Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 128, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

PERIODIC ORBITS WITH SMALL ANGULAR MOMENTUM FOR SINGULAR RADIALLY SYMMETRIC SYSTEMS

FANG-FANG LIAO

ABSTRACT. We study radially symmetric systems with a repulsive singularity, and show the existence of periodic orbits with small angular momentum. Our proofs are based on the use of topological degree theory.

1. INTRODUCTION

During the previous years, Fonda and his coworkers have studied the periodic, subharmonic and quasi-periodic orbits for the radially symmetric system

$$\ddot{x} + f(t,|x|)\frac{x}{|x|} = 0, \quad x \in \mathbb{R}^2 \setminus \{0\}, \tag{1.1}$$

in a systematic way, where f is a real function, T-periodic in t and is defined on $\mathbb{R} \times (0,+\infty)$, so that 0 may be a singularity. See [5, 6, 7, 8, 9, 10].

Here we recall one result proved by Fonda and Toader for the system

$$\ddot{x} + c(t)\frac{x}{|x|^{\omega+1}} = e(t)\frac{x}{|x|}, \quad x \in \mathbb{R}^2 \setminus \{0\}, \tag{1.2}$$

where $\omega > 0$ and $c, e \in L^1_{\rm loc}(\mathbb{R})$ are T-periodic.

Corollary 1.1. [7, Corollary 1.1] *Assume that $\omega \geq 1$, $\bar{e} = \frac{1}{T}\int_0^T e(t)dt < 0$ and*

$$c_1 \leq c(t) \leq c_2 < 0, \quad \text{for a.e. } t \in \mathbb{R},$$

for some negative constants c_1 and c_2. Then there exists $k_1 \geq 1$ such that for every integer $k \geq k_1$, equation (1.2) has a periodic solution $x_k(t)$ with minimal period kT, which makes exactly one revolution around the origin in the period time kT. Moreover, there exists a constant $C > 0$ such that, for every $k \geq k_1$,

$$\frac{1}{C} < |x_k(t)| < C, \quad \text{for every } t \in \mathbb{R}, \lim_{k\to\infty} \mu_k = 0,$$

where μ_k denotes the angular momentum associated to x_k.

On the other hand, Chu, Li and Siegmund in the recent paper [3] studied the system

$$\ddot{x} + l^2 x = f(t,|x|)\frac{x}{|x|}, \quad x \in \mathbb{R}^2 \setminus \{0\}, \tag{1.3}$$

2000 *Mathematics Subject Classification.* 34C25.
Key words and phrases. Periodic solutions; singular radially symmetric systems; topological degree theory.

Submitted February 7, 2014. Published June 6, 2014.

where $0 < l < \pi/T$ and the function $f : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is continuous, T-periodic in the time variable t. Based on topological degree theory, they proved the following result.

Theorem 1.2. [3, Main result] *Assume that the following condition is satisfied*

(H) *There exist continuous T-periodic functions $b, \hat{b} \geq 0$ with positive mean and a constant $\alpha > 0$ such that*
$$\frac{b(t)}{r^\alpha} \leq f(t, r) \leq \frac{\hat{b}(t)}{r^\alpha}, \quad \text{for all } t \in \mathbb{R} \text{ and } r > 0.$$

Then system (1.3) *has two distinct families of periodic orbits with the following distinct behavior: one rotates around the origin with small angular momentum, and the other one rotates around the origin with large angular momentum and large amplitude.*

Motivated by those works mentioned above, in this work, we study the system
$$\ddot{x} + l^2 x = (f(t, |x|) + e(t))\frac{x}{|x|}, \quad x \in \mathbb{R}^2 \setminus \{0\}, \tag{1.4}$$
where $0 < l < \pi/T$ is a constant, $e \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ and the function $f : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is continuous, T-periodic in the time variable t with period $T > 0$. Therefore, $f(t, r)$ may be singular at $r = 0$. We look for solutions $x(t) \in \mathbb{R}^2$ which never attain the singularity in the sense that
$$x(t) \neq 0, \quad \text{for every } t \in \mathbb{R}. \tag{1.5}$$
We write the solutions of (1.4) in polar coordinates
$$x(t) = \rho(t)(\cos\varphi(t), \sin\varphi(t)).$$
Then we have the collisionless orbits if $\rho(t) > 0$ for every t. Moreover, equation (1.4) is equivalent to the system
$$\begin{gathered} \ddot{\rho} + l^2\rho = \frac{\mu^2}{\rho^3} + f(t, \rho) + e(t), \\ \rho^2\dot{\varphi} = \mu. \end{gathered} \tag{1.6}$$
In our analysis, we will use such an equivalent system.

Finally we remark that the existence of periodic solutions for singular differential equations has been studied by many researchers. Usually, the proof is based on either a variational approach [12, 16] or topological methods, starting with the pioneering paper of Lazer and Solimini [14]. From then on, the method of upper and lower solutions [15], degree theory [21], fixed point theorems [11, 18, 19] and a nonlinear alternative principle of Leray-Schauder type [2, 13] have been widely applied.

Throughout this paper, for a function $\psi \in C[0, T]$, we use the symbols $\psi_* = \min_t \psi(t)$ and $\psi^* = \max_t \psi(t)$.

2. Preliminaries and main results

Let us define the function $\gamma : \mathbb{R} \to \mathbb{R}$ by
$$\gamma(t) = \int_0^T G(t, s)e(s)ds,$$

which is the unique T-periodic solution of

$$\ddot{x} + l^2 x = e(t), \tag{2.1}$$

where

$$G(t,s) = \begin{cases} \frac{\sin l(t-s)+\sin l(T-t+s)}{2l(1-\cos lT)}, & 0 \le s \le t \le T, \\ \frac{\sin l(s-t)+\sin l(T-s+t)}{2l(1-\cos lT)}, & 0 \le t \le s \le T, \end{cases} \tag{2.2}$$

is the Green function. See [11]. Since we assume that $l \in (0, \pi/T)$, we know that $G(t,s) > 0$ for $0 \le s, t \le T$, and we denote

$$m = \min_{0 \le s,t \le T} G(t,s), \quad M = \max_{0 \le s,t \le T} G(t,s), \quad \sigma = m/M.$$

A direct calculation shows that $\sigma = \cos(lT/2) \in (0,1)$. Note that, since both $x(t) = \int_0^T G(t,s)ds$ and $x(t) \equiv \frac{1}{l^2}$ are T-periodic solution of (2.1) with $e(t) \equiv 1$, by uniqueness, we obtain

$$\int_0^T G(t,s)ds = \frac{1}{l^2}.$$

The above facts have been used in [11]. The main result of this paper reads as follows.

Theorem 2.1. *Let $l \in (0, \pi/T)$. Assume that the following conditions are satisfied:*

- (H1) *for each constant $L > 0$, there exists a continuous nonnegative function ϕ_L with positive mean such that $f(t,r) \ge \phi_L(t)$ for all $t \in [0,T]$ and $r \in (0,L]$.*
- (H2) *there exist continuous, non-negative functions $g(r)$ and $k(t)$ such that*
$$0 \le f(t,r) \le k(t)g(r), \quad \text{for all } (t,r) \in [0,T] \times (0,+\infty),$$
where $g(r) > 0$ is non-increasing.
- (H3) *there exists a positive number $R > 0$ large enough and a positive constant α such that*
$$K^* g(\sigma R + \gamma_*) + \sigma^{-\alpha}\Phi_* < R,$$
where
$$K(t) = \int_0^T G(t,s)k(s)ds, \quad \Phi(t) = \int_0^T G(t,s)\phi_{R+\gamma^*}(s)ds.$$

Then for any function e with $\gamma_* \ge 0$, there exists $K \ge 1$ such that for every integer $k \ge K$, system (1.4) has a periodic solution $x_k(t)$ with minimal period kT, which makes exactly one revolution around the origin in the period time kT. Moreover, $\lim_{k\to\infty} \mu_k = 0$ and there exists a constant $C > 0$ such that

$$\frac{1}{C} < |x_k(t)| < C, \quad \text{for every } t \in \mathbb{R} \text{ and every } k \ge K.$$

As an application of Theorem 2.1, we consider the system

$$\ddot{x} + l^2 x = \frac{c(t)x}{|x|^{\omega+1}} + e(t)\frac{x}{|x|}, \quad x \in \mathbb{R}^2 \setminus \{0\}. \tag{2.3}$$

Corollary 2.2. *Assume that $0 < l < \pi/T$, $\omega > 0$, and c is a continuous non-negative T-periodic function with positive mean. Then for any T-periodic function $e \in L^1_{\rm loc}(\mathbb{R})$ with $\gamma_* \ge 0$, system (2.3) has a family of periodic orbits $\{x_k\}$ with angular momentum $\{\mu_k\}$ satisfying $\lim_{k\to\infty} \mu_k = 0$.*

Proof. Let us take

$$f(t,r)=\frac{c(t)}{r^{\omega}}, r>0.$$

Then (H1) and (H2) hold by taking

$$\phi_L(t)=\frac{c(t)}{L^{\omega}},\quad k(t)=c(t),\quad g(r)=\frac{1}{r^{\omega}}.$$

Let

$$C^*=\sup_{t\in[0,T]}\int_0^T G(t,s)c(s)ds,\quad C_*=\inf_{t\in[0,T]}\int_0^T G(t,s)c(s)ds>0.$$

Note that

$$\begin{aligned}\Phi_*&=\inf_{t\in[0,T]}\int_0^T G(t,s)\phi_{R+\gamma^*}(s)ds\\&=\inf_{t\in[0,T]}\int_0^T G(t,s)\frac{c(s)}{(R+\gamma^*)^{\omega}}ds\\&=\frac{C_*}{(R+\gamma^*)^{\omega}}.\end{aligned}$$

Then condition (H3) becomes

$$\frac{C^*}{(\sigma R+\gamma_*)^{\omega}}+\frac{C_*}{\sigma^{\alpha}(R+\gamma^*)^{\omega}}<R,$$

for some positive constant R, which is obvious since $\omega>0$. Now the proof is finished by applying Theorem 2.1. □

Remark 2.3. We remark that Corollary 1.1 and Corollary 2.2 are different. In fact, Corollary 2.2 is applicable to the case of a strong singularity as well as the case of a weak singularity because we only need $\omega>0$, while we required that $\omega\geq 1$ in Corollary 1.1. Moreover, condition imposed on e in Corollary 1.1 and that in Corollary 2.2 are also different.

3. Proof of Theorem 2.1

We will consider the equivalent system (1.6). If x is a T-radially periodic solution of (1.4), then ρ must be a T-periodic solution of the first equation of (1.6). Thus we consider the boundary value problem

$$\begin{gathered}\ddot\rho+l^2\rho=f(t,\rho)+\frac{\mu^2}{\rho^3}+e(t),\\ \rho(0)=\rho(T),\quad \dot\rho(0)=\dot\rho(T).\end{gathered}\tag{3.1}$$

To prove that (3.1) has a T-periodic solution, we first consider

$$\begin{gathered}\ddot\rho+l^2\rho=f(t,\rho+\gamma)+\frac{\mu^2}{(\rho+\gamma)^3},\\ \rho(0)=\rho(T),\quad \dot\rho(0)=\dot\rho(T),\end{gathered}\tag{3.2}$$

and show that (3.2) has a T-periodic solution ρ satisfying $\rho(t)+\gamma(t)>0$ for $t\in[0,T]$. If this is true, it is easy to see that $\tilde\rho(t)=\rho(t)+\gamma(t)$ will be a positive T-periodic solution of (3.1), since

$$\ddot{\tilde\rho}+l^2\tilde\rho=\ddot\rho+l^2\rho+\ddot\gamma+l^2\gamma$$

$$
\begin{aligned}
&= f(t,\rho+\gamma) + \frac{\mu^2}{(\rho+\gamma)^3} + e(t) \\
&= f(t,\tilde{\rho}) + \frac{\mu^2}{\tilde{\rho}^3} + e(t).
\end{aligned}
$$

We consider first (3.2) for the case $\mu = 0$, which corresponds to the boundary value problem

$$
\begin{gathered}
\ddot{\rho} + l^2\rho = f(t,\rho+\gamma), \\
\rho(0) = \rho(T), \quad \dot{\rho}(0) = \dot{\rho}(T).
\end{gathered} \tag{3.3}
$$

We will prove several preliminary results. To do this, let us define the truncated functions $f_n : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by

$$
f_n(t,\rho) = \begin{cases} f(t,\rho), & \text{if } \rho \ge 1/n, \\ f(t,1/n), & \text{if } \rho \le 1/n, \end{cases}
$$

and consider the family of boundary-value problems

$$
\begin{gathered}
\ddot{\rho} + l^2\rho = f_n(t,\rho+\gamma) + \frac{l^2}{n}, \\
\rho(0) = \rho(T), \quad \dot{\rho}(0) = \dot{\rho}(T).
\end{gathered} \tag{3.4}
$$

To use the topological degree theory, we also consider the homotopy equations

$$
\begin{gathered}
\ddot{\rho} + l^2\rho = f_n^\lambda(t,\rho+\gamma) + \frac{l^2}{n}, \\
\rho(0) = \rho(T), \quad \dot{\rho}(0) = \dot{\rho}(T),
\end{gathered} \tag{3.5}
$$

where $\lambda \in [0,1]$ and

$$
f_n^\lambda(t,\rho) = \lambda f_n(t,\rho) + (1-\lambda)\frac{c}{\rho^\alpha},
$$

where $\alpha > 0$ is fixed and the constant c is chosen as

$$
c = \frac{\Phi_* R^\alpha}{\omega_*}.
$$

Lemma 3.1 ([3, Lemma 3.2])**.** *Assume that ρ is a T-periodic solution of* (3.5)*, then*

$$
\min_{t\in\mathbb{R}} \rho(t) \ge \sigma\|\rho\|,
$$

for every $\lambda \in [0,1]$ and $n \in \mathbb{N}$.

Lemma 3.2. *There exists a constant $C > 0$ such that if $\lambda \in [0,1]$ and ρ is a T-periodic solution of* (3.5)*, then*

$$
\frac{1}{C} < \rho(t) < C, \quad |\dot{\rho}(t)| < C
$$

for every $t \in [0,T]$ and $n \ge n_0$ with some $n_0 \in \mathbb{N}$.

Proof. Since (H3) holds, we can choose $n_0 \in \{1,2,\dots\}$ such that $\frac{1}{n_0} < \sigma R + \gamma_*$ and

$$
K^* g(\sigma R + \gamma_*) + \sigma^{-\alpha}\Phi_* + \frac{1}{n_0} < R.
$$

Let $N_0 = \{n_0, n_0+1, \dots\}$. Fix $n \in N_0$.

We claim that any possible solution $\rho(t)$ of (3.5) must satisfy $\rho(t) < R$, i.e., $\|\rho\| < R$. Otherwise, assume that ρ is a solution of (3.5) such that $\|\rho\| \geq R$. By Lemma 3.1,

$$\rho(t) \geq \sigma\|\rho\| \geq \sigma R.$$

On the other hand, using the fact

$$\rho(t) = \int_0^T G(t,s) f_n^\lambda(s, \rho(s) + \gamma(s)) ds + \frac{1}{n},$$

we have $\rho(t) \geq 1/n$ for all $t \in \mathbb{R}$. Since $\gamma_* \geq 0$, we have

$$f_n^\lambda(t, \rho(t) + \gamma(t)) = f^\lambda(t, \rho(t) + \gamma(t)) = \lambda f(t, \rho) + (1-\lambda)\frac{c}{\rho^\alpha}.$$

Using conditions (H2) and (H3), we have

$$\begin{aligned}
\rho(t) &= \int_0^T G(t,s) f_n^\lambda(s, \rho(s) + \gamma(s)) ds + \frac{1}{n} \\
&= \int_0^T G(t,s)\Big(\lambda f(s, \rho(s) + \gamma(s)) + (1-\lambda)\frac{c}{\rho^\alpha(s)}\Big) ds + \frac{1}{n} \\
&\leq \int_0^T G(t,s)\Big(\lambda k(s) g(\rho(s) + \gamma(s)) + (1-\lambda)\frac{c}{\rho^\alpha(s)}\Big) ds + \frac{1}{n} \\
&\leq \int_0^T G(t,s)\Big(k(s) g(\sigma R + \gamma_*) + \frac{c}{(\sigma R)^\alpha}\Big) ds + \frac{1}{n} \\
&\leq g(\sigma R + \gamma_*) K^* + \frac{c\omega^*}{(\sigma R)^\alpha} + \frac{1}{n_0} \\
&= K^* g(\sigma R + \gamma_*) + \sigma^{-\alpha}\Phi_* + \frac{1}{n_0} < R,
\end{aligned}$$

which is a contradiction, and the claim is proved.

On the other hand, using condition (H1), we have

$$\begin{aligned}
\rho(t) &= \int_0^T G(t,s) f_n^\lambda(t, \rho(s) + \gamma(s)) ds + \frac{1}{n} \\
&\geq \int_0^T G(t,s)\Big(\lambda \phi_{R+\gamma^*}(s) + (1-\lambda)\frac{c}{\rho^\alpha(s)}\Big) ds \\
&\geq \lambda \Phi_* + \frac{(1-\lambda)c\omega_*}{R^\alpha} \\
&\geq \Phi_* =: \delta > 0.
\end{aligned}$$

Next we prove that $|\dot\rho(t)| \leq M$ for some constant $M > 0$. To this end, by the boundary condition $\rho(0) = \rho(T)$, we have $\dot\rho(t_0) = 0$ for some $t_0 \in [0, T]$. Integrating (3.5) from 0 to T, we obtain

$$l^2 \int_0^T \rho(t) dt = \int_0^T \Big(f_n^\lambda(t, \rho(t) + \gamma(t)) + \frac{l^2}{n}\Big) dt.$$

Then

$$\begin{aligned}
\|\dot\rho\| &= \max_{0 \leq t \leq T} |\dot\rho(t)| = \max_{0 \leq t \leq T} \Big|\int_{t_0}^t \ddot\rho(s) ds\Big| \\
&= \max_{0 \leq t \leq T} \Big|\int_{t_0}^t \Big(f_n^\lambda(s, \rho(s) + \gamma(s)) + \frac{l^2}{n} - l^2\rho(s)\Big) ds
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^T \Big(f_n^\lambda(s,\rho(s)+\gamma(s)) + \frac{l^2}{n} + l^2\rho(s)\Big)ds \\
&= 2l^2 \int_0^T \rho(t)dt \\
&< 2l^2 RT = M.
\end{aligned}
$$

The proof is complete when we take $C = \max\{R, M, 1/\delta\}$. □

Now let us define the operators:

$$
\begin{gathered}
L : D(L) \subset C^1[0,T] \to L^1(0,T), \\
D(L) = \{\rho \in W^{2,1}(0,T) : \rho(0) = \rho(T), \dot{\rho(0)} = \dot{\rho(T)}\}, \\
L\rho = \ddot{\rho} + l^2\rho, \\
N_n^\lambda : [0,\infty) \times C^1[0,T] \to L^1(0,T), \\
N_n^\lambda(\mu,\rho)(t) = f_n^\lambda(t,\rho(t)+\gamma(t)) + \frac{\mu^2}{(\rho(t)+\gamma(t))^3} + \frac{l^2}{n}.
\end{gathered}
$$

It is clear that the equation

$$
\ddot{\rho} + l^2\rho = f_n^\lambda(t,\rho(t)+\gamma(t)) + \frac{\mu^2}{(\rho(t)+\gamma(t))^3} + \frac{l^2}{n} \tag{3.6}
$$

is equivalent to the operator equation

$$
L\rho = N_n^\lambda(\mu,\rho).
$$

Since L is invertible, we have

$$
\rho - L^{-1}N_n^\lambda(\mu,\rho) = 0. \tag{3.7}
$$

Define

$$
\Omega = \{\rho \in C^1[0,T] : \frac{1}{C} < \rho(t) < C \text{ and } |\dot{\rho}(t)| < C \text{ for every } t \in [0,T]\big\},
$$

where C is the constant given by Lemma 3.2. Obviously, Ω is an open subset of $C^1[0,T]$. By Lemma 3.2,

$$
\rho - L^{-1}N_n^\lambda(0,\rho) = 0
$$

has no solutions ρ on $\partial\Omega$.

Lemma 3.3. *There exist a constant $M > 0$ such that for each $\mu \in [0,M]$, there exists no solution of* (3.7) *on the boundary of* Ω.

Proof. On the contrary, assume that there exist sequences $\{\mu_m\}$ and $\{\rho_m\}$ such that

$$
\mu_m \to 0 \quad (m \to +\infty), \quad \rho_m \in \partial\Omega, \quad \rho_m = L^{-1}N_n^\lambda(\mu_m,\rho_m).
$$

Since $\{\rho_m\}$ is uniformly bounded, $\{N_n^\lambda(\mu_m,\rho_m)\}$ is bounded in $L^1(0,T)$. Since

$$
L^{-1} : L^1(0,T) \to C^1[0,T]
$$

is a compact operator, there exists a subsequence $\{\rho_{m_i}\}$ of $\{\rho_m\}$ such that

$$
L^{-1}N_n^\lambda(\mu_{m_i},\rho_{m_i}) \to \bar{\rho}, \quad i \to +\infty,
$$

for some $\bar{\rho}$. Hence

$$
\rho_{m_i} \to \bar{\rho}, \quad i \to +\infty.
$$

Since $\partial\Omega$ is closed, we have $\bar\rho \in \partial\Omega$. Moreover,

$$\bar\rho = L^{-1}N_n^{\lambda}(0,\bar\rho),$$

which implies that $\bar\rho$ solves (3.5). It is a contradiction since (3.5) has no solution on $\partial\Omega$. □

Lemma 3.4. *For $\mu \in [0, M]$ and $n \in N_0$, there exists a positive T-periodic solution of equation*

$$\ddot\rho + l^2\rho = f_n(t,\rho+\gamma) + \frac{\mu^2}{(\rho+\gamma)^3} + \frac{l^2}{n}. \tag{3.8}$$

Proof. We need to compute the degree for (3.8). By Lemma 3.3, the degree is the same for equations

$$\ddot\rho + l^2\rho = f_n^{\lambda}(t,\rho+\gamma) + \frac{\mu^2}{(\rho+\gamma)^3} + \frac{l^2}{n}, \tag{3.9}$$

for every $\lambda \in [0,1]$. Therefore, we consider the equation (3.9) with $\lambda = 0$, that is the equation

$$\ddot\rho + l^2\rho = \frac{c}{\rho^{\alpha}} + \frac{\mu^2}{(\rho+\gamma)^3} + \frac{l^2}{n},$$

which is equivalent to the system

$$\dot Y = F_n(Y),$$

with $Y = (\rho, u)$ and

$$F_n(Y) = \Big(u, -l^2\rho + \frac{\mu^2}{(\rho+\gamma)^3} + \frac{c}{\rho^{\alpha}} + \frac{l^2}{n}\Big).$$

It is easy to know that F_n has a unique zero (ρ_0, u_0) and the determinant of the Jacobian matrix satisfies $|J_{F_n}(\rho_0,u_0)| > 0$. By [1, Theorem 1], the Leray-Schauder degree of $I - L^{-1}N_n^{\lambda}(\mu,\cdot)$ is equal to the Brouwer degree of F_n, i.e.,

$$d_L(I - L^{-1}N_n^{\lambda}(\mu,\cdot,),\Omega,0) = d_B(F_n,(\frac{1}{C},C)\times(-C,C)) = 1,$$

which implies that (3.8) has a T-periodic solution ρ_n for any fixed $n \in \mathbb{N}$. □

Up to now, for each $n \in N_0$, we have proved that the system

$$\begin{gathered}\ddot\rho + l^2\rho = f_n(t,\rho+\gamma) + \frac{\mu^2}{(\rho+\gamma)^3} + \frac{l^2}{n}, \\ (\rho+\gamma)^2\dot\varphi = \mu,\end{gathered} \tag{3.10}$$

has a solution (μ_n, ρ_n). By Lemma 3.2, $\{\rho_n\}$ is a bounded and equi-continuous family on $[0,T]$. The Arzela-Ascoli Theorem guarantees that $\{\rho_n\}$ has a subsequence $\{\rho_{n_i}\}_{i\in\mathbb{N}}$, converging uniformly on $[0,T]$ to a function $\rho \in C[0,T]$. Without loss of generality, we can assume that $\mu_{n_i} \to \mu$ as $i \to \infty$. Moreover ρ_{n_i} satisfies the integral equation

$$\rho_{n_i}(t) = \int_0^T G(t,s)f(s,\rho_{n_i}(s)+\gamma(s))ds + \frac{\mu_{n_i}^2}{(\rho_{n_i}(t)+\gamma(t))^3} + \frac{1}{n_i}.$$

Let $i \to \infty$, we arrive at

$$\rho(t) = \int_0^T G(t,s)f(s,\rho(s)+\gamma(s))ds + \frac{\mu^2}{(\rho(t)+\gamma(t))^3}.$$

Therefore, we have constructed the solution (ρ, μ) of the system

$$\begin{gathered}\ddot{\rho} + l^2\rho = f(t, \rho+\gamma) + \frac{\mu^2}{(\rho+\gamma)^3}, \\ (\rho+\gamma)^2\dot{\varphi} = \mu.\end{gathered} \tag{3.11}$$

By the global continuation principle of Leray-Schauder [20, Theorem 14. C], there is a connected set $\mathcal{C}$, contained in $[0, M] \times \Omega$, which connects $\{0\} \times \Omega$ with $\{M\} \times \Omega$, whose elements (μ, ρ) are solutions of system (3.11). As a result, we have also obtained a connected set $\tilde{\mathcal{C}}$, contained in $[0, M] \times \tilde{\Omega}$, which connects $\{0\} \times \tilde{\Omega}$ with $\{M\} \times \tilde{\Omega}$, whose elements $(\mu, \tilde{\rho})$ are solutions of system (1.6), where $\tilde{\rho} = \rho + \gamma$ and $\tilde{\Omega}$ is deduced from Ω with the constant C being changed as another constant $\tilde{C}$.

Let us define the function $\Phi : \tilde{\mathcal{C}} \to \mathbb{R}$ by

$$\Phi(\mu, \tilde{\rho}) \mapsto \int_0^T \frac{\mu}{\tilde{\rho}^2(t)} dt.$$

It is continuous and defined on a compact and connected domain, so its image is a compact interval. Since $\Phi(0, \tilde{\rho}) = 0$ and Φ is not identically zero, this interval is of the type $[0, \bar{\theta}]$ for some $\bar{\theta} > 0$.

Lemma 3.5. *For every $\theta \in (0, \bar{\theta}]$, there exist $(\mu, \tilde{\rho}, \varphi)$ satisfying system (1.6), for which $(\mu, \tilde{\rho}) \in \mathcal{C}$ and*

$$\tilde{\rho}(t+T) = \tilde{\rho}(t), \quad \varphi(t+T) = \varphi(t) + \theta,$$

for every $t \in \mathbb{R}$.

Proof. Given $\theta \in (0, \bar{\theta}]$, there exist $(\mu, \tilde{\rho}) \in \tilde{\mathcal{C}}$, such that

$$\Phi(\mu, \tilde{\rho}) = \theta.$$

Obviously, the first equation in (1.6) is satisfied and $\tilde{\rho}$ is T-perodic. Define

$$\varphi(t) = \int_0^t \frac{\mu}{\tilde{\rho}^2(s)} ds,$$

which satisfies the second equation in (1.6). Moreover,

$$\varphi(t+T) - \varphi(t) = \int_t^{t+T} \frac{\mu}{\tilde{\rho}^2(s)} ds = \int_0^T \frac{\mu}{\tilde{\rho}^2(s)} ds = \theta.$$

□

Now we are in a position to complete the proof of Theorem 2.1. For every $\theta \in (0, \bar{\theta}]$, the solution of system (1.6) found in Lemma 3.5 provides a solution of (1.4) such that

$$x(t+T) = x(t)e^{i\theta}, \quad \text{for every } t \in \mathbb{R}.$$

In particular, if $\theta = 2\pi/k$ for some integer $k \geq 1$, then $x(t)$ is periodic with minimal period kT and rotates exactly once around the origin in the period time kT since $\varphi(t + kT) = \varphi(t) + 2\pi$. Hence, for every integer $k \geq 2\pi/\bar{\theta}$, we have such a kT-periodic solution, which we denote by $x_k(t)$. Let $(\tilde{\rho}_k(t), \varphi_k(t))$ be its polar coordinates, and μ_k be its angular momentum. By the above construction, $(\mu_k, \tilde{\rho}_k, \varphi_k)$ satisfy system (1.6), $(\mu_k, \tilde{\rho}_k) \in \mathcal{C}$, and

$$\int_0^T \frac{\mu_k}{\tilde{\rho}_k^2(t)} dt = \frac{2\pi}{k}. \tag{3.12}$$

Since $\rho_k \in \Omega$, we have

$$\frac{2\pi}{k} = \int_0^T \frac{\mu_k}{\tilde{\rho}_k^2(t)} dt > \frac{\mu_k T}{C^2},$$

which implies that $\lim_{k\to\infty} \mu_k = 0$. The proof is thus finished.

Acknowledgments. The author would like to show her appreciation to the anonymous referee for the valuable suggestions and comments, which improve the original version of this article.

References

[1] A. Capietto, J. Mawhin, F. Zanolin; *Continuation theorems for periodic perturbations of autonomous systems*, Trans. Amer. Math. Soc. 329 (1992), 41-72.

[2] J. Chu, P. J. Torres, M. Zhang; *Periodic solutions of second order non-autonomous singular dynamical systems*, J. Differential Equations, 239 (2007), 196-212.

[3] J. Chu, S. Li, S. Siegmund; *Periodic orbits of singular radially symmetric perturbations of Hill's equations*, preprint.

[4] M. A. del Pino, R. F. Manásevich; *Infinitely many T-periodic solutions for a problem arising in nonlinear elasticity*, J. Differential Equations, 103 (1993), 260-277.

[5] A. Fonda, R. Toader; *Periodic orbits of radially symmetric Keplerian-like systems: A topological degree approach*, J. Differential Equations, 244 (2008), 3235-3264.

[6] A. Fonda, R. Toader; *Radially symmetric systems with a singularity and asymptotically linear growth*, Nonlinear Anal. 74 (2011), 2485-2496.

[7] A. Fonda, R. Toader; *Periodic orbits of radially symmetric systems with a singularity: the repulsive case*, Adv. Nonlinear Stud. 11 (2011), 853-874.

[8] A. Fonda, R. Toader; *Periodic solutions of radially symmetric perturbations of Newtonian systems*, Proc. Amer. Math. Soc. 140 (2012), 1331-1341.

[9] A. Fonda, R. Toader, F. Zanolin; *Periodic solutions of singular radially symmetric systems with superlinear growth*, Ann. Mat. Pura Appl. 191 (2012), 181-204.

[10] A. Fonda, A. J. Ureña; *Periodic, subharmonic, and quasi-periodic oscillations under the action of a central force*, Discrete Contin. Dyn. Syst. 29 (2011), 169-192.

[11] D. Franco, J. R. L. Webb; *Collisionless orbits of singular and nonsingular dynamical systems*, Discrete Contin. Dyn. Syst. 15 (2006), 747-757.

[12] P. Habets, L. Sanchez; *Periodic solution of some Liénard equations with singularities*, Proc. Amer. Math. Soc. 109 (1990), 1135-1144.

[13] D. Jiang, J. Chu, M. Zhang; *Multiplicity of positive periodic solutions to superlinear repulsive singular equations*, J. Differential Equations, 211 (2005), 282-302.

[14] A. C. Lazer, S. Solimini; *On periodic solutions of nonlinear differential equations with singularities*, Proc. Amer. Math. Soc. 99 (1987), 109-114.

[15] I. Rachunková, M. Tvrdý, I. Vrkoč; *Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems*, J. Differential Equations, 176 (2001), 445-469.

[16] M. Ramos, S. Terracini; *Noncollision periodic solutions to some singular dynamical systems with very weak forces*, J. Differential Equations, 118 (1995), 121-152.

[17] J. Ren, Z. Cheng, S. Siegmund; *Positive periodic solution for Brillouin electron beam focusing system*, Discrete Contin. Dyn. Syst. Ser. B. 16(2011), 385-392.

[18] P. J. Torres; *Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem*, J. Differential Equations, 190 (2003), 643-662.

[19] P. J. Torres; *Non-collision periodic solutions of forced dynamical systems with weak singularities*, Discrete Contin. Dyn. Syst. 11 (2004), 693-698.

[20] E. Zeidler; *Nonlinear functional analysis and its applications*, vol. 1, Springer, New York, Heidelberg, 1986.

[21] M. Zhang; *Periodic solutions of equations of Ermakov-Pinney type*, Adv. Nonlinear Stud. 6 (2006), 57-67.

Fang-Fang Liao
Department of Mathematics, College of Science, Hohai University, Nanjing 210098, China
E-mail address: liaofangfang8178@sina.com, hohai_deds@126.com