

## MINIMIZATION OF ENERGY INTEGRALS ASSOCIATED WITH THE $p$ -LAPLACIAN IN $\mathbb{R}^N$ FOR REARRANGEMENTS

SHANMING JI, JINGXUE YIN, RUI HUANG

ABSTRACT. In this article, we establish the existence of minimizers for energy integrals associated with the  $p$ -Laplacian in  $\mathbb{R}^N$  with the admissible set being a rearrangement class of a given function. Some representation formulae of the minimizers are also stated.

### 1. INTRODUCTION

In this article, we study the optimization problems of minimizing the energy integrals associated with the  $p$ -Laplacian equation

$$-\Delta_p u = f - h, \quad x \in \mathbb{R}^N, \quad (1.1)$$

$$\lim_{|x| \rightarrow +\infty} u(x) = 0. \quad (1.2)$$

Here  $1 < p < N$ ,  $\Delta_p$  stands for the usual  $p$ -Laplacian; i.e.,  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ . Let  $f_0, h \in L^\infty(\mathbb{R}^N)$  be fixed nonnegative functions with compact supports, and let  $\mathcal{R}$  be the class of rearrangements of  $f_0$  with compact support; that is,  $\mathcal{R} = \{f \in L^\infty(\mathbb{R}^N); |\{x; f(x) \geq \alpha\}| = |\{x; f_0(x) \geq \alpha\}|, \forall \alpha \in \mathbb{R}, \operatorname{supp} f \text{ is bounded}\}$ , where  $|\cdot|$  is the Lebesgue measure. For  $\lambda \geq 0$  and  $f$  varying in  $\mathcal{R}$ , we define the energy functional with ( $\lambda > 0$ ) or without ( $\lambda = 0$ ) penalty as

$$\Psi_\lambda(f) = \int_{\mathbb{R}^N} |\nabla u_f|^p dx + \lambda \int_{\mathbb{R}^N} g f dx, \quad (1.3)$$

where  $u_f$  is the solution of problem (1.1)–(1.2),  $g \in C^2(\mathbb{R}^N)$  is the penalty function,  $\lim_{|x| \rightarrow +\infty} g(x) = +\infty$  and  $\Delta_p g \geq \sigma^{p-1}$  for some constant  $\sigma > 0$ . The optimization problem (1.4) is to find the minimizer for energy integral  $\Psi_\lambda(f)$ , namely,

$$\min_{f \in \mathcal{R}} \Psi_\lambda(f). \quad (1.4)$$

The optimization problems of maximizing or minimizing convex functionals over the set of rearrangements of a given function have been investigated by many authors. In such problems, the theory of rearrangements and functionals on rearrangements established by Burton [2, 3] has proved to be a crucial tool in addressing questions such as existence, uniqueness and symmetry of optimal solutions. This

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theory has already been applied to shape optimization problems of membranes, solid and fluid mechanics, eigenvalue optimization problems of some differential operators and so on, see [7] and references therein.

In recent years, a great deal of attention has been devoted to optimization problems where the cost functionals are the energy integrals associated with elliptic equations. For problems in bounded domains, numerous elliptic operators have been studied, including the Laplacian [2, 5, 6],  $p$ -Laplacian [4, 10] and some semi-linear operators [8]. For example, Marras [10] studied the minimization problem of energy integral  $\Psi_0(f)$  associated with the  $p$ -Laplacian on bounded domain

$$\begin{aligned} -\Delta_p u &= f, & x \in \Omega, \\ u &= 0, & x \in \partial\Omega, \end{aligned}$$

where  $p > 1$ ,  $f \in \mathcal{R}$ . There are also some works dealing with elliptic operators in unbounded domains. Bahrami and Fazli [1] considered the minimization problem of energy integral

$$\Phi_\lambda(f) = \frac{1}{2} \int_{\mathbb{R}^3} f u_f dx + \lambda \int_{\mathbb{R}^3} g f dx,$$

where  $u_f$  is the solution of Poisson's equation

$$\begin{aligned} -\Delta u &= f - 2h, & x \in \mathbb{R}^3, \\ \lim_{|x| \rightarrow +\infty} u(x) &= 0, \end{aligned} \tag{1.5}$$

where  $f \in \mathcal{R}$ ,  $h \in L^\infty(\mathbb{R}^3)$ ,  $g \in C^2(\mathbb{R}^3)$ ,  $\lim_{|x| \rightarrow +\infty} g(x) = +\infty$ ,  $\Delta g \geq c > 0$  and  $\lambda \geq 0$ .

We mention here some details of the previous works. The weakly sequentially continuity of the functional  $\Psi_\lambda(f)$  on space  $L^q(\Omega)$  for  $q \geq 1$  and bounded domain  $\Omega$ , is essential in the proof of [10] and other works when applying Burton's theory of rearrangements. However, the continuity is generally not true on unbounded domains due to the general loss of compact imbedding of Sobolev spaces on unbounded domains, especially on the whole space. Thus the authors in [1] investigate the problem on bounded domains to solve the optimization problem on unbounded domains.

We are interested in the extension of the work of Bahrami and Fazli [1] to the nonlinear diffusion case. As a matter of fact, the  $p$ -Laplacian arises in various physical contexts: non-Newtonian fluids, reaction diffusion problems, nonlinear elasticity, electrochemical machining, elastic-plastic torsional creep, etc., see [4] and references therein.

We state here our main results of existence and representation formulae of minimizers for problem (1.4) in the case  $\lambda > 0$  and  $\lambda = 0$  respectively.

**Theorem 1.1.** *The optimization problem (1.4) has a solution for*

$$\lambda > \lambda_0 \equiv \frac{p'}{\sigma} \|f_0\|_\infty^{\frac{1}{p-1}}.$$

Moreover, if  $f_\lambda \in \mathcal{R}$  is a solution of (1.4) and  $u_{f_\lambda}$  is the solution of problem (1.1)–(1.2) corresponding to  $f_\lambda$ , then there exists a decreasing function  $\varphi_\lambda$  such that

$$f_\lambda = \varphi_\lambda \circ (p' u_{f_\lambda} + \lambda g),$$

almost everywhere in  $\mathbb{R}^N$ .

**Theorem 1.2.** *Let  $f_0$  and  $h$  be as introduced above. There exists a constant  $\kappa = \kappa(N, p) \in (0, \frac{1}{2}]$  depending only on  $N$  and  $p$ , such that if  $\|f_0\|_\infty < \|h\|_{-\infty; \text{supp } h}$ ,  $\text{supp } h \subset B_{r_h}$  with  $r_h > 0$  appropriately large, and*

$$|\text{supp } f_0| \leq \kappa \left( \frac{\|h\|_{-\infty; \text{supp } h}}{\|h\|_\infty} \right)^{\frac{p}{p-1}} \left( \frac{|\text{supp } h|}{|B_{r_h}|} \right)^{\frac{N-p}{N(p-1)}} |\text{supp } h|, \quad (1.6)$$

*then the optimization problem (1.4) with  $\lambda = 0$  has a solution. Moreover, if  $\hat{f} \in \mathcal{R}$  is a solution of (1.4) with  $\lambda = 0$  and  $u_{\hat{f}}$  is the solution of problem (1.1)–(1.2) corresponding to  $\hat{f}$ , then there exists a decreasing function  $\hat{\varphi}$  such that*

$$\hat{f} = \hat{\varphi} \circ u_{\hat{f}},$$

*almost everywhere in  $\mathbb{R}^N$ .*

The crucial point of the proofs, compared with the linear diffusion case, lies in the estimates on the different contributions of the two opposed-signed functions  $f$  and  $-h$  to the solution. In the previous work [1], the classical theory of linear elliptic equations was applied, namely, the explicit expression of solutions based on the superposition principle is feasible and effective in deriving the estimates on solutions of linear elliptic equations. However, such a method is inapplicable in the current problem due to the nonlinearity of the  $p$ -Laplacian. It turns out to be more difficult as we attempt to estimate the different contributions of the two opposed-signed functions. To overcome those difficulties, we use the De Giorgi and Moser iteration techniques to derive estimates in quasilinear case and we take advantage of the representation formulas of the problem on bounded domains since they provide additional correlation between the solution and the free term.

The organization of this paper is as follows. Section 2 is devoted to the basic notations and some preliminary results, especially some fundamental estimates. Then we will discuss the case with ( $\lambda > 0$ ) and without ( $\lambda = 0$ ) penalty in Section 3 and Section 4 respectively.

## 2. DEFINITIONS AND PRELIMINARY RESULTS

Throughout this paper, we assume that  $1 < p < N$ , where  $N$  is the spatial dimension,  $p' = \frac{p}{p-1}$  the conjugate exponent of  $p$ ,  $p_* = \frac{Np}{N-p}$  the Sobolev conjugate exponent of  $p$  and  $p'_* = \frac{p_*}{p_*-1}$ . The measure of a Lebesgue measurable set  $A \subset \mathbb{R}^N$  is denoted by  $|A|$ . By  $B_r(x)$  we denote the ball centered at  $x \in \mathbb{R}^N$  with radius  $r$ ; if the center is the origin, we write  $B_r$  for simplicity. Constant  $\omega_N$  stands for the measure of the unit ball in  $\mathbb{R}^N$ .

Let us begin with the usual concept of rearrangement. Denote by  $L_\alpha(f)$  the level set of a measurable function  $f$  at height  $\alpha$ ; that is  $L_\alpha(f) = \{x \in \mathbb{R}^N; f(x) = \alpha\}$ . The strong support of a nonnegative function  $f$  is defined as  $\text{supp } f = \{x \in \mathbb{R}^N; f(x) > 0\}$ . Furthermore, we define

$$\|f\|_{-\infty; \text{supp } f} = \sup\{M \geq 0; f(x) \geq M, \text{ a.e. in } \text{supp } f\}.$$

When  $f$  and  $g$  are nonnegative measurable functions that vanish outside sets of finite measure in  $\mathbb{R}^N$ , we say  $f$  is a rearrangement of  $g$  whenever

$$|\{x \in \mathbb{R}^N; f(x) \geq \alpha\}| = |\{x \in \mathbb{R}^N; g(x) \geq \alpha\}|, \quad \forall \alpha \geq 0.$$

Now fix  $f_0 \in L^\infty(\mathbb{R}^N)$  being a measurable nonnegative function vanishing outside a set of finite measure. As defined in Section 1,  $\mathcal{R}$  denotes the set of all rearrangements on  $\mathbb{R}^N$  of  $f_0$  with bounded support. The subset of  $\mathcal{R}$  containing functions vanishing outside the ball  $B_r$  is denoted by  $\mathcal{R}(r)$ ; here we assume  $\omega_N r^N \geq |\text{supp } f_0|$  in order that  $\mathcal{R}(r)$  is nonempty. The weak closure in  $L^{p'}(B_r)$  of  $\mathcal{R}(r)$  is denoted by  $\overline{\mathcal{R}(r)^w}$ .

Henceforth we may regard a function  $f \in L^q(\mathbb{R}^N)$  as a function in  $L^q(B_r)$  by restricting its domain; we can also regard a function  $f \in L^q(B_r)$  as its zero extension in  $L^q(\mathbb{R}^N)$  when necessary for  $1 \leq q \leq +\infty$ .

To solve the optimization problems (1.4), we first need to consider the similar problems whose admissible sets are nested subsets of  $\mathcal{R}$ . We define minimizing problems (2.1) as follows:

$$\min_{f \in \mathcal{R}(r)} \Psi_\lambda(f). \quad (2.1)$$

The sets of solutions of (1.4) and (2.1) are denoted by  $S_\lambda$  and  $S_\lambda(r)$  respectively.

In the following part of this section we state and prove some lemmas which are essential in our analysis. We begin with some results about properties of rearrangement classes and the relative variational problems proved by Burton in [3].

**Lemma 2.1.** *For  $r > 0$  satisfying  $\omega_N r^N \geq |\text{supp } f_0|$  and  $q \geq 1$ , we have*

- (i)  $\|f\|_q = \|f_0\|_q$ , for  $f \in \mathcal{R}(r)$ ;
- (ii)  $\overline{\mathcal{R}(r)^w}$  is weakly sequentially compact in  $L^q(B_r)$ ;
- (iii)  $\overline{\mathcal{R}(r)^w} = \{f \in L^1(B_r); \int_0^s f^\Delta dt \leq \int_0^s f_0^\Delta dt, 0 < s \leq \omega_N r^N, \int_{B_r} f dx = \int_{B_r} f_0 dx\}$ , where  $f^\Delta$  is a decreasing function on the interval  $(0, \omega_N r^N)$  satisfying
 
$$|\{s \in (0, \omega_N r^N); f^\Delta(s) \geq \alpha\}| = |\{x \in B_r; f(x) \geq \alpha\}|, \quad \forall \alpha > 0.$$

**Remark** From the representation of  $\overline{\mathcal{R}(r)^w}$  in (iii), we find that the weak closure of  $\mathcal{R}(r)$  in  $L^{p'}(B_r)$  is actually the weak closure in  $L^q(B_r)$  for  $1 \leq q < +\infty$ . Combining (i) and the property of weak closure we have

$$\|f\|_q \leq \|f_0\|_q, \quad \forall f \in \overline{\mathcal{R}(r)^w}, \quad 1 \leq q \leq +\infty. \quad (2.2)$$

The following two lemmas are simple variations of [3, Lemma 2.15 and Theorem 3.3].

**Lemma 2.2** ([3, Lemma 2.15]). *Let  $T : L^{p'}(B_r) \rightarrow \mathbb{R}$  be the linear functional defined as  $T(f) = \int_{B_r} f v dx$  for  $r > 0$ ,  $\omega_N r^N \geq |\text{supp } f_0|$  and  $v \in L^p(B_r)$ . If  $\hat{f}$  is a minimizer of  $T$  relative to  $\overline{\mathcal{R}(r)^w}$  and*

$$|L_\alpha(v) \cap \text{supp } \hat{f}| = 0, \quad \forall \alpha \in \mathbb{R},$$

*we have  $\hat{f} \in \mathcal{R}(r)$  and  $\hat{f} = \varphi \circ v$  a.e. in  $B_r$ , for a decreasing function  $\varphi$ .*

**Lemma 2.3** ([3, Theorem 3.3]). *Let  $\Psi : L^{p'}(B_r) \rightarrow \mathbb{R}$  be a weakly sequentially continuous and Gâteaux differentiable functional.*

- (i) *There exists a minimizer for  $\Psi$  relative to  $\overline{\mathcal{R}(r)^w}$ .*
- (ii) *If  $f^*$  is a minimizer for  $\Psi$  relative to  $\overline{\mathcal{R}(r)^w}$  and the Gâteaux differential of  $\Psi$  at  $f^*$  is  $\Psi'(f^*) \in L^p(B_r)$ ,  $f^*$  is a minimizer for the functional  $\langle \cdot, \Psi'(f^*) \rangle$  relative to  $\overline{\mathcal{R}(r)^w}$ .*

The following Sobolev’s inequality plays an important role in our estimates. For more details, see [9].

**Lemma 2.4** ([9, Theorem 7.10]). *For  $1 < p < N$  and  $p_* = \frac{Np}{N-p}$ , we have*

$$\|u\|_{p_*} \leq C_0 \|\nabla u\|_p, \quad \forall u \in W^{1,p}(\mathbb{R}^N), \tag{2.3}$$

where  $C_0 = C_0(N, p)$  is a constant depending only on  $N$  and  $p$ .

**Remark** Invoking usual approximations, we see that this estimate is also valid provided  $u \in L^{p_*}(\mathbb{R}^N)$ ,  $\nabla u \in L^p(\mathbb{R}^N; \mathbb{R}^N)$ .

Henceforth, we assume  $r > 0$ ,  $\omega_N r^N \geq |\text{supp } f_0|$  and  $\lambda \geq 0$ . For  $f \in L^{p'}(\mathbb{R}^N)$ , we consider the problem (1.1)–(1.2). It is a classical result of variational theory that such a problem has a unique solution  $u \in W \equiv \{w \in W_{\text{loc}}^{1,p}(\mathbb{R}^N); w \in L^{p_*}(\mathbb{R}^N), \nabla w \in L^p(\mathbb{R}^N; \mathbb{R}^N)\}$  satisfying

$$\begin{aligned} \sup_{v \in W} \int_{\mathbb{R}^N} (p(f-h)v - |\nabla v|^p) dx &= \int_{\mathbb{R}^N} (p(f-h)u - |\nabla u|^p) dx \\ &= (p-1) \int_{\mathbb{R}^N} |\nabla u|^p dx, \end{aligned} \tag{2.4}$$

$$\int_{\mathbb{R}^N} (f-h)v dx = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx, \quad \forall v \in W. \tag{2.5}$$

**Lemma 2.5.** *Let  $u$  be the solution of problem (1.1)–(1.2) corresponding to  $f \in L^{p'}(\mathbb{R}^N)$ . We have*

$$\|\nabla u\|_p \leq C_0^{\frac{1}{p-1}} (\|f_0\|_{p'_*} + \|h\|_{p'_*})^{\frac{1}{p-1}}, \tag{2.6}$$

$$\|u\|_{p_*} \leq C_0^{\frac{p}{p-1}} (\|f_0\|_{p'_*} + \|h\|_{p'_*})^{\frac{1}{p-1}}, \tag{2.7}$$

where  $C_0$  is the constant in (2.3).

*Proof.* From (2.5), we apply Hölder’s inequality to find

$$\int_{\mathbb{R}^N} |\nabla u|^p dx = \int_{\mathbb{R}^N} (f-h)u dx \leq \|f-h\|_{p'_*} \|u\|_{p_*}.$$

Combining this inequality with Sobolev’s inequality (2.3), we get the results.  $\square$

**Lemma 2.6.** *The functional  $\Psi_\lambda$  defined in (1.3) is weakly sequentially continuous and Gâteaux differentiable in  $L^{p'}(B_r)$  with derivative  $p'u_f + \lambda g \in L^p(B_r)$  at  $f \in L^{p'}(B_r)$ , where  $u_f$  is the solution of problem (1.1)–(1.2) corresponding to  $f$ .*

*Proof.* It suffices to prove that the functional  $I(f) \equiv \int_{\mathbb{R}^N} |\nabla u_f|^p dx$  is weakly sequentially continuous and Gâteaux differentiable in  $L^{p'}(B_r)$  with derivative  $p'u_f \in L^p(B_r)$  at  $f \in L^{p'}(B_r)$ . Let  $f_n \rightharpoonup f$  in  $L^{p'}(B_r)$  and  $u_{f_n}, u_f$  be the solutions of the problem (1.1)–(1.2) corresponding to  $f_n, f$  respectively. Using (2.4), we have

$$\begin{aligned} &(p-1)I(f) + \int_{\mathbb{R}^N} p(f_n - f)u_f dx \\ &= \int_{\mathbb{R}^N} (p(f_n - h)u_f - |\nabla u_f|^p) dx \leq (p-1)I(f_n) \\ &= \int_{\mathbb{R}^N} (p(f-h)u_{f_n} - |\nabla u_{f_n}|^p) dx + \int_{\mathbb{R}^N} p(f_n - f)u_{f_n} dx \\ &\leq (p-1)I(f) + \int_{\mathbb{R}^N} p(f_n - f)u_{f_n} dx. \end{aligned} \tag{2.8}$$

By assumption, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (f_n - f)u_f dx = \lim_{n \rightarrow \infty} \int_{B_r} (f_n - f)u_f dx = 0. \quad (2.9)$$

Let us prove that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (f_n - f)u_{f_n} dx = \lim_{n \rightarrow \infty} \int_{B_r} (f_n - f)u_{f_n} dx = 0. \quad (2.10)$$

From (2.6), (2.7) and  $\|f\|_{p'_*, \mathbb{R}^N} = \|f\|_{p'_*, B_r} \leq |B_r|^{\frac{1}{N}} \|f\|_{p', B_r}$  for  $f \in L^{p'}(B_r)$ , we see that the norms  $\|\nabla u_{f_n}\|_{p; \mathbb{R}^N}$ ,  $\|u_{f_n}\|_{p^*, \mathbb{R}^N}$  and  $\|u_{f_n}\|_{1, p; B_r}$  are bounded by constants independent of  $n$ . Therefore, we can choose a subsequence of  $\{u_{f_n}\}$  denoted by  $\{u_{f_{n_k}}\}$  and a function  $w \in W$ , such that  $\{u_{f_{n_k}}\}$  converges weakly in  $L^{p^*}(\mathbb{R}^N)$  and strongly in  $L^p(B_r)$  to  $w$ ,  $\{\nabla u_{f_{n_k}}\}$  converges weakly in  $L^p(\mathbb{R}^N; \mathbb{R}^N)$  to  $\nabla w$ . From

$$\int_{\mathbb{R}^N} (f_{n_k} - f)u_{f_{n_k}} dx = \int_{\mathbb{R}^N} (f_{n_k} - f)w dx + \int_{\mathbb{R}^N} (f_{n_k} - f)(u_{f_{n_k}} - w) dx,$$

and

$$\left| \int_{\mathbb{R}^N} (f_{n_k} - f)(u_{f_{n_k}} - w) dx \right| \leq \|f_{n_k} - f\|_{p'; B_r} \|u_{f_{n_k}} - w\|_{p; B_r},$$

the limit (2.10) is valid for a subsequence  $\{n_k\}$ . Combining this with (2.8)–(2.9), we deduce

$$\lim_{k \rightarrow \infty} I(f_{n_k}) = I(f). \quad (2.11)$$

We claim that the function  $w$  is actually  $u_f$ , which is a fixed function independent of the choice of subsequence  $\{n_k\}$ , to show that the sequence  $\{u_{f_n}\}$  itself converges and equality (2.10) is valid. Indeed, from

$$\begin{aligned} (p-1)I(f_{n_k}) &= \int_{\mathbb{R}^N} (p(f_{n_k} - h)u_{f_{n_k}} - |\nabla u_{f_{n_k}}|^p) dx, \\ \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} (f_{n_k} - h)u_{f_{n_k}} dx &= \int_{\mathbb{R}^N} (f - h)w dx, \end{aligned}$$

and the classical result

$$\liminf_{k \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_{f_{n_k}}|^p dx \geq \int_{\mathbb{R}^N} |\nabla w|^p dx, \quad (2.12)$$

using (2.11) and (2.4), we get

$$(p-1)I(f) \leq \int_{\mathbb{R}^N} (p(f-h)w - |\nabla w|^p) dx \leq (p-1)I(f). \quad (2.13)$$

By the uniqueness of the maximizer of  $\int_{\mathbb{R}^N} (p(f-h)v - |\nabla v|^p) dx$  in  $W$ , we have  $w = u_f$ . Thus (2.8)–(2.10) yield the weak continuity.

Let  $z \in L^{p'}(B_r)$  and  $\{t_n\}$  be a positive sequence such that  $\lim_{n \rightarrow \infty} t_n = 0$ . Taking  $f_n = f + t_n z$  in the inequality (2.8), we find

$$\int_{\mathbb{R}^N} p' u_f z dx \leq \frac{I(f + t_n z) - I(f)}{t_n} \leq \int_{\mathbb{R}^N} p' u_{f_n} z dx.$$

As already observed,  $\{u_{f_n}\}$  converges to  $u_f$  strongly in  $L^p(B_r)$ . Therefore,

$$\lim_{n \rightarrow \infty} \frac{I(f + t_n z) - I(f)}{t_n} = \int_{\mathbb{R}^N} p' u_f z dx.$$

Since the sequence  $\{t_n\}$  and the function  $z$  are arbitrary, it follows that  $I(f)$  is Gâteaux differentiable with derivative  $p'u_f$ .  $\square$

Note that the functional  $\Psi_\lambda$  is not weakly sequentially continuous in  $L^{p'}(\mathbb{R}^N)$ .

**Lemma 2.7.** *Let  $u$  be the solution of the problem (1.1)–(1.2) corresponding to  $f \in \overline{\mathcal{R}(r)^w}$ . We have*

$$\|u\|_{\infty; \mathbb{R}^N} \leq C_1(N, p) \|f - h\|_{\infty}^{\frac{N-p}{Np-N+p}} \|f - h\|_{p'_*}^{\frac{p^2}{(p-1)(Np-N+p)}}, \tag{2.14}$$

where  $C_1(N, p)$  is a constant depending only on  $N$  and  $p$ .

*Proof.* For any  $k > 0$ , take  $v = (u - k)_+ \in W$  in (2.5). We deduce

$$\int_{\mathbb{R}^N} |\nabla v|^p dx \leq \int_{\mathbb{R}^N} |f - h| |v| dx.$$

By Sobolev’s inequality and Hölder’s inequality, we have

$$\|v\|_{p_*; A(k)}^p \leq C_0^p \int_{A(k)} |f - h| |v| dx \leq C_0^p \|v\|_{p_*; A(k)} \|f - h\|_{p'_*; A(k)},$$

where  $A(k) = \{x \in \mathbb{R}^N; u(x) > k\}$ . Therefore,

$$\|v\|_{p_*; A(k)}^{p-1} \leq C_0^p \|f - h\|_{p'_*; A(k)} \leq C_0^p \|f - h\|_{\infty} |A(k)|^{1/p'_*}.$$

Combining this with

$$\|v\|_{p_*; A(k)} \geq \|v\|_{p_*; A(h)} \geq (h - k) |A(h)|^{1/p_*}, \quad \forall h > k > 0,$$

we have

$$|A(h)| \leq \left( \frac{C_0^{p'} \|f - h\|_{\infty}^{\frac{1}{p-1}}}{h - k} \right)^{p_*} |A(k)|^{\frac{p_*-1}{p-1}}, \quad \forall h > k > 0. \tag{2.15}$$

By iteration, we see that  $|A(k_0 + d)| = 0$  for  $k_0 > 0$ ,

$$d = C_0^{p'} \|f - h\|_{\infty}^{\frac{1}{p-1}} 2^{\frac{(p_*-1)(N-p)}{p^2}} |A(k_0)|^{\frac{p'}{N}}.$$

From estimate (2.7), we see that

$$k_0 |A(k_0)|^{\frac{1}{p_*}} \leq \|u\|_{p_*} \leq C_0^{p'} \|f - h\|_{p'_*}^{\frac{1}{p-1}}.$$

Hence

$$u \leq k_0 + d \leq k_0 + 2^N C_0^{p' + \frac{p'2p_*}{N}} \|f - h\|_{\infty}^{\frac{1}{p-1}} \frac{\|f - h\|_{p'_*}^{\frac{p'p_*}{N(p-1)}}}{k_0^{\frac{p'p_*}{N}}}.$$

Let  $\alpha = \frac{p'p_*}{N} = \frac{p^2}{(N-p)(p-1)}$ ,  $A = 2^N C_0^{p'2p_*} \|f - h\|_{\infty}^{\frac{1}{p-1}} \|f - h\|_{p'_*}^{\frac{p'p_*}{N(p-1)}}$  and  $k_0 = (\alpha A)^{\frac{1}{\alpha+1}}$ . We get  $u \leq (\alpha^{\frac{1}{\alpha+1}} + \alpha^{-\frac{\alpha}{\alpha+1}}) A^{\frac{1}{\alpha+1}}$ . By considering  $-u$  instead of  $u$ , we complete the proof.  $\square$

In Section 4, the case  $\lambda = 0$ , more precise estimates are required to demonstrate our result. We begin with an estimate on the lower bound of the energy functional  $\int_{\mathbb{R}^N} |\nabla u|^p dx$ .

**Lemma 2.8.** *Let  $u$  be the solution of the problem (1.1)–(1.2) corresponding to  $f \in \overline{\mathcal{R}(r)^w}$  and  $\text{supp } h \subset B_{r_h}$ ,  $\|f_0\|_1 \leq \|h\|_1$ . We have*

$$\|\nabla u\|_p \geq C_2(N, p)r_h^{-\frac{N-p}{p(p-1)}} (\|h\|_1 - \|f_0\|_1)^{\frac{1}{p-1}}, \tag{2.16}$$

where  $C_2(N, p)$  is a constant depending only on  $N$  and  $p$ .

*Proof.* From (2.4), it suffices to prove that there exists  $v \in W$  such that

$$\int_{\mathbb{R}^N} (p(f-h)v - |\nabla v|^p) dx \geq C(N, p)r_h^{-\frac{N-p}{p-1}} (\|h\|_1 - \|f_0\|_1)^{\frac{p}{p-1}},$$

for a constant  $C(N, p)$  depending only on  $N$  and  $p$ . We verify that the function  $v(x) \equiv -k \min\{(\frac{r_h+a-|x|}{a})_+, 1\} \in W$  fulfills the conditions for some specially selected positive constants  $k$  and  $a$ . Indeed, noticing the signs of  $f$ ,  $h$  and  $v$ , we have

$$\begin{aligned} \int_{\mathbb{R}^N} (p(f-h)v - |\nabla v|^p) dx &\geq kp(\|h\|_1 - \|f_0\|_1) - \omega_N \left(\frac{k}{a}\right)^p (r_h + a)^N \\ &= kp(\|h\|_1 - \|f_0\|_1) - \omega_N k^p \frac{N^N r_h^{N-p}}{p^p (N-p)^{N-p}} \\ &= \frac{(p-1)p^{\frac{p'}{p-1}}(N-p)^{\frac{N-p}{p-1}}}{\omega_N^{\frac{1}{p-1}} N^{\frac{N}{p-1}}} r_h^{-\frac{N-p}{p-1}} (\|h\|_1 - \|f_0\|_1)^{\frac{p}{p-1}}, \end{aligned}$$

for  $a = \frac{p}{N-p}r_h$  and  $k^{p-1} = \frac{p^p(N-p)^{N-p}(\|h\|_1 - \|f_0\|_1)}{\omega_N N^N r_h^{N-p}}$ . □

Next we deduce the local boundedness of solutions by the Moser iteration technique.

**Lemma 2.9.** *Let  $u$  be the solution of the problem (1.1)–(1.2) corresponding to nonnegative function  $f \in L^{p'}(B_r)$ ,  $v = (-u)_+$  and  $\text{supp } h \subset B_{r_h}$ . There holds*

$$\|v\|_{\infty; B_{R/2}(x_0)} \leq C_3(N, p) \left( \frac{1}{R^N} \int_{B_R(x_0)} |v|^{p^*} dx \right)^{1/p^*}, \tag{2.17}$$

for any  $x_0 \in \mathbb{R}^N$  and  $R > 0$  provided  $B_R(x_0) \cap B_{r_h} = \emptyset$ , where  $C_3(N, p)$  is a constant depending only on  $N$  and  $p$ .

*Proof.* For  $0 < \rho < \rho' \leq R$ , let  $\eta(x)$  be a cut-off function  $\eta \in C_0^\infty(B_{\rho'}(x_0))$ , satisfying  $0 \leq \eta \leq 1$ ,  $\eta(x) = 1$  on  $B_\rho(x_0)$ ,  $\eta(x) = 0$  on  $\mathbb{R}^N \setminus B_{\rho'}(x_0)$  and  $|\nabla \eta(x)| \leq \frac{2}{\rho' - \rho}$ . We write  $B_R = B_R(x_0)$  in this proof for the sake of convenience.

Choose  $\eta^p v^s$  as a test function in (2.5) for  $s \geq 1$  and set  $q = s + p - 1$ . We have

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla (\eta^p v^s) dx = \int_{\mathbb{R}^N} (f-h)\eta^p v^s dx = \int_{B_R} f \eta^p v^s dx \geq 0,$$

or

$$- \int_{B_R} \eta^p |\nabla v|^{p-2} \nabla v \cdot \nabla (v^s) dx - \int_{B_R} v^s |\nabla v|^{p-2} \nabla v \cdot \nabla (\eta^p) dx \geq 0.$$

Therefore, using Young's inequality, we deduce

$$\begin{aligned} &\int_{B_R} \eta^p |\nabla (v^{\frac{q}{p}})|^p dx \\ &= \frac{q^p}{sp^p} \int_{B_R} \eta^p |\nabla v|^{p-2} \nabla v \cdot \nabla (v^s) dx \leq \frac{q^p}{sp^p} \int_{B_R} |\nabla v|^{p-1} |\nabla (\eta^p)| v^s dx \end{aligned}$$



$$\begin{aligned} &\leq \frac{q^p}{sp^{p-1}} \int_{B_R} \eta^{p-1} |\nabla \eta| |\nabla v|^{p-1} v^s dx = \frac{q}{s} \int_{B_R} \eta^{p-1} |\nabla(v^{\frac{q}{p}})|^{p-1} |\nabla \eta| v^{\frac{q}{p}} dx \\ &\leq \frac{p-1}{p} \int_{B_R} \eta^p |\nabla(v^{\frac{q}{p}})|^p dx + \frac{q^p}{ps^p} \int_{B_R} |\nabla \eta|^p v^q dx. \end{aligned}$$

Hence we obtain

$$\int_{B_R} \eta^p |\nabla(v^{\frac{q}{p}})|^p dx \leq \frac{q^p}{s^p} \int_{B_R} |\nabla \eta|^p v^q dx \leq p^p \int_{B_R} |\nabla \eta|^p v^q dx, \quad \forall q \geq p.$$

Combining this with Sobolev's inequality (2.3), we have

$$\left( \int_{B_R} \eta^{\frac{Np}{N-p}} v^{\frac{Nq}{N-p}} dx \right)^{\frac{N-p}{N}} \leq C_0^p \int_{B_R} |\nabla(\eta v^{\frac{q}{p}})|^p dx \leq (4pC_0)^p \int_{B_R} |\nabla \eta|^p v^q dx.$$

It follows that

$$\left( \frac{1}{R^N} \int_{B_\rho} v^{\frac{Nq}{N-p}} dx \right)^{\frac{N-p}{N}} \leq (8pC_0)^p \left( \frac{1}{R^{N-p}(\rho' - \rho)^p} \int_{B_{\rho'}} v^q dx \right).$$

Denote  $\rho_k = \frac{R}{2}(1 + \frac{1}{2^k})$ ,  $k = 0, 1, \dots$  and choose  $q = p_*(\frac{N}{N-p})^k$ ,  $\rho = \rho_{k+1}$ ,  $\rho' = \rho_k$ . Since  $\frac{N}{N-p} > 1$ , invoking iterations we see that (2.17) is valid.  $\square$

There are difficulties in carrying out an estimate independent of  $r$  on the corresponding solution of (1.1)–(1.2) due to the fact that  $f$  varies in  $\overline{\mathcal{R}(r)}^w$ . Hence we introduce the following comparison principle.

**Lemma 2.10.** *Let  $u_f$  and  $u_0$  be the solutions of the problem (1.1)–(1.2) corresponding to  $f \in \overline{\mathcal{R}(r)}^w$  and  $f = 0$  respectively. There holds*

$$u_f(x) \geq u_0(x), \quad \text{a.e. in } \mathbb{R}^N.$$

*Proof.* From (2.5), we see that

$$\int_{\mathbb{R}^N} (|\nabla u_f|^{p-2} \nabla u_f - |\nabla u_0|^{p-2} \nabla u_0) \cdot \nabla \varphi dx = \int_{\mathbb{R}^N} f \varphi dx, \quad \forall \varphi \in W.$$

Choosing  $\varphi = (u_0 - u_f)_+ \in W$ , we obtain

$$\int_A (|\nabla u_f|^{p-2} \nabla u_f - |\nabla u_0|^{p-2} \nabla u_0) \cdot (\nabla u_f - \nabla u_0) dx = - \int_A f \varphi dx \leq 0,$$

where  $A = \{x \in \mathbb{R}^N; u_f(x) \leq u_0(x)\}$ . Thus  $\nabla \varphi \equiv 0$  and  $\varphi \equiv 0$  from Sobolev's inequality.  $\square$

Now we could give a locally lower bound of the solution independent of  $f$  and  $r$ .

**Lemma 2.11.** *Let  $u$  be the solution of the problem (1.1)–(1.2) corresponding to  $f \in \overline{\mathcal{R}(r)}^w$ . For any  $\varepsilon > 0$ , there exists  $r_\varepsilon > 0$  depending only on  $N$ ,  $p$ ,  $h$  and  $\varepsilon$ , such that*

$$u(x) \geq -\varepsilon, \quad \forall x \in \mathbb{R}^N \setminus B_{r_\varepsilon}.$$

*Proof.* Let  $u_0$  be as defined in Lemma 2.10, which is independent of  $f$  and  $r$ . Using the similar method in the proof of Lemma 2.10, we demonstrate  $u_0 \leq 0$  in  $\mathbb{R}^N$ . Utilizing Lemma 2.9, we find

$$\|u_0\|_{\infty; B_{1/2}(x_0)} \leq C_3(N, p) \|u_0\|_{p_*; B_1(x_0)},$$

provided  $B_1(x_0) \cap \text{supp } h = \emptyset$ . Let  $r \geq r_h + 1$  and  $|x_0| \geq r$ , where  $r_h > 0$  satisfies  $\text{supp } h \subset B_{r_h}$ . From (2.7), we see that  $\|u_0\|_{p_*; \mathbb{R}^N} \leq C_0' \|h\|_{p_*'}^{\frac{1}{p_*}}$ . It follows

$\|u_0\|_{\infty; B_{1/2}(x_0)} \leq \varepsilon$  provided  $|x_0|$  is large enough. By applying Lemma 2.11, we complete the proof.  $\square$

### 3. THE CASE $\lambda > 0$

First we are concerned with the existence of minimizers for the energy functional in bounded domains, then we will show that a solution valid in a sufficiently large bounded domain is in fact valid in the whole space.

**Lemma 3.1.** *Let  $\lambda \geq 0$ ,  $r > 0$  and  $\omega_N r^N \geq |\text{supp } f_0|$ .*

- (i) *The functional  $\Psi_\lambda$  attains its minimum relative to  $\overline{\mathcal{R}(r)^w}$ .*
- (ii) *If  $f_{r,\lambda}$  is a minimizer for  $\Psi_\lambda$  relative to  $\overline{\mathcal{R}(r)^w}$ ,  $f_{r,\lambda}$  is a solution of the variational problem*

$$\min_{f \in \overline{\mathcal{R}(r)^w}} \int_{\mathbb{R}^N} f(p'u_{f_{r,\lambda}} + \lambda g) dx,$$

where  $u_{f_{r,\lambda}}$  is the solution of (1.1)–(1.2) corresponding to  $f_{r,\lambda}$ .

The above lemma is a simple consequence of Lemma 2.3 and Lemma 2.6.

**Lemma 3.2.** *Let  $\lambda > \lambda_0 \equiv \frac{p'}{\sigma} \|f_0\|_{\infty}^{\frac{1}{p-1}}$ . If  $f_{r,\lambda}$  is a minimizer of  $\Psi_\lambda$  relative to  $\overline{\mathcal{R}(r)^w}$  and  $\psi_{r,\lambda} = p'u_{f_{r,\lambda}} + \lambda g$ , we have*

$$|L_\alpha(\psi_{r,\lambda}) \cap \text{supp } f_{r,\lambda}| = 0, \quad \forall \alpha \in \mathbb{R}.$$

*Proof.* We argue by contradiction. Suppose there exists  $\hat{\alpha} \in \mathbb{R}$  such that  $|S_{\hat{\alpha}}| > 0$ ,  $S_{\hat{\alpha}} = L_{\hat{\alpha}}(\psi_{r,\lambda}) \cap \text{supp } f_{r,\lambda} \subset B_r$ . We have  $\psi_{r,\lambda} = p'u_{f_{r,\lambda}} + \lambda g = \hat{\alpha}$ , a.e. in  $S_{\hat{\alpha}}$ . Therefore,

$$\|f\|_{\infty} \geq f - h = -\Delta_p u_{f_{r,\lambda}} = \Delta_p \left(\frac{\lambda}{p'} g\right) \geq \left(\frac{\lambda}{p'}\right)^{p-1} \sigma^{p-1} > \|f_0\|_{\infty}, \quad \text{a.e. in } S_{\hat{\alpha}},$$

in the sense of distribution, which contradicts to (2.2). This completes the proof.  $\square$

**Lemma 3.3.** *Let  $\lambda_0$  be as defined in the lemma above and  $\lambda > \lambda_0$ ,  $\omega_N r^N \geq |\text{supp } f_0|$ . The set of solutions of the variational problem (2.1) denoted by  $S_\lambda(r)$  is nonempty. If  $f_{r,\lambda} \in S_\lambda(r)$ , we have*

$$f_{r,\lambda} = \varphi_{r,\lambda} \circ (p'u_{f_{r,\lambda}} + \lambda g), \quad (3.1)$$

almost everywhere in  $B_r$  for a decreasing function  $\varphi_{r,\lambda}$ .

*Proof.* From Lemma 3.1, there exists  $f_{r,\lambda} \in \overline{\mathcal{R}(r)^w}$ , which is a minimizer of  $\Psi_\lambda$  relative to  $\overline{\mathcal{R}(r)^w}$ . By Lemma 3.2, the level sets of  $\psi_{r,\lambda} = p'u_{f_{r,\lambda}} + \lambda g$  on  $\text{supp } f_{r,\lambda}$  have zero measure. Utilizing Lemma 3.1 (ii) and Lemma 2.2, we see that  $f_{r,\lambda} \in \overline{\mathcal{R}(r)^w}$  solves the variational problem (2.1) and has the representation (3.1). As already shown in the proof, the minimum for  $\Psi_\lambda$  relative to  $\overline{\mathcal{R}(r)^w}$  actually equals the minimum relative to  $\mathcal{R}(r)$  under the assumption of this lemma. Thus for any  $f_{r,\lambda} \in S_\lambda(r)$ ,  $f_{r,\lambda}$  has a representation as (3.1) for some  $\varphi_{r,\lambda}$ .  $\square$

We have proved that the variational problem (2.1) has a solution for  $\lambda > \lambda_0$  and  $\omega_N r^N \geq |\text{supp } f_0|$ . Now we will show that if  $r$  is chosen large enough, it ceases to have any influence on the variational problem (2.1).

**Lemma 3.4.** *Let  $\lambda > \lambda_0$ . There exists  $r_\lambda > 0$  satisfying  $\omega_N r_\lambda^N \geq |\text{supp } f_0|$  such that for any  $r \geq r_\lambda$  and  $f_{r,\lambda} \in S_\lambda(r)$ , we have  $\text{supp } f_{r,\lambda} \subset B_{r_\lambda}$ .*

*Proof.* Let  $r_a > 0$ ,  $\omega_N r_a^N > |\text{supp } f_0| = |\text{supp } f_{r,\lambda}|$ . From estimates (2.2) and (2.14), we see that  $\|u_{f_{r,\lambda}}\|_{\infty; \mathbb{R}^N}$  is bounded by a constant depending on  $N, p, \|f_0\|_\infty, \|h\|_\infty, |\text{supp } f_0|, |\text{supp } h|$  and independent of  $r, \lambda$ . Since  $\lambda > \lambda_0, g \in C^2(\mathbb{R}^N)$  and  $\lim_{|x| \rightarrow +\infty} g(x) = +\infty$ , we can find a constant  $r_\lambda \geq r_a$  such that

$$p' u_{f_{r,\lambda}}(x) + \lambda g(x) \geq p' u_{f_{r,\lambda}}(z) + \lambda g(z), \quad \forall x \in \mathbb{R}^N \setminus B_{r_\lambda}, z \in \overline{B_{r_a}}. \tag{3.2}$$

Using the representation (3.1) of  $f_{r,\lambda}$  in  $B_r$ , the decreasing property of  $\varphi_{r,\lambda}$  and the fact  $\text{supp } f_{r,\lambda} \subset B_r$ , we deduce

$$0 \leq f_{r,\lambda}(x) \leq \inf_{|z| \leq r_a} f_{r,\lambda}(z), \quad \forall x \in \mathbb{R}^N \setminus B_{r_\lambda}.$$

By the assumption of  $r_a$ , we get  $\inf_{|z| \leq r_a} f_{r,\lambda}(z) = 0$  since  $|B_{r_a} \setminus \text{supp } f_{r,\lambda}| > 0$ . It follows  $\text{supp } f_{r,\lambda} \subset B_{r_\lambda}$ . □

Now we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* Let  $\lambda > \lambda_0, r \geq r_\lambda$  and  $f_{r,\lambda} \in S_\lambda(r)$ . From Lemma 3.4, we have  $\text{supp } f_{r,\lambda} \subset B_{r_\lambda}$ . Therefore,  $f_{r,\lambda} \in \mathcal{R}(r_\lambda) \subset \mathcal{R}(r)$ . It shows that the minimum of  $\Psi_\lambda$  relative to  $\mathcal{R}(r)$  is attained at and only at some points in subset  $\mathcal{R}(r_\lambda)$  for  $r \geq r_\lambda$ . Since  $\mathcal{R} = \bigcup_{r \geq r_\lambda} \mathcal{R}(r)$ , we obtain  $S_\lambda = S_\lambda(r_\lambda) = S_\lambda(r)$  for  $r \geq r_\lambda$ . It follows (1.4) has a solution. To prove the last part of this theorem, for any  $r \geq r_\lambda$  and  $f_\lambda \in S_\lambda = S_\lambda(r)$ , we have by applying Lemma 3.3

$$f_\lambda = \varphi_{r,\lambda} \circ (p' u_{f_\lambda} + \lambda g), \quad \text{a.e. in } B_r,$$

for a decreasing function  $\varphi_{r,\lambda}$ . We can use the similar method in the proof of (3.2) to choose  $r \geq r_\lambda$  and  $C_\lambda \in \mathbb{R}$  such that

$$p' u_{f_\lambda}(x) + \lambda g(x) \geq C_\lambda = \sup_{z \in \overline{B_{r_\lambda}}} (p' u_{f_\lambda}(z) + \lambda g(z)), \quad \forall x \in \mathbb{R}^N \setminus B_r.$$

Noticing that  $\text{supp } f_\lambda \subset B_{r_\lambda}$ , we have that  $\varphi_{r,\lambda}(t) = 0$  for  $t \in [C_\lambda, C'_\lambda]$ , and  $C'_\lambda = \sup_{z \in \overline{B_r}} (p' u_{f_\lambda}(z) + \lambda g(z)) \geq C_\lambda$ . Now define

$$\varphi_\lambda(t) = \begin{cases} \varphi_{r,\lambda}(t), & t \leq C_\lambda, \\ 0, & t > C_\lambda. \end{cases}$$

Clearly  $\varphi_\lambda$  is a decreasing function and  $f_\lambda = \varphi_\lambda \circ (p' u_{f_\lambda} + \lambda g)$  a.e. in  $\mathbb{R}^N$ . □

#### 4. THE CASE $\lambda = 0$

To derive the existence result in this case, we need some additional conditions on  $f$  and  $h$ . Similarly, we first deduce the following lemma in bounded domains.

**Lemma 4.1.** *Suppose  $\|f_0\|_\infty < \|h\|_{-\infty; \text{supp } h}$ ,  $r > 0$  and  $\omega_N r^N \geq |\text{supp } f_0|$ . Let  $f_r$  be a minimizer of  $\Psi_0$  relative to  $\overline{\mathcal{R}(r)}^w$  and  $u_{f_r}$  be the solution of the problem (1.1)–(1.2) corresponding to  $f_r$ . We have*

$$|L_\alpha(u_{f_r}) \cap \text{supp } f_r| = 0, \quad \forall \alpha \in \mathbb{R}.$$

*Proof.* We argue by contradiction. Suppose there exists  $\hat{\alpha} \in \mathbb{R}$  such that  $|A_{\hat{\alpha}}| > 0$ ,  $A_{\hat{\alpha}} = L_{\hat{\alpha}}(u_{f_r}) \cap \text{supp } f_r \subset B_r$ . We have  $u_{f_r} = \hat{\alpha}$  a.e. in  $A_{\hat{\alpha}}$ . Hence  $-\Delta_p u_{f_r} = f - h = 0$  a.e. in  $A_{\hat{\alpha}}$ , in the sense of distributions. Since  $A_{\hat{\alpha}} \subset \text{supp } f_r$ , we find  $f > 0$  in  $A_{\hat{\alpha}}$ , which follows  $h > 0$  a.e. in  $A_{\hat{\alpha}}$  and  $A_{\hat{\alpha}} \subset \text{supp } h$ . Thus  $\|h\|_{-\infty; \text{supp } h} \leq \|f\|_{\infty} \leq \|f_0\|_{\infty}$  from (2.2), which is a contradiction to our assumption.  $\square$

**Lemma 4.2.** *The set of solutions of the variational problem (2.1) with  $\lambda = 0$  denoted by  $S_0(r)$  is nonempty under the assumption of the lemma above. Moreover, if  $f_r \in S_0(r)$ , we have*

$$f_r = \varphi_r \circ u_{f_r}, \quad \text{a.e. in } B_r, \tag{4.1}$$

for a decreasing function  $\varphi_r$ .

*Proof.* Utilizing Lemma 3.1, Lemma 4.1 and Lemma 2.2, we obtain the required results by using the similar method in the proof of Lemma 3.3.  $\square$

**Lemma 4.3.** *There exists a constant  $\kappa = \kappa(N, p) \in (0, \frac{1}{2}]$  depending only on  $N$  and  $p$ , such that if  $\|f_0\|_{\infty} < \|h\|_{-\infty; \text{supp } h}$ ,  $\text{supp } h \subset B_{r_h}$  and*

$$|\text{supp } f_0| \leq \kappa \left( \frac{\|h\|_{-\infty; \text{supp } h}}{\|h\|_{\infty}} \right)^{\frac{p}{p-1}} \left( \frac{|\text{supp } h|}{|B_{r_h}|} \right)^{\frac{N-p}{N(p-1)}} |\text{supp } h|, \tag{4.2}$$

we have  $\text{supp } f_r \subset B_{r_0}$  for any  $r \geq r_0$  and  $f_r \in S_0(r)$ , where  $r_0 \geq r_h$  is a constant independent of  $r$  and  $f_r$ .

*Proof.* From the representation of  $f_r$  in (4.1) and the decreasing property of  $\varphi_r$ , we see that

$$\sup_{x \in \text{supp } f_r} u_{f_r}(x) = s_0 \leq \inf_{z \in B_r \setminus \text{supp } f_r} u_{f_r}(z). \tag{4.3}$$

Using (2.5), we calculate

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla u_{f_r}|^p dx \\ &= \int_{\mathbb{R}^N} (f_r - h) u_{f_r} dx \\ &= \int_{\text{supp } f_r \setminus \text{supp } h} f_r u_{f_r} dx - \int_{\text{supp } h \setminus \text{supp } f_r} h u_{f_r} dx + \int_{\text{supp } f_r \cap \text{supp } h} (f_r - h) u_{f_r} dx \\ &\leq s_0 \|f_r\|_{1; \text{supp } f_r \setminus \text{supp } h} - s_0 \|h\|_{1; \text{supp } h \setminus \text{supp } f_r} + \|u_{f_r}\|_{\infty} \|f_r - h\|_{\infty} |\text{supp } f_r|. \end{aligned} \tag{4.4}$$

By assumption, for any  $\kappa \leq \frac{1}{2}$ , utilizing (2.2) and (2.14), we have

$$\begin{aligned} \|f_r\|_{1; \text{supp } f_r \setminus \text{supp } h} &\leq \|f_r\|_1 \leq \|f_0\|_1 \leq \|f_0\|_{\infty} |\text{supp } f_0| \\ &< \|h\|_{-\infty; \text{supp } h} (|\text{supp } h| - |\text{supp } f_r|) \leq \|h\|_{1; \text{supp } h \setminus \text{supp } f_r}, \end{aligned} \tag{4.5}$$

$$\|f_0\|_1 \leq \|f_0\|_{\infty} |\text{supp } f_0| < \frac{1}{2} \|h\|_{-\infty; \text{supp } h} |\text{supp } h| \leq \frac{1}{2} \|h\|_1, \tag{4.6}$$

$$\begin{aligned} & \|u_{f_r}\|_{\infty} \|f_r - h\|_{\infty} |\text{supp } f_r| \\ &\leq C_1 \|f_r - h\|_{\infty}^{\frac{Np}{Np-N+p}} \|f_r - h\|_{p'_*}^{\frac{p^2}{(p-1)(Np-N+p)}} |\text{supp } f_r| \\ &\leq C_1 2^{\frac{p^2}{(p-1)(Np-N+p)}} \|h\|_{\infty}^{\frac{Np}{Np-N+p}} \|h\|_{p'_*}^{\frac{p^2}{(p-1)(Np-N+p)}} |\text{supp } f_0| \\ &\leq C_1 2^{\frac{p^2}{(p-1)(Np-N+p)}} \|h\|_{\infty}^{\frac{p}{p-1}} |\text{supp } h|^{\frac{p}{N(p-1)}} |\text{supp } f_0|, \end{aligned} \tag{4.7}$$

where  $C_1 = C_1(N, p)$  is the constant in (2.14). From the assumption and the estimates (2.16), (4.6), we deduce

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_{f_r}|^p dx &\geq C_2^p r_h^{-\frac{N-p}{p-1}} (\|h\|_1 - \|f_0\|_1)^{\frac{p}{p-1}} \geq C_2^p r_h^{-\frac{N-p}{p-1}} \left(\frac{1}{2}\|h\|_1\right)^{\frac{p}{p-1}} \\ &\geq 2^{-p'} C_2^p r_h^{-\frac{N-p}{p-1}} (\|h\|_{-\infty; \text{supp } h} |\text{supp } h|)^{\frac{p}{p-1}}, \end{aligned} \tag{4.8}$$

where  $C_2 = C_2(N, p)$  is the constant in (2.16). Let

$$\kappa = \min\left\{\frac{1}{2}, \frac{C_2^p \omega_N^{\frac{N-p}{N(p-1)}}}{2 \cdot 2^{p'} \cdot 2^{\frac{p^2}{(p-1)(Np-N+p)}} C_1}\right\}.$$

Combining (4.2), (4.4)–(4.5), (4.7)–(4.8), we obtain

$$\begin{aligned} s_0 &\leq -\frac{C_2^p \|h\|_1^{p'} r_h^{-\frac{N-p}{p-1}}}{2 \cdot 2^{p'} (\|h\|_{1; \text{supp } h \setminus \text{supp } f_r} - \|f_r\|_{1; \text{supp } f_r \setminus \text{supp } h})} \\ &\leq -\frac{C_2^p}{2 \cdot 2^{p'}} \|h\|_1^{\frac{1}{p-1}} r_h^{-\frac{N-p}{p-1}} \equiv -\delta, \end{aligned}$$

where  $\delta > 0$  is a constant independent of  $r$  and  $f_r$ .

For  $\varepsilon = \frac{1}{2}\delta$ , applying Lemma 2.14, we find that there exists  $r_0 \geq r_h$  independent of  $r$  and  $f_r$  such that

$$u_{f_r}(x) \geq -\frac{1}{2}\delta > -\delta \geq s_0 = \sup_{x \in \text{supp } f_r} u_{f_r}(x), \quad \forall x \in \mathbb{R}^N \setminus B_{r_0}.$$

It follows  $\text{supp } f_r \subset B_{r_0}$ . □

*Proof of Theorem 1.2.* The first part of this theorem can be proved by using the similar method in the proof of Theorem 1.1. It follows  $S_0 = S_0(r_0) = S_0(r)$  for  $r \geq r_0$ . To prove the last part of this theorem, for any  $\hat{f} \in S_0 = S_0(r_0)$ , we have from (4.1)

$$\hat{f} = \varphi_{r_0} \circ u_{\hat{f}}, \quad \text{a.e. in } B_{r_0},$$

for a decreasing function  $\varphi_{r_0}$ . Combining this with (4.3), we obtain  $\varphi_{r_0}(t) \geq 0$  for  $t \leq s_0$  and  $\varphi_{r_0}(t) = 0$  for  $t \in [s_0, s'_0]$ ,  $s'_0 = \sup_{x \in B_{r_0}} u_{\hat{f}}(x)$ . Now define

$$\hat{\varphi}(t) = \begin{cases} \varphi_{r_0}(t), & t \leq s_0, \\ 0, & t > s_0. \end{cases}$$

Clearly,  $\hat{\varphi}$  is a decreasing function and  $\hat{f} = \hat{\varphi} \circ u_{\hat{f}}$  a.e. in  $\mathbb{R}^N$ . □

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SHANMING JI

SCHOOL OF MATHEMATICAL SCIENCES, SOUTH CHINA NORMAL UNIVERSITY, GUANGZHOU 510631, CHINA

*E-mail address:* sam@m.scnu.edu.cn

JINGXUE YIN

SCHOOL OF MATHEMATICAL SCIENCES, SOUTH CHINA NORMAL UNIVERSITY, GUANGZHOU 510631, CHINA

*E-mail address:* yjx@scnu.edu.cn

RUI HUANG

SCHOOL OF MATHEMATICAL SCIENCES, SOUTH CHINA NORMAL UNIVERSITY, GUANGZHOU 510631.  
DEPARTMENT OF MATHEMATICS, SOUTH CHINA UNIVERSITY OF TECHNOLOGY, GUANGZHOU 510640, CHINA

*E-mail address:* huang@scnu.edu.cn