads to a contradiction, which proves that $h(t) < \overline{h}(t)$ for $0 \le t \le T$ in the case $\overline{h}(0) > h(0)$. The general case can be established through approximation (we also can refer to [8, Lemma 5.1]). Since $h(t) \le \overline{h}(t)$ for $0 \le t \le T$, we have $(u(t,x), v(t,x) \le (\overline{u}(t,x), \overline{v}(t,x))$ in $[0,T] \times (0, h(t))$. □

Remark 4.3. The pair $(\overline{u}, \overline{v}, \overline{h})$ in Lemma 4.2 is usually called an upper solution of (1.1). We can define a lower solution by reversing all the inequalities in the obvious places. Moreover, one can easily prove an analogue of Lemma 4.2 for lower solutions.

Next we present some conditions so that the global fast solution is possible.

Theorem 4.4. *If $pq > rs$, then the free boundary problem* (1.1) *admits a global fast solution, provided the initial data is suitably small and h_0 is suitably small.*

Proof. It suffices to construct the suitable global supersolution. Inspired by [10], we define

$$\begin{gathered}\sigma(t) = 2h_0(2 - e^{-\gamma t}), \quad t \ge 0;\\ V(y) = \cos\big(\frac{\pi}{2}y\big), \quad 0 \le y \le 1;\\ w(t,x) = z(t,x) = \varepsilon e^{-\alpha t} V\big(\frac{x}{\sigma(t)}\big), \quad t \ge 0,\ 0 \le x \le \sigma(t),\end{gathered}$$

where γ, α and $\varepsilon > 0$ are to be chosen later. Direct computation yields

$$\begin{aligned}&w_t - d_1 w_{xx} - w(a_1 - b_1 w^r + z^p)\\ &\ge \varepsilon e^{-\alpha t}[-\alpha V + d_1 V \sigma^{-2}(\frac{\pi}{2})^2 - V(a_1 - b_1 w^r + z^p)]\\ &\ge \varepsilon e^{-\alpha t} V\big[-\alpha + \big(\frac{\pi}{2}\big)^2 \frac{d_1}{16 h_0^2} - a_1 - \varepsilon^p\big],\end{aligned}$$

and

$$z_t - d_2 z_{xx} - z(a_2 + z^q - b_2 z^s) \ge \varepsilon e^{-\alpha t} V\big[-\alpha + \big(\frac{\pi}{2}\big)^2 \frac{d_2}{16 h_0^2} - a_2 - \varepsilon^q\big],$$

for all $t > 0$ and $0 < x < \sigma(t)$. On the other hand, we have $\sigma'(t) = 2\gamma h_0 e^{-\gamma t} > 0$ and $-w_x(t, \sigma(t)) = -z_x(t, \sigma(t)) < 2\varepsilon \sigma^{-1}(t) e^{-\alpha t}$. Now we choose h_0 that satisfies

$$\begin{gathered}a_i \le \big(\frac{\pi}{2}\big)^2 \frac{d_i}{64 h_0^2},\ i = 1, 2; \quad \alpha = \gamma = \min\big\{\big(\frac{\pi}{2}\big)^2 \frac{d_1}{64 h_0^2}, \big(\frac{\pi}{2}\big)^2 \frac{d_2}{64 h_0^2}\big\},\\ \varepsilon = \min\Big\{\Big(\big(\frac{\pi}{2}\big)^2 \frac{d_1}{64 h_0^2}\Big)^{1/p}, \Big(\big(\frac{\pi}{2}\big)^2 \frac{d_2}{64 h_0^2}\Big)^{1/q}, \frac{8 h_0^2 \gamma}{\mu\pi(1+\rho)}\Big\}.\end{gathered}$$

Then we have

$$\begin{gathered}w_t - d_1 w_{xx} \ge w(a_1 - b_1 w^r + z^p), \quad t > 0,\ 0 < x < \sigma(t),\\ z_t - d_2 z_{xx} \ge z(a_2 + w^q - b_2 z^s), \quad t > 0,\ 0 < x < \sigma(t),\end{gathered}$$

$$
\begin{gathered}
w(t,0) \geq 0, \quad z(t,0) \geq 0, \\
w(t,x) = z(t,x) = 0, \quad \sigma'(t) > -\mu(\frac{\partial w}{\partial x} + \rho \frac{\partial z}{\partial x}), \quad t > 0,\ x = \sigma(t), \\
\sigma(0) = 2h_0 > h_0.
\end{gathered}
$$

By Lemma 4.2, one can show that $h(t) < \sigma(t)$, as long as the solution exists $u(t,x) < w(t,x)$, $v(t,x) < z(t,x)$ for $0 \leq x \leq h(t)$. In particular, it follows from Lemma 4.2 that (u,v,h) exists globally and $\lim_{t\to\infty} h(t) < \infty$. □

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