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BIFURCATION OF TRAVELING WAVE SOLUTIONS OF A GENERALIZED $K(n,n)$ EQUATION

XIAOSHAN ZHAO, GUANHUA ZHAO, LINPING PENG

Abstract. In this article, a generalized $K(n,n)$ equation is studied by the qualitative theory of bifurcations and the method of dynamical systems. The result shows the existence of the different kinds of traveling solutions of the generalized $K(n,n)$ equation, including solitary waves, kink waves, periodic wave and compacton solutions, which depend on different parametric ranges. Moreover, various sufficient conditions to guarantee the existence of the above traveling solutions are provided under different parameters conditions.

1. Introduction

The well-known $K(m,n)$ equation [10] takes the form

$$u_t + a(u^n)_x + (u^m)_{xxx} = 0, \quad n > 1. \tag{1.1}$$

which generates the so termed compactons: solitary waves with exact compact support. Compactons are defined as solitons with finite wavelengths or solitons free of exponential tails [1, 3, 4, 6, 7, 8, 9, 10, 11, 12, 14, 15]. Wazwaz [13] used the tanh and sine-cosine method to study the generalized $K(n,n)$ equation given by

$$u_t + \alpha(u^n)_x + \beta(u^{2n}(u^{-n})_{xx})_x = 0, \quad n > 1. \tag{1.2}$$

In this paper, we aim to consider the bifurcation behavior of the traveling wave solutions of equation (1.2) in the parameter space and obtain bifurcations of traveling solutions under different parameter conditions. Let $u(x,t) = \psi(\xi)$, $\xi = x - ct$, where c is wave speed. Substituting the above traveling transformation into equation (1.2) and integrating once, we have

$$n\beta\psi^{n-1}\psi_{\xi\xi} - n(n+1)\beta\psi^{n-2}\psi_\xi^2 - \alpha\psi^n + c\psi - g = 0, \tag{1.3}$$

where g is an integral constant. equation (1.3) is equivalent to the two dimensional system as follows

$$\begin{gathered} d\psi/d\xi = y, \\ dy/d\xi = (n(n+1)\psi^{n-2}y^2 + \rho(\psi^n - \mu\psi + \nu))/(n\psi^{n-1}), \end{gathered} \tag{1.4}$$

where $\rho = \alpha/\beta$, $\mu = c/\alpha, \nu = g/\alpha$. And the system (1.4) has the first integral

$$H(\psi, y) = -1/2\psi^{-2(n+1)}y^2 - \rho\psi^{-3n}(1/(2n)\psi^n + \mu/(1-3n)\psi + \nu/(3n)) = h. \tag{1.5}$$

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In fact, system (1.4) is a three-parameter planar dynamical system depending on the parameter group (ρ, μ, ν). Because the phase orbits defined by the vector fields of system (1.4) determine types of traveling wave solutions of equation (1.2), we should consider the bifurcations of phase portraits of system (1.4) in the phase plane (ψ, y) as the parameters ρ, μ, ν change. Since only physical models with bounded traveling waves are meaningful, we just focus on the bounded solutions of system (1.4).

The paper is organized as follows. In Section 2, we discuss bifurcation curves and phase portraits of system (1.4). In Section 3, we show the existence of solitary and periodic kink wave and compacton solutions of (1.2). In Section 4, we present some exact explicit solutions for (1.2).

2. Bifurcation set and phase portraits of system (1.4)

In this section, we will study all phase portraits and bifurcation set of system (1.4) in the parameter space. Let $d\xi = n\psi^{n-1}d\omega$. Thus, system (1.4) becomes

$$\begin{gathered} d\psi/d\omega = n\psi^{n-1}y, \\ dy/d\omega = n(n+1)\psi^{n-2}y^2 + \rho(\psi^n - \mu\psi + \nu). \end{gathered} \tag{2.1}$$

Except for the straight line $\psi = 0$, systems (1.4) and (2.1) have the same first integral as (1.5). Note that for a fixed h, equation (1.5) determines a set of invariant curves of system (2.1), which contains more different branches of curves. As h changes, (1.5) defines different families of orbits of system (2.1) with different dynamical behavior. We assume that (ψ_i, y_i) is an equilibrium point of system (1.4). At this point, the determinant of the linearized system of system (2.1) has the form

$$J(\psi_i, y_i) = n^3(n+1)\psi_i^{2n-4}y_i^2 - n\rho\psi_i^{n-1}(n\psi_i^{n-1} - \mu). \tag{2.2}$$

By the bifurcation theory of dynamical system [2, 5], we know that if $J(\psi_i, y_i) > 0$ ($or < 0$) , then equilibrium point (ψ_i, y_i) is a center (or saddle point); if $J(\psi_i, y_i) = 0$) and the Poincare index of (ψ_i, y_i) is zero, then the equilibrium point (ψ_i, y_i) is a cusp point. Denote $h_i = H(\psi_i, y_i)$, $h = H(\psi, y)$ defined by (1.5), $M_i(\psi_i, 0)$ and $M(\psi, 0)$ are equilibrium points of system (1.4) and $\psi_i < \psi_{i+1}$. $N_\pm(0, \pm 1/6\sqrt{-6\rho\nu})$ are the equilibrium points on the straight line $\psi = 0$. From the above qualitative analysis, we can obtain the bifurcation curves and phase portraits with the aid of mathematical software Maple.

2.1. **Bifurcation set and phase portraits of system** (2.1) **when** $n = 2$**.**

In this case, there are two bifurcation curves on the (μ, ν)-plane

$$\Pi_1 : \nu = 0, \quad \Pi_2 : \nu = \frac{1}{4}\mu^2, \tag{2.3}$$

which divide the (μ, ν)-parameter plane into four different subregions (see Figure 1).

By using the above facts to do qualitative analysis, we obtain the following results.

Proposition 2.1. *(see Figure 2) Suppose that* $\rho > 0$.

(1) For $(\mu, \nu) \in \Pi_1^\pm$, *system* (2.1) *has two heteroclinic orbits connecting to the saddle points* $M_1(\psi_1, 0)$ *and* $O(0, 0)$. *In addition, there is a family of homoclinic orbits to* O *for* $h \in (-\infty, h_1)$.

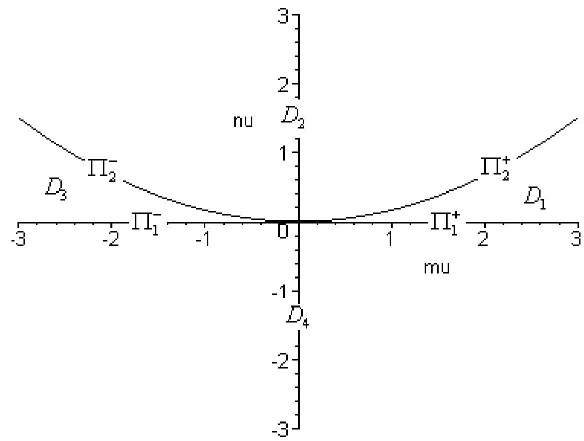

FIGURE 1. Bifurcation set and curves of system (2.1) for $n = 2$. Where $\Pi_1^{\pm} = \{(\mu, \nu)|\mu > 0(\mu < 0), \nu = 0\}$, $\Pi_2^{\pm} = \{(\mu, \nu)|\mu > 0(\mu < 0), \nu = \frac{1}{4}\mu^2\}$, $D_1 : 0 < \Pi_1^+ < \nu < \Pi_2^+$, $D_2 : \nu > \Pi_2 > 0$, $D_3 : \Pi_1^- < \nu < \Pi_2^-$, $D_4 : \nu < \Pi_1$.

(2)For $(\mu, \nu) \in (D_1 \cup D_3)$, system (2.1) has a homoclinic orbit connecting to the saddle point $M_2(\psi_2, 0)$ (or $M_1(\psi_1, 0)$). In addition, there is a family of periodic orbits surrounding the center $M_1(or M_2)$ for $h \in (h_1, h_2)$ $(or h \in (h_2, h_1))$.

(3) For $(\mu, \nu) \in \Pi_2^{\pm}$, system (2.1) has a cusp point.

(4) For $(\mu, \nu) \in D_4$, system (2.1) has four heteroclinic orbits connecting to the saddle points $M_1(\psi_1, 0)$, $M_2(\psi_2, 0)$ and $N_{\pm}(0, \pm 1/6\sqrt{-6\rho\nu})$ respectively. In addition, there are two families of heteroclinic orbits to $N_{\pm}(0, \pm 1/6\sqrt{-6\rho\nu})$ for $h \in (-\infty, h_1)$.

Proposition 2.2. *(see Figure 3). Suppose that $\rho < 0$.*

(1) For $(\mu, \nu) \in \Pi_1^{\pm}$, system (2.1) has a homoclinic orbit connecting to the saddle point $O(0, 0)$. In addition, there are a family of homoclinic orbits to O and a family of periodic orbits surrounding the center M_1 $(or M_2)$ respectively for $h \in (h_1, 0)$.

(2) For $(\mu, \nu) \in (D_1 \cup D_3)$, system (2.1) has two heteroclinic orbits connecting to the saddle points $h \in (-\infty, h_1)$ and M_1 (or M_2) respectively. In addition, there are a family of periodic orbits surrounding the center M_1 (or M_2), a family of heteroclinic orbits connecting to the saddle points $N_{\pm}(0, \pm 1/6\sqrt{-6\rho\nu})$ for $h \in (h_2, 0)$ (or $h \in (h_1, 0)$) and a family of heteroclinic orbits connecting to the saddle points $N_{\pm}(0, \pm 1/6\sqrt{-6\rho\nu})$ for $h \in (-\infty, h_1)$ (or $h \in (-\infty, h_2)$) respectively. If $H(\psi, 0) = h$ (here $M(\psi, 0)$ is the saddle point) defined by (1.5) has a zero ψ^* satisfying $0 < \psi < \psi^*$ (or $0 > \psi > \psi^*$), there exist a homoclinic orbit connecting to the saddle point M and three heteroclinic orbits connecting to the saddle points M and $N_{\pm}(0, \pm 1/6\sqrt{-6\rho\nu})$ respectively. Furthermore, there are a family of periodic orbits surrounding the center M_2 $(or M_1)$ and a family of heteroclinic orbits to $N_{\pm}(0, \pm 1/6\sqrt{-6\rho\nu})$ for $h \in (h_2, h_1)$ (or $h \in (h_1, h_2)$) respectively.

(3) For $(\mu, \nu) \in \Pi_2^{\pm}$, system (2.1) has four heteroclinic orbits connecting to the saddle points $N_{\pm}(0, \pm 1/6\sqrt{-6\rho\nu})$ and a cusp point $M(\psi, 0)$ respectively. In addition, there are two families of heteroclinic orbits to $N_{\pm}(0, \pm 1/6\sqrt{-6\rho\nu})$ for $h \in (-\infty, h_1)$ and $h \in (h_1, 0)$ respectively.

(4) For $(\mu, \nu) \in D_4$*, system* (2.1) *has two centers* M_1 *and* M_2*. When* $h \in (h_2, 0)$ *(or* $h \in (h_1, 0)$*), there are two families of periodic orbits surrounding the centers* M_2 *and* M_1 *respectively.*

By the above analysis, we have the following phase portraits of system (1.4) under different parametric conditions shown in figures 2 and 3. they are mae with the help of mathematical software Maple.

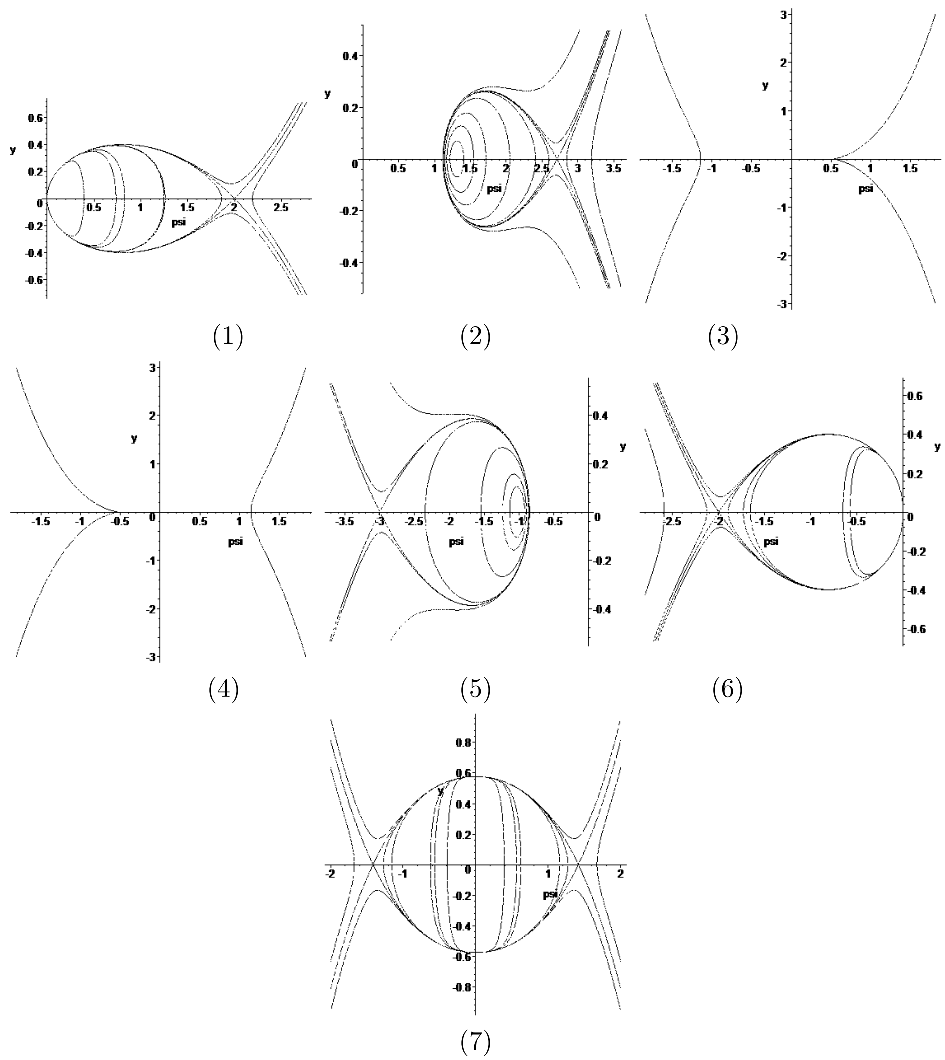

FIGURE 2. Phase portraits of (2.1) when $n = 2$ for $\rho > 0$. (1) $(\mu, \nu) \in \Pi_1^+$; (2) $(\mu, \nu) \in D_1$; (3) $(\mu, \nu) \in \Pi_2^+$; (4) $(\mu, \nu) \in \Pi_2^-$; (5) $(\mu, \nu) \in D_3$; (6) $(\mu, \nu) \in \Pi_1^-$; (7) $(\mu, \nu) \in D_4$.

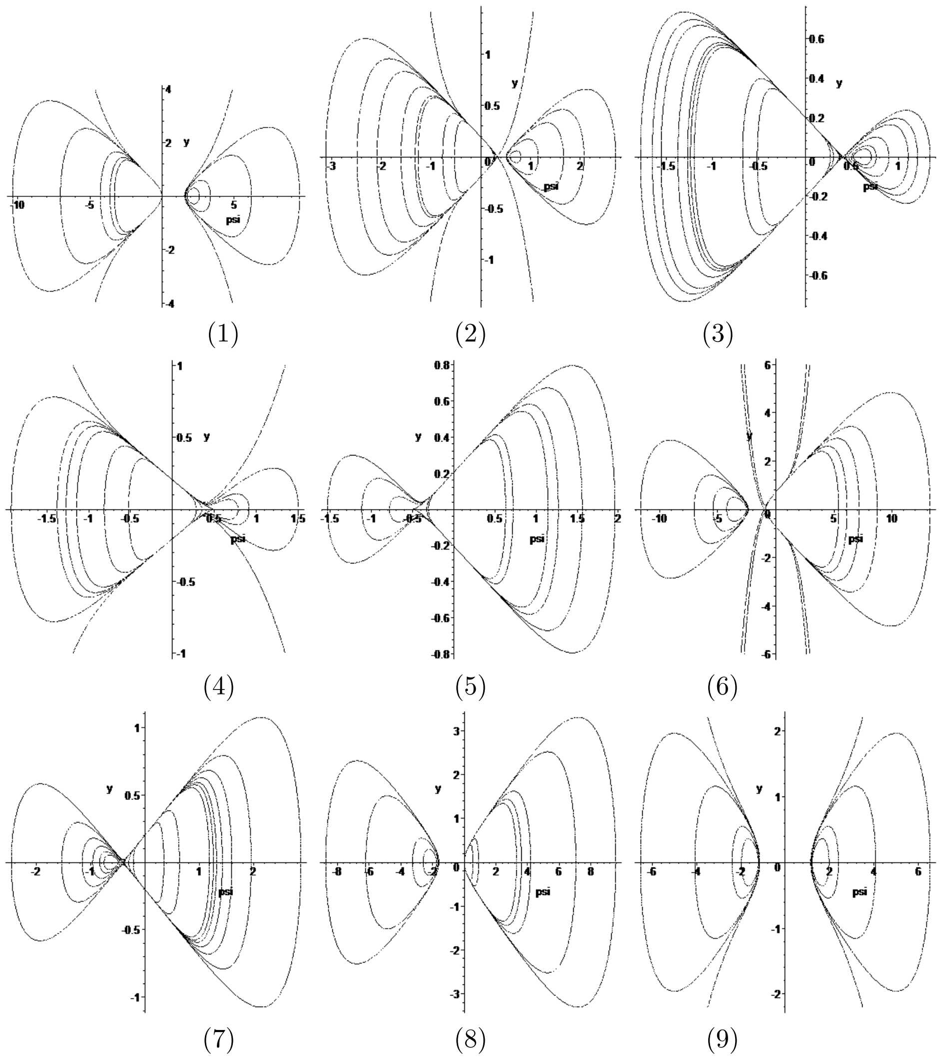

Figure 3. Phase portraits of (2.1) when $n = 2$ for $\rho < 0$. (1) $(\mu, \nu) \in \Pi_1^+$; (2) $(\mu, \nu) \in D_1$; (3) $(\mu, \nu) \in D_1$; (4) $(\mu, \nu) \in \Pi_2^+$; (5) $(\mu, \nu) \in \Pi_2^-$; (6) $(\mu, \nu) \in D_3$; (7) $(\mu, \nu) \in D_3$; (8) $(\mu, \nu) \in \Pi_1^-$; (9) $(\mu, \nu) \in D_4$.

2.2. **Bifurcation set and phase portraits of system** (2.1) **when** $n = 2k$ ($k > 1$)**.** In this case, there are two bifurcation curves on the (μ, ν)-parameter plane:

$$\Gamma_1 : \nu = 0, \quad \Gamma_2 : \nu = [(\frac{1}{2k})^{\frac{1}{2k-1}} - (\frac{1}{2k})^{\frac{2k}{2k-1}}]\mu^{\frac{2k}{2k-1}}, \tag{2.4}$$

which divide the (μ, ν)-parameter plane into four different subregions (see Figure 4).

By applying the above facts to do qualitative analysis, we obtain the following results.

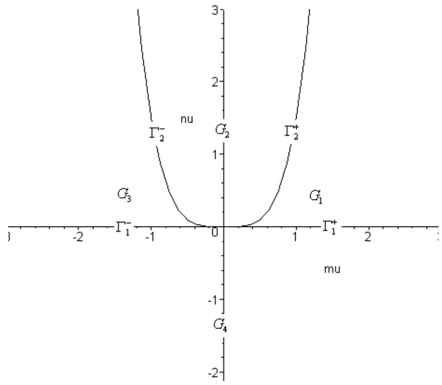

FIGURE 4. Bifurcation set and curves of system (2.1) for $n = 2k, k > 1$. Where $\Gamma_1^{\pm} = \{(\mu,\nu)|\mu > 0(\mu < 0), \nu = 0\}$, $\Gamma_2^{\pm} = \{(\mu,\nu)|\mu > 0(\mu < 0), \nu = [\frac{1}{(2k)^{\frac{1}{2k-1}}} - \frac{1}{(2k)^{\frac{2k}{2k-1}}}]\mu^{\frac{2k}{2k-1}}\}$, $G_1 : 0 < \Gamma_1^+ < \nu < \Gamma+_2$, $G_2 : \nu > \Gamma_2 > 0$, $G_3 : \Gamma_1^- < \nu < \Gamma_2^-$, $G_4 : \nu < \Gamma_1$.

Proposition 2.3. *Suppose that $\rho > 0$ (see Figure 5).*

(1) For $(\mu,\nu) \in \Gamma_1^{\pm}$, system (2.1) has one saddle point. system (2.1) has two heteroclinic orbits connecting to the saddle points $M_1(\psi_1, 0)$ and $O(0,0)$. In addition, there is a family of homoclinic orbits to O for $h \in (-\infty, h_1)$.

(2) For $(\mu,\nu) \in (G_1 \cup G_3)$, system (2.1) has a homoclinic orbit connecting to the saddle point $M_2(\psi_2, 0)$ (or $M_1(\psi_1, 0)$). In addition, there is a family of periodic orbits surrounding the center M_1 (or M_2) for $h \in (h_1, h_2)$ (or $h \in (h_2, h_1)$).

(3) For $(\mu,\nu) \in \Gamma_2^{\pm}$, system (2.1) has a cusp point.

(4) For $(\mu,\nu) \in G_4$, system (2.1) has two saddle points.

Proposition 2.4. *Suppose that $\rho < 0$ (see Figure 6).*

(1) For $(\mu,\nu) \in \Gamma_1^{\pm}$, system (2.1) has a center M_1. In addition, there are a family of periodic orbits surrounding the center M_1 for $h \in (h_1, 0)$.

(2) For $(\mu,\nu) \in (G_1 \cup G_3)$, system (2.1) has one saddle point and one center. In addition, there are a family of periodic orbits surrounding the center for $h \in (h_2, 0)$ (or $h \in (h_1, 0)$). If $H(\psi, 0) = h$ (here $M(\psi, 0)$ is the saddle point) defined by (1.5) has a zero ψ^* satisfying $0 < \psi < \psi^*$ (or $0 > \psi > \psi^*$), there exists a homoclinic orbit connecting to the saddle point M , and there are a family of periodic orbits surrounding the center for $h \in (h_2, h_1)$ (or $h \in (h_1, h_2)$).

(3) For $(\mu,\nu) \in \Gamma_2^{\pm}$, system (2.1) has a cusp point $M(\psi, 0)$.

(4) For $(\mu,\nu) \in G_4$, system (2.1) has two centers M_1 and M_2. When $h \in (h_1, 0)$ (or $h \in (h_2, 0)$), there are two families of periodic orbits surrounding the centers M_1 and M_2 respectively.

By means of the above analysis, we have the following phase portraits of system (2.1) under different parametric conditions shown in figures 5 and 6. they were made with the aid of Mathematical software Maple.

2.3. **Bifurcation set and phase portraits of system** (2.1) **when** $n = 2k + 1$ **(**$k \geq 1$**).** In this case, there are two bifurcation curves on the (μ,ν)-parameter

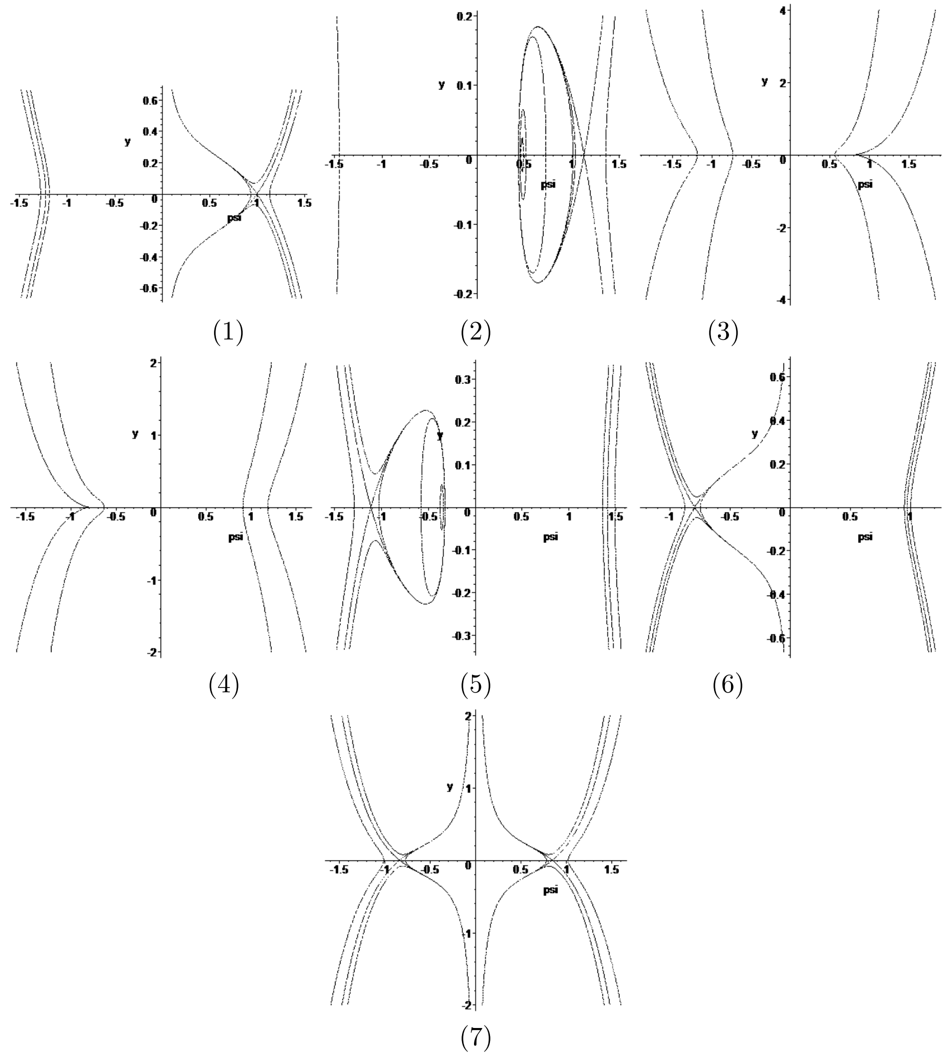

FIGURE 5. Phase portraits of (2.1) when $n = 2k$, $k > 1$ for $\rho > 0$. (1) $(\mu, \nu) \in \Gamma_1^+$; (2) $(\mu, \nu) \in G_1$; (3) $(\mu, \nu) \in \Gamma_2^+$; (4) $(\mu, \nu) \in \Gamma_2^-$; (5) $(\mu, \nu) \in G_3$; (6) $(\mu, \nu) \in \Gamma_1^-$; (7) $(\mu, \nu) \in G_4$.

plane:

$$\Upsilon_1 : \nu = 0, \quad \Upsilon_2 : \nu = [(\frac{1}{2k+1})^{\frac{1}{2k}} - (\frac{1}{2k+1})^{\frac{2k+1}{2k}}]\mu^{\frac{2k+1}{2k}}, \tag{2.5}$$

which divide the (μ, ν)-parameter plane into four different subregions (see Figure 7).

By using the above facts to do qualitative analysis, we obtain the following results.

Proposition 2.5. *Suppose that $\rho > 0$ (see Figure 8).*

(1) For $(\mu, \nu) \in \Upsilon_1^-$, system (2.1) has two saddle points on the axis of abscissa and two saddle points on the axis of ordinates. And there exist four heteroclinic

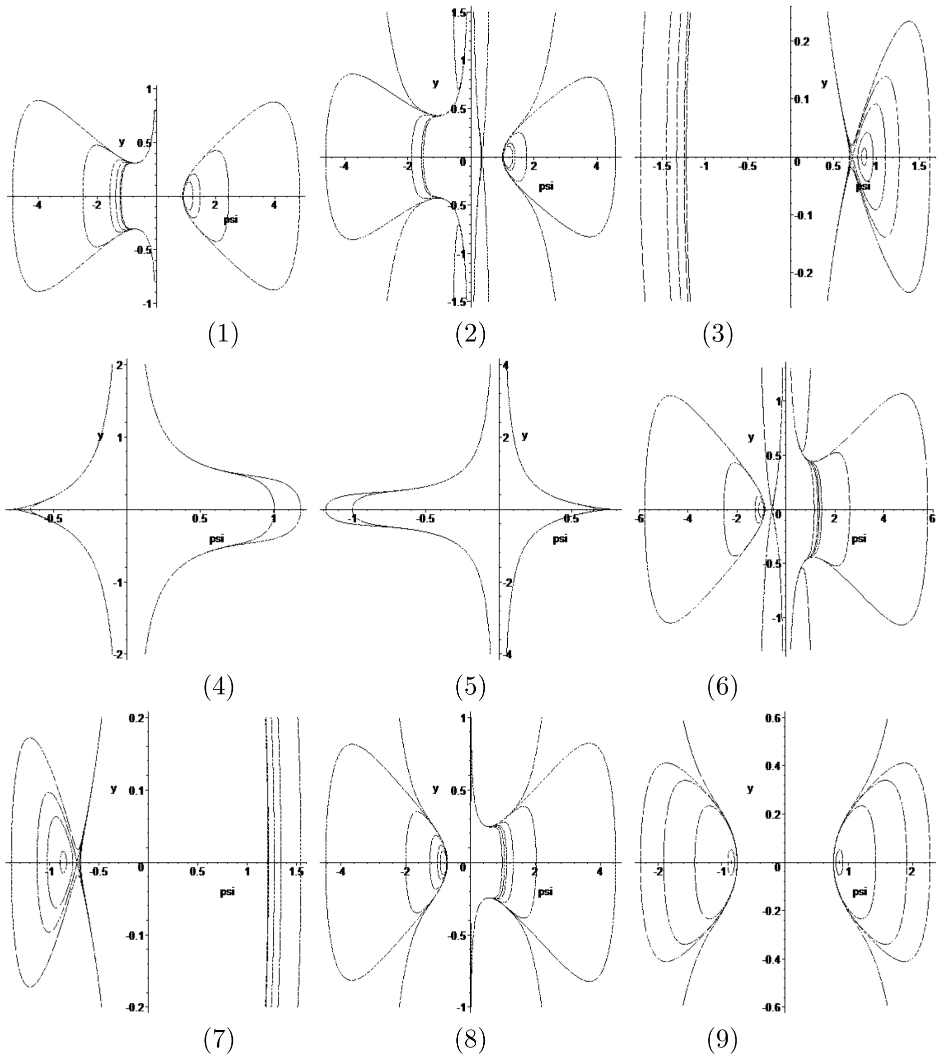

FIGURE 6. Phase portraits of (2.1) when $n = 2k$, $k > 1$ for $\rho < 0$. (1) $(\mu, \nu) \in \Gamma_1^+$; (2) $(\mu, \nu) \in G_1$; (3) $(\mu, \nu) \in G_1$; (4) $(\mu, \nu) \in \Gamma_2^+$; (5) $(\mu, \nu) \in \Gamma_2^-$; (6) $(\mu, \nu) \in G_3$; (7) $(\mu, \nu) \in G_3$; (8) $(\mu, \nu) \in \Gamma_{-1}$; (9) $(\mu, \nu) \in G_4$.

orbits connecting to the saddle points $M_1(\psi_1, 0)$, $M_2(\psi_2, 0)$ and $N_{\pm}(0, \pm 1/6\sqrt{-6\rho\nu})$ respectively, for $h = h_1$ (or h_2).

(2) For $(\mu, \nu) \in (Z_1 \cup Z_4)$, system (2.1) has two saddle points $M_1(\psi_1, 0)$, $M_3(\psi_3, 0)$, and one center $M_2(\psi_2, 0)$. And there exist a homoclinic orbit connecting to the saddle point M_1 or M_3 for $h = h_1$ (or h_3) and a family of periodic orbits surrounding the center M_2 for $h \in (h_2, h_1)$ (or$h \in (h_2, h_3)$).

(3) For $(\mu, \nu) \in \Upsilon_2^{\pm}$, system (2.1) has a saddle point and a cusp point.

(4) For $(\mu, \nu) \in (Z_2 \cup Z_3)$, system (2.1) has a saddle point.

Proposition 2.6. *(see Figure 9). Suppose that $\rho < 0$*

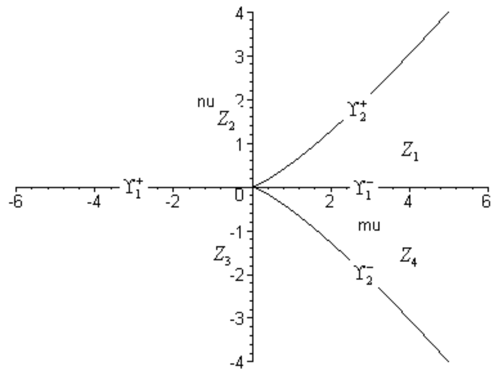

Figure 7. Bifurcation set and curves of (2.1) for $n = 2k+1 (k \geq 1)$. $\Upsilon_1^{\pm} = \{(\mu,\nu) | \mu > 0 (\mu < 0), \nu = 0\}$, $\Upsilon_2^{\pm} = \{(\mu,\nu) | \mu > 0 (\mu < 0), \nu = [\frac{1}{(2k+1)^{\frac{1}{2k}}} - \frac{1}{(2k+1)^{\frac{2k+1}{2k}}}]\mu^{\frac{2k+1}{2k}}\}$, $Z_1 : 0 < \Upsilon_1^+ < \nu < \Upsilon_2^+$, $Z_2 : \nu > \Upsilon_2^+ > 0$, $Z_3 : \nu < \Upsilon_2^- < 0$, $Z_4 : \Upsilon_2^- < \nu < \Upsilon_1^+$.

(1) For $(\mu,\nu) \in \Upsilon_1^+$, system (2.1) has two centers. In addition, there are two families of periodic orbits surrounding the centers M_1 (orM_2) for $h \in (h_1, 0)$, respectively.

(2) For $(\mu,\nu) \in (Z_1 \cup Z_4)$, system (2.1) has one saddle point and two centers. In addition, there are two families of periodic orbits surrounding the two centers for $h \in (h_1, 0)$ and $h \in (h_3, 0)$ respectively. If $H(\psi, 0) = h$ (here $M(\psi, 0)$ is the saddle point) defined by (1.5) has a zero ψ^* satisfying $0 < \psi^* < \psi$ (or $0 > \psi^* > \psi$), there exists a homoclinic orbit connecting to the saddle point M.

(3) For $(\mu,\nu) \in \Upsilon_2^{\pm}$, system (2.1) has a cusp point and a center. When $h \in (h_1, 0)$ (or $h \in (h_2, 0)$), there exist a family of periodic orbits surrounding the center.

(4) For $(\mu,\nu) \in (Z_2 \cup Z_3)$, system (2.1) has one center. When $h \in (h_1, 0)$, there is a family of periodic orbits surrounding the center.

From the above analysis, we have the following phase portraits of system (1.4) under different parametric conditions shown in figures 8 and 9.

3. Existence of traveling wave solutions of equation (1.2))

In this section, we consider the existence of smooth and non-smooth solitary traveling wave and periodic traveling wave solutions of equation (1.2). Obviously, the system (1.4) has the same orbits as the system (2.1), except for $\psi = 0$. The transformation of variables $d\xi = n\psi^{n-1} d\omega$ only derives the difference of the parametric representations of orbits of the systems (1.4) and (2.1) when $\psi = 0$. If an orbit of (2.1) has no intersection point with the straight line $\psi = 0$, then, it is well defined in (1.4). It follows that the profile defined by this orbit on the (ψ, y)-plane is smooth. If an orbit of (2.1) has intersection point with the straight line $\psi = 0$, then it is not defined in (1.4). It follows that the profile defined by this orbit on the (ψ, y)-plane may be non-smooth.

According to the previous discussions, we deduce the following conclusions from figures 2, 3, 5, 6, 8 and 9.

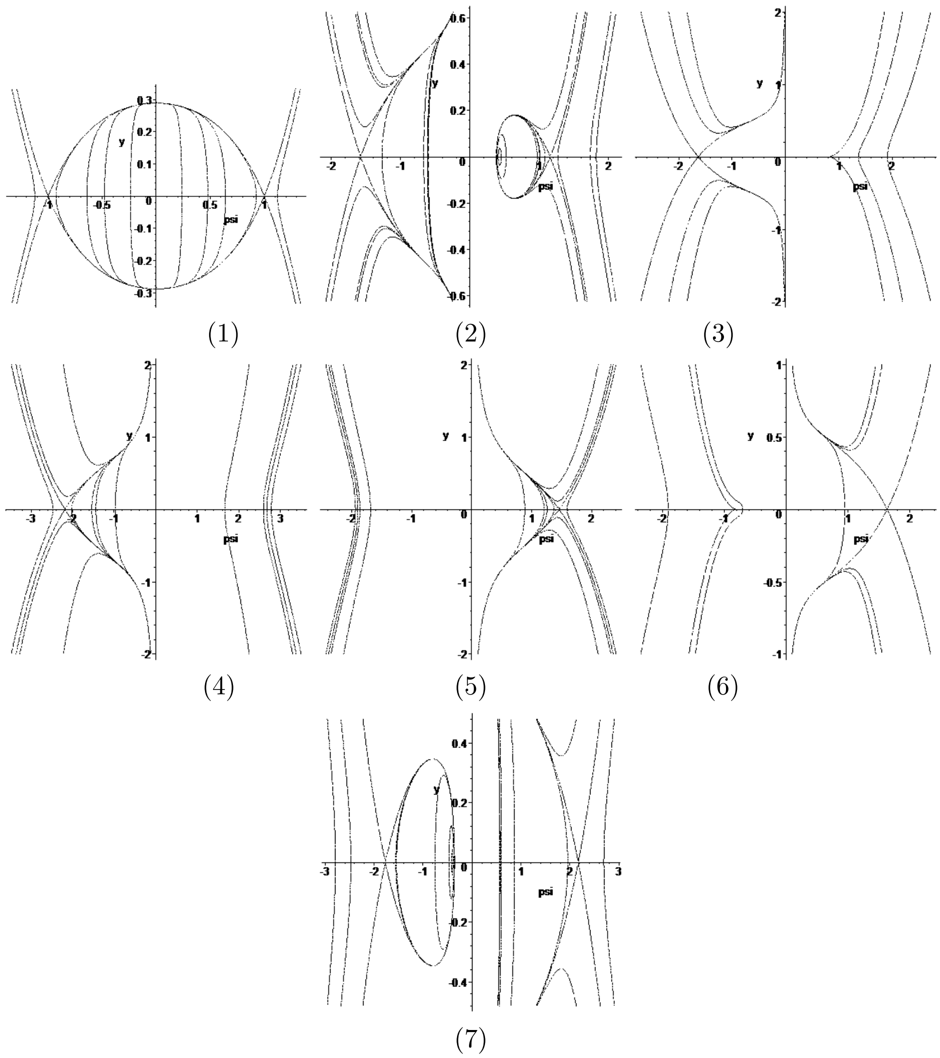

FIGURE 8. Phase portraits of (2.1) when $n = 2k+1, k \geq 1$ for $\rho > 0$. (1) $(\mu,\nu) \in \Upsilon_1^+$; (2) $(\mu,\nu) \in Z_1$; (3) $(\mu,\nu) \in \Upsilon_2^+$; (4) $(\mu,\nu) \in Z_2^-$; (5) $(\mu,\nu) \in Z_3$; (6) $(\mu,\nu) \in \Upsilon_2^-$; (7) $(\mu,\nu) \in Z_4$.

Theorem 3.1. *(see figures 2 and 3) Suppose that $n = 2$.*

(1) For $(\mu,\nu) \in \Pi_1^\pm$. When $\rho > 0$, equation (1.2) has a couple of kink and anti-kink wave solutions for $h = h_1(h_2)$, and has a family of uncountably infinite many compactons solutions for $h \in (-\infty, h_1)$ (or $h \in (-\infty, h_2)$). And when $\rho < 0$, equation (1.2) has a compacton solution for $h = h_1$, and has a family of uncountably infinite many compactons solutions and a family of smooth periodic wave solutions whose amplitudes tend to ∞ for $h \in (h_1, 0)$.

(2) For $(\mu,\nu) \in (D_1 \cup D_3)$. When $\rho > 0$, equation (1.2) has a smooth solitary wave solutions with valley (peak) form for $h = h_2(h_1)$, and has a family of uncountably infinite many smooth periodic wave solutions for $h \in (h_1, h_2)$ (or $h \in (h_2, h_1)$).

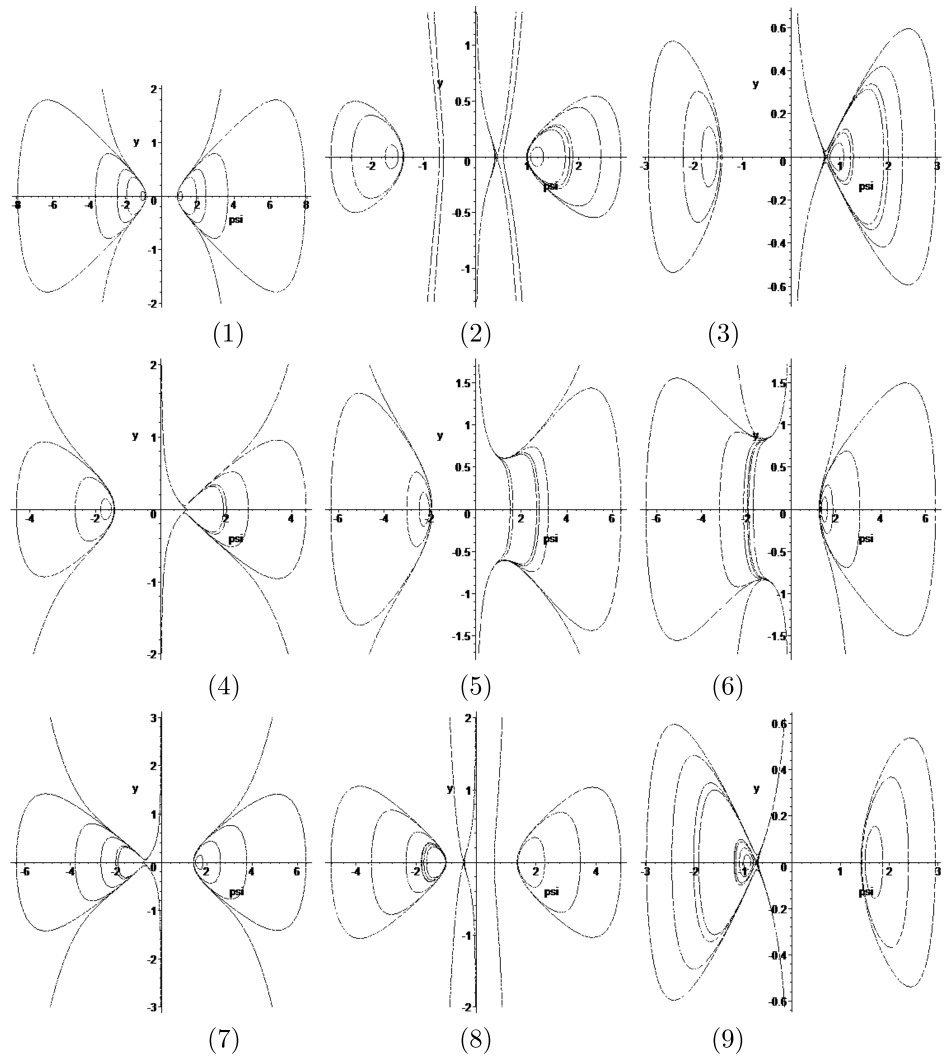

FIGURE 9. Phase portraits of (2.1) when $n = 2k+1, k \geq 1$ for $\rho < 0$. (1) $(\mu, \nu) \in \Upsilon_1^+$; (2) $(\mu, \nu) \in Z_1$; (3) $(\mu, \nu) \in Z_1$; (4) $(\mu, \nu) \in \Upsilon_2^+$; (5) $(\mu, \nu) \in Z_2$; (6) $(\mu, \nu) \in Z_3$; (7) $(\mu, \nu) \in \Upsilon_2^-$; (8) $(\mu, \nu) \in Z_4$; (9) $(\mu, \nu) \in Z_4$.

And when $\rho < 0$, equation (1.2) *has a smooth solitary wave solutions with valley (peak) form for $h = h_1$ (or h_2), and has a family of uncountably infinite many smooth periodic wave solutions and a family of uncountably infinite many solitary wave solutions with valley (peak) wave solutions for $h \in (h_1, 0)$ (or $h \in (h_2, 0)$) respectively and has a family of uncountably infinite many periodic wave solutions for $h \in (-\infty, h_1)$ (or $h \in (-\infty, h_2)$); if $H(\psi, 0) = h$ (here $M(\psi, 0)$ is the saddle point) defined by* (1.5) *has a zero ψ^* satisfying $0 < \psi < \psi^*$ (or $0 > \psi > \psi^*$, then equation* (1.2) *has a couple of solitary wave solutions with peak and valley form for $h = h_2(h_1)$, and has a family of uncountably infinite many smooth periodic wave* *solutions and a family of uncountably infinite many solitary wave solutions with peak (valley) form for* $h \in (h_2, 0)$ *(or* $h \in (h_1, 0)$*) respectively, and has a family of uncountably infinite many periodic wave solutions form for* $h \in (-\infty, h_1)$ *or* $h \in (-\infty, h_2)$*.*

(3) For $(\mu, \nu) \in \Pi_2^{\pm}$*. When* $\rho < 0$*, equation* (1.2) *has a solitary wave solution with valley (or peak) form for* $h = h_1$*, and has a family of uncountably infinite many periodic wave solutions for* $h \in (h_1, 0)$*, and has a family of uncountably infinite many periodic wave solutions form for* $h \in (-\infty, h_1)$*.*

(4) For $(\mu, \nu) \in D_4$*, When* $\rho > 0$*, equation* (1.2) *has a couple of kink and anti-kink wave solutions for* $h = h_2(h_1)$*, and has a family of uncountably infinite many periodic wave solutions for* $h \in (-\infty, h_1)$*. And when* $\rho < 0$*, equation* (1.2) *has two families of uncountably infinite many smooth periodic wave solutions for* $h \in (h_2, 0)$*, and their amplitudes tend to* ∞ *for* $h \to 0$*.*

Theorem 3.2. *(see figures 5 and 6). Suppose that* $n = 2k$*.*

(1) For $(\mu, \nu) \in \Gamma_1^{\pm}$*. When* $\rho < 0$*, equation* (1.2) *has a family of uncountably infinite many periodic solutions for* $h \in (h_1, 0)$*, and their amplitudes tend to* ∞ *for* $h \to 0$*.*

(2) For $(\mu, \nu) \in G_1$*. When* $\rho > 0$*, equation* (1.2) *has a smooth solitary wave solutions with valley form for* $h = h_2$ *, and has a family of uncountably infinite many smooth periodic solutions for* $h \in (h_1, h_2)$*. And when* $\rho < 0$*, equation* (1.2) *has a family of uncountably infinite many smooth periodic solutions for* $h \in (h_2, 0)$*, and their amplitudes tend to* ∞ *for* $h \to 0$*; if* $H(\psi_1, 0) = h$ *(here* $M(\psi_1, 0)$ *is the saddle point) defined by* (1.5) *has a zero* ψ^* *satisfying* $0 < \psi_1 < \psi^*$*, equation* (1.2) *has a smooth solitary wave solutions with peak form for* $h = h_1$ *and a family of uncountably infinite many smooth periodic solutions for* $h \in (h_2, h_1)$*.*

(3) For $(\mu, \nu) \in G_3$*. When* $\rho > 0$*, equation* (1.2) *has a smooth solitary wave solutions with peak form for* $h = h_1$*, and has a family of uncountably infinite many smooth periodic solutions for* $h \in (h_2, h_1)$*. And when* $\rho < 0$*, equation* (1.2) *has a family of uncountably infinite many smooth periodic solutions for* $h \in (h_1, 0)$*, and their amplitudes tend to* ∞ *for* $h \to 0$*; if* $H(\psi_2, 0) = h$ *(here* $M(\psi_2, 0)$ *is the saddle point) defined by* (1.5) *has a zero* ψ^* *satisfying* $0 > \psi_2 > \psi^*$*, equation* (1.2) *has a solitary wave solutions with valley form for* $h = h_2$ *and a family of uncountably infinite many smooth periodic solutions for* $h \in (h_1, h_2)$*.*

(4) For $(\mu, \nu) \in G_4$*. When* $\rho < 0$*, equation* (1.2) *has two families of uncountably infinite many smooth periodic solutions for* $h \in (h_1, 0)$*, and their amplitudes tend to* ∞ *for* $h \to 0$*.*

Theorem 3.3. *(see figures 8 and 9). Suppose that* $n = 2k + 1, k \geq 1$*.*

(1) For $(\mu, \nu) \in \Upsilon_1^{+}$*. When* $\rho > 0$*, equation* (1.2) *has a couple of kink and anti-kink wave solutions for* $h = h_2$ *(or* h_1*), and has a family of uncountably infinite many periodic wave solutions for* $h \in (h_1, +\infty)$*. And when* $\rho < 0$*, equation* (1.2) *has two families of uncountably infinite many smooth periodic wave solutions for* $h \in (h_1, 0)$*, and their amplitudes tend to* ∞ *for* $h \to 0$*.*

(2) For $(\mu, \nu) \in (Z_1 \cup Z_4)$*. When* $\rho > 0$*, equation* (1.2) *has a smooth solitary wave solutions with valley (peak) form for* $h = h_3$ *(or* h_1*), and has a family of uncountably infinite many smooth periodic solutions for* $h \in (h_2, h_3)$*(or* $h \in (h_2, h_1)$*). And when* $\rho < 0$*, equation* (1.2) *has two families of uncountably infinite many smooth periodic solutions for* $h \in (h_2, 0)$ *and* $h \in (h_3, 0)$ *respectively, and their amplitudes tend to* ∞ *for* $h \to 0$*; if* $H(\psi_2, 0) = h$ *(here* $M(\psi_2, 0)$ *is the saddle point) defined by* (1.5)

has a zero ψ^* satisfying $0 < \psi_2 < \psi^*$ (or $0 > \psi_2 > \psi^*$), equation (1.2) *has a smooth solitary wave solutions with peak (or valley) form for $h = h_2$ and two families of uncountably infinite many smooth periodic solutions for $h \in (h_3, h_2)$ and $h \in (h_1, 0)$ respectively.*

(3) For $(\mu, \nu) \in \Upsilon_2^{\pm}$. When $\rho < 0$, equation (1.2) *has a family of uncountably infinite many smooth periodic solutions for $h \in (h_2, 0)$ (or $h \in (h_1, 0)$), and their amplitudes tend to ∞ for $h \to 0$.*

(4) For $(\mu, \nu) \in (Z_2 \cup Z_3)$. When $\rho < 0$, equation (1.2) *has two families of uncountably infinite many smooth periodic solutions for $h \in (h_1, 0)$, and their amplitudes tend to ∞ for $h \to 0$.*

4. Exact traveling wave solutions of equation (1.2)

In this section, to further reveal above results, we provide some exact solutions of equation (1.2) for $n = 2$ by the bifurcation theory. We denote that $h = H(\psi, 0)$ and $M(\psi, 0)$ are equilibrium points of (1.4).

1. The case of $\rho > 0$ and $h = 0$:

(1) When $\nu = 0$, corresponding to figures 2(1) and 2(6) respectively, equation (1.2) has the following compacton solutions

$$u(x,t) = \begin{cases} \frac{4}{5}\mu \cos[\frac{1}{4}\sqrt{\rho}(x-ct)], & |x-ct| < \frac{2\pi}{\sqrt{\rho}}, \\ 0, & \text{otherwise.} \end{cases}$$

(2) When $\mu = 0$ and $\nu < 0$, corresponding to Figure 2(7), equation (1.2) has the following periodic solutions

$$u(x,t) = \pm\sqrt{\frac{-2\nu}{3}} \sin[\frac{\sqrt{\rho}}{2}(x-ct)].$$

2. The case of $\rho < 0$ and $h = 0$:

(1) When $\nu = 0$, corresponding to figures 3(7) and 3(8), equation (1.2) has the following explicit formula of solitary patterns solutions

$$u(x,t)) = \frac{2}{5}\mu\{1 + \cosh[\frac{\sqrt{-\rho}}{2}(x-ct)]\}.$$

(2) When $\mu = 0$ and $\nu < 0$, corresponding to Figure 3(9), equation (1.2) has the following solitary patterns solutions

$$u(x,t)) = \sqrt{\frac{-2\nu}{3}} \cosh[\frac{\sqrt{-\rho}}{2}(x-ct)].$$

conclusion. Employing the bifurcation theory of nonlinear dynamic system, we have studied the bifurcations and dynamic behaviors of traveling wave solutions of equation 1.2. The obtained results show that equation (1.2) has infinite many periodic wave, solitary wave, kink wave and compacton solutions under some parameters' conditions. Therefore, the results in this work clearly demonstrate the effect of the purely nonlinear dispersion and the qualitative change made in the genuinely nonlinear phenomenon.

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References

[1] M. J. Ablowitz, P. A. Clarkson; *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991.

[2] A. A. Andronov, E. A. Leontovich, J. I. Gordon; *Theory of bifurcations of dynamical systems on a plane*, Wiley, New York, 1973.

[3] R. T. Benjamin, J. L. Bona, J. J. Mahony; *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. R Soc. London, 272 (1972), 47-78.

[4] Z. Feng; *Computations of soliton solutions and periodic solutions for the focusing branch of the nonlinear dispersive $K(n,n)$ equations in higher-dimensional spaces*, Appl. Math. Comput. 182 (2006), 781-790.

[5] J. Guckenheimer and P. J. Holmes; *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, NewYork, 1983.

[6] W. Hereman, M. Takaoka; *Solitary wave solutions of nonlinear evolution and wave equations using a direct method and MACSYMA*, J. Phys. A 23 (1990), 4805-4822.

[7] J. B. Li, Z. R. Liu; *Smooth and non-smooth travelling waves in a nonlinearly dispersive equation*, Appl. Math. Model., 25 (2000), 41-56.

[8] J. B. Li, J. W. Shen; *Travelling wave solutions in a model of the helix polypeptide chains*, Chaos, Solitons and Fractals, 20 (2004), 827-41.

[9] Y. A. Li, P. J. Olver; *Convergence of solitary-wave solutions in a perturbed bi-Hamiltonian dynamoical system II: complex analytic behavior and convergence to non-analytic solutions*,Discrete Contin. Dyn. Syst. 4 (1998), 159-91.

[10] P. Rosenau, J. M. Hyman; *Compactons: solitons with finite wavelengths*, Phys. Rev. Lett., 70(5) (1993), 564-567.

[11] M. Wadati; *The modified Kortweg-de Vries equation*, J. Phys. Soc. Japan., 34 (1973), 1289-1296.

[12] A. M. Wazwaz; *Compactons dispersive structures for variants of the $K(n,n)$ and the KP equations*, Chaos Soliton and Fractals, 13(5) (2002), 1053-1062.

[13] A. M. Wazwaz; *Explicit traveling wave solutions of variants of the $K(n,n)$ and the $ZK(n,n)$ equations with compact and noncompact structures*, Appl. Math. Comput., 173 (2006), 213-230.

[14] Z. Y. Yan; *New compacton-like and solitary patterns to nonlinear wave equations with linear dispersion terms*, Nonlinear Anal., 64 (2006), 901-909.

[15] N. J. Zabusky, M. D. Kruskal; *Interaction of solitons in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Lett., 15 (1965), 240-243.

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