Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 149, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

LIPSCHITZ STABILITY FOR DEGENERATE PARABOLIC SYSTEMS

IDRISS BOUTAAYAMOU, ABDELKARIM HAJJAJ, LAHCEN MANIAR

Abstract. In this article, we study an inverse problem for weakly degenerate coupled parabolic systems with one force. We establish Lipschitz stability for the source term from measurements of one component of the solution at a positive time and on a subset of the space domain. The key ingredient is the derivation of a Carleman-type estimate.

1. Introduction

The null controllability and inverse problems of parabolic equations and parabolic coupled systems have attracted much interest in the previous years; see [3, 4, 5, 6, 7, 14, 15, 16, 17, 19, 20, 22, 24, 25, 26]. The main result in these article is the development of adequate Carleman estimates, which is a crucial tool to obtain observability inequalities and Lipschitz stability for term sources, initial data, potentials and diffusion coefficients. The above systems are considered to be non-degenerate. In other words, the diffusion coefficients are uniformly coercive. The case of degenerate coefficients at the boundary is also considered in several papers by developing adequate Carleman estimates. The null controllability of degenerate parabolic equations is studied in [11, 12, 13, 23], and the null controllability of coupled degenerate parabolic systems in [1, 2, 10, 21]. While, the inverse problem for degenerate parabolic equations is studied in [9, 27, 28, 29]. In this article, we consider the Lipschitz stability of an inverse problem for the linear coupled degenerate parabolic systems with two different diffusion coefficients.

$$\begin{gathered}
u_t - (x^{\alpha_1} u_x)_x + b_{11}(x)u + b_{12}(x)v = F, \quad (t,x) \in Q, \\
v_t - (x^{\alpha_2} v_x)_x + b_{21}(x)u + b_{22}(x)v = 0, \quad (t,x) \in Q, \\
u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0, \quad t \in (0,T), \\
u(0,x) = u_0(x),\ v(0,x) = v_0(x), \quad x \in (0,1),
\end{gathered} \tag{1.1}$$

where $u_0,\ v_0 \in L^2(0,1)$, $\alpha_1, \alpha_2 \in (0,1)$, $T > 0$ fixed, $Q := (0,T) \times (0,1)$ and $b_{ij} \in L^\infty(0,1)$, $i,j = 1,2$. For $t_0 \in (0,T)$ given, let $Q_{t_0}^T = (t_0,T) \times (0,1)$ and

---

2000 *Mathematics Subject Classification.* 35K65.
Key words and phrases. Degenerate parabolic system; Carleman estimate; Lipschitz stability; inverse problem.

Submitted October 4, 2013. Published June 26, 2014.

$T' := \frac{T+t_0}{2}$. For a given $C_0 > 0$, we denote by $S(C_0)$ the space

$$S(C_0) := \{F \in H^1(0,T;L^2(0,1)) : |F_t(t,x)| \leqslant C_0 |F(T',x)|,\ (t,x) \in Q\}.$$

Our purpose is to determine F, belonging to the space $S(C_0)$, from the measurements $(x^{\alpha_1} u_x)_x(T',\cdot)$ and additional observations of the component u.

The main ingredient to obtain Lipschitz stability is Carleman estimates. We prove a Carleman estimates for the coupled system (1.1), similar to the one obtained in [9] but with different weight functions, that are necessary for the case of different exponents α_1 and α_2. Having the Carleman estimates in hand, we follow the method developed in [7, 5, 20] to obtain the Lipschitz stability results.

If we restrict ourselves to the particular case $F \in \{rf : f \in L^2(0,1)\}$ for some given function r, uniqueness results can be shown for the system (1.1) as an immediate consequence of our Lipschitz stability results, see [7, 20, 25].

To prove our Carleman estimates, we use the following Hardy-Poincaré inequality proved in [2]

$$\int_0^1 x^{\kappa-2} v^2 dx \leqslant \frac{4}{(1-\kappa)^2} \int_0^1 x^{\kappa} v_x^2 dx,$$

for $\kappa < 1$ and v locally absolutely continuous on $[0,1]$, continuous at 0 and satisfying $v(0) = 0$ and $\int_0^1 x^\kappa v_x^2 dx < \infty$.

This paper is organised as follows: in Section 2, we discuss the well-posedness of the problem (1.1). Then, we establish different Carleman estimates for parabolic equations and parabolic systems (1.1) and the last section treats the Lipschitz stability of F.

2. Well-Posedness of system

To study the well-posedness of (1.1), we introduce the following weighted spaces, for $0 < \alpha < 1$:

$$\begin{aligned} H^1_\alpha(0,1) := \Big\{ & u \in L^2(0,1) : u \text{ abs. cont. in } [0,1], \\ & x^{\alpha/2} u_x \in L^2(0,1),\ u(0) = u(1) = 0 \Big\} \end{aligned}$$

with the norm $\|u\|^2_{H^1_\alpha} := \|u\|^2_{L^2(0,1)} + \|x^{\alpha/2} u_x\|^2_{L^2(0,1)}$ and

$H^2_\alpha(0,1) := \{u \in H^1_\alpha(0,1) : x^\alpha u_x \in H^1(0,1)\}$, $\|u\|^2_{H^2_\alpha} := \|u\|^2_{H^1_\alpha} + \|(x^\alpha u_x)_x\|^2_{L^2(0,1)}$.

We recall from [8, 11] that for $i = 1,2$, the operator $(A_i, D(A_i))$ defined by $A_i u := (x^{\alpha_i} u_x)_x$, $u \in D(A_i) = H^2_{\alpha_i}(0,1)$ is closed self-adjoint, negative with dense domain in $L^2(0,1)$. In the Hilbert space $\mathbb{H} := L^2(0,1) \times L^2(0,1)$, system (1.1) can be transformed into the Cauchy problem

$$\begin{gathered} X'(t) = \mathcal{A}X(t) - BX(t) + h(t), \quad t \in (0,T), \\ X(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \end{gathered}$$

where $X(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$, $\mathcal{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, $D(\mathcal{A}) = D(A_1) \times D(A_2)$, $h(t) = \begin{pmatrix} F(t) \\ 0 \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Since $\mathcal{A}$ is a diagonal operator and B is a bounded perturbation of $\mathcal{A}$, the following wellposedness and regularity results hold.

Proposition 2.1. *(i) The operator $\mathcal{A}$ generates a contraction strongly continuous semigroup $(T(t))_{t\geqslant 0}$.*

(ii) For all $(u_0, v_0) \in H^2_{\alpha_1} \times H^2_{\alpha_2}$ and $F \in H^1(0,T;L^2(0,1))$, problem (1.1) has a unique solution $(u,v) \in C([0,T], H^2_{\alpha_1} \times H^2_{\alpha_2}) \cap C^1(0,T;\mathbb{H})$.

(iii) For all $F \in L^2(Q)$, $u_0, v_0 \in L^2(0,1)$, and $\varepsilon \in (0,T)$, there exists a unique mild solution $(u,v) \in X_T := H^1([\varepsilon,T],\mathbb{H}) \cap L^2(\varepsilon,T;H^2_{\alpha_1} \times H^2_{\alpha_2})$ of (1.1) satisfying

$$\|(u,v)\|_{X_T} \leqslant C_T\Big(\|(u_0,v_0)\|^2_{\mathbb{H}} + \|(F,G)\|^2_{\mathbb{H}}\Big).$$

Moreover, for $F \in H^1(0,T;L^2(0,1))$ and $\varepsilon \in (0,T)$,

$$(u,v) \in C\big([\varepsilon,T], H^2_{\alpha_1} \times H^2_{\alpha_2}\big) \cap C^1\big(\varepsilon,T;\mathbb{H}\big).$$

3. Carleman estimates

The main goal of this section is to establish a Carleman estimate for a degenerate parabolic single equation with a boundary observation on the right hand side. Then, we will deduce the one for the degenerate system (1.1) with locally distributed observations of (u,v).

Some of these estimates were obtained in [2] for a null controllability purpose. In the forthcoming theorems we will prove additional estimates on u and u_t, that are crucial to prove Lipschitz stability results.

As in [2], we introduce the following weight functions which will be used throughout the paper

$$\begin{gathered}
\varphi(t,x) := \theta(t)p(x), \quad \theta(t) := \frac{1}{(t-t_0)^4(T-t)^4}, \\
p(x) := \lambda\left(x^{2-\beta} - d\right), \quad \eta(t) := T + t_0 - 2t,
\end{gathered} \tag{3.1}$$

where $t_0 > 0$ is a fixed initial time, $T > 0$ is a final time, $d > 1$ and β is a constant that will be chosen later.

3.1. Carleman estimates for parabolic equations.

Consider the parabolic equation

$$\begin{gathered}
y_t - (x^\alpha y_x)_x + b(x)y = f(t,x), \quad (t,x) \in Q, \\
y(t,0) = y(t,1) = 0, \quad t \in (0,T), \\
y(0,x) = y_0(x), \quad x \in (0,1)
\end{gathered} \tag{3.2}$$

We assume that $\alpha \in (0,1)$, $b \in L^\infty(0,1)$ and $f \in L^2(Q)$, and state the first Carleman estimate for smooth initial data.

Theorem 3.1. *For all $T > 0$ and $\beta \in [\alpha, 1)$, there exist two positive constants C and s_0 such that for all $s \geqslant s_0$, the solution y of (3.2) with $y_0 \in H^1_\alpha(0,1)$ satisfies*

$$\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s^3\theta^3 x^{2+2\alpha-3\beta} y^2 + s\theta x^{2\alpha-\beta} y_x^2 + \frac{1}{s\theta} y_t^2 + s\theta^{3/2}|\eta p| y^2\Big) e^{2s\varphi(t,x)}\,dt\,dx \\
&\leqslant C\Big(\int_{Q_{t_0}^T} f^2(t,x) e^{2s\varphi(t,x)}\,dt\,dx + \int_{t_0}^T s\theta(t) y_x^2(t,1) e^{2s\varphi(t,1)} dt\Big).
\end{aligned} \tag{3.3}$$

Proof. For $s > 0$ and a solution y of (3.2), the function $w := e^{s\varphi} y$ satisfies

$$\underbrace{-(x^\alpha w_x)_x - s\varphi_t w - s^2 x^\alpha \varphi_x^2 w}_{P_s^+ w} + \underbrace{w_t + 2sx^\alpha \varphi_x w_x + s(x^\alpha \varphi_x)_x w}_{P_s^- w} = \underbrace{fe^{s\varphi} - bw}_{f_s}.$$

By [2, Theorem 3.2], we have

$$\begin{aligned} &\|P_s^+ w\|^2 + \|P_s^- w\|^2 + \int_{Q_{t_0}^T} (s\theta x^{2\alpha-\beta} w_x^2 + s^3\theta^3 x^{2+2\alpha-3\beta} w^2)\,dt\,dx \\ &\leqslant C\Big(\int_{Q_{t_0}^T} f^2 e^{2s\varphi}\,dt\,dx + \int_{t_0}^T s\theta y_x^2(t,1) e^{2s\varphi(t,1)}\,dt \Big). \end{aligned} \tag{3.4}$$

On one hand we have

$$\begin{aligned} \operatorname{sgn}(\eta)\theta^{1/4} w P_s^+ w = 4s\theta^{3/2}|\eta| p w^2 - s^2\lambda^2(2-\beta)^2 \operatorname{sgn}(\eta)\theta^{9/4} x^{2+\alpha-2\beta} w^2 \\ - \operatorname{sgn}(\eta)\theta^{1/4} w (x^\alpha w_x)_x, \end{aligned}$$

where $\operatorname{sgn}(\eta)$ denotes the sign function of η. So, integrating by parts and using Young and Hardy-Poincaré inequalities and $\beta \in [\alpha, 1)$, for s large, we obtain the following inequalities

$$\begin{aligned} &\int_{Q_{t_0}^T} s\theta^{3/2} |\eta p| w^2 \\ &\leqslant \frac{1}{4}\int_{Q_{t_0}^T} \theta^{1/4} w P_s^+ w + \frac{\lambda^2(2-\beta)^2}{4} \int_{Q_{t_0}^T} s^2\theta^{9/4} x^{2+\alpha-2\beta} w^2 + \frac{1}{4}\int_{Q_{t_0}^T} \theta^{1/4} x^\alpha w_x^2 \\ &\leqslant \frac{1}{8}\|P_s^+ w\|^2_{L^2(Q_{t_0}^T)} + \frac{1}{8}\int_{Q_{t_0}^T} \theta^{1/2} w^2 + \int_{Q_{t_0}^T} (s^3\theta^3 x^{2+2\alpha-3\beta} w^2 + s\theta x^{2\alpha-\beta} w_x^2), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \int_{Q_{t_0}^T} \theta^{1/2} w^2\,dt\,dx &= \int_{Q_{t_0}^T} (\theta^{1/4} x^{\alpha-\frac{\beta}{2}-1} w)(\theta^{1/4} x^{1-\alpha+\frac{\beta}{2}} w)\,dt\,dx \\ &\leqslant \frac{1}{2}\int_{Q_{t_0}^T} (\theta^{1/2} x^{2\alpha-\beta-2} w^2 + \theta^{1/2} x^{2-2\alpha+\beta} w^2)\,dt\,dx \\ &\leqslant \int_{Q_{t_0}^T} (s\theta x^{2\alpha-\beta} w_x^2 + s^3\theta^3 x^{2+2\alpha-3\beta} w^2)\,dt\,dx. \end{aligned} \tag{3.6}$$

On the other hand we have

$$\frac{1}{\sqrt{s\theta}} P_s^- w = \frac{1}{\sqrt{s\theta}} w_t + 2\lambda(2-\beta)\sqrt{s\theta} x^{1+\alpha-\beta} w_x + \lambda(2-\beta)(1+\alpha-\beta)\sqrt{s\theta} x^{\alpha-\beta} w.$$

Therefore, using Hardy-Poincaré inequality and $\beta \in [\alpha, 1)$, we obtain

$$\begin{aligned} \int_{Q_{t_0}^T} \frac{1}{s\theta} w_t^2\,dt\,dx &\leqslant C\Big(\|P_s^- w\|^2 + \int_{Q_{t_0}^T} (s\theta x^{2+2\alpha-2\beta} w_x^2 + s\theta x^{2\alpha-2\beta} w^2)\,dt\,dx \Big) \\ &\leqslant C\Big(\|P_s^- w\|^2 + \int_{Q_{t_0}^T} (s\theta x^{2\alpha-\beta} w_x^2 + s\theta x^{2\alpha-\beta-2} w^2)\,dt\,dx \Big) \\ &\leqslant C\Big(\|P_s^- w\|^2 + \int_{Q_{t_0}^T} s\theta x^{2\alpha-\beta} w_x^2\,dt\,dx \Big). \end{aligned} \tag{3.7}$$

Hence, combining (3.4)-(3.7), it follows that

$$\int_{Q_{t_0}^T} \Big(s^3\theta^3 x^{2+2\alpha-3\beta} w^2 + s\theta x^{2\alpha-\beta} w_x^2 + \frac{1}{s\theta} w_t^2 + s\theta^{3/2} |\eta p| w^2 \Big)\,dt\,dx$$

$$\leqslant C\Big(\int_{Q_{t_0}^T} f^2 e^{2s\varphi(t,x)}\,dt\,dx + \int_{t_0}^T s\theta(t) y_x^2(t,1) e^{2s\varphi(t,1)} dt\Big).$$

Finally, the definition of w yields

$$w = ye^{s\varphi}, \quad y_x^2 e^{2s\varphi} \leqslant 2w_x^2 + cs^2\theta^2 x^{2-2\beta} w^2, \quad y_t^2 e^{2s\varphi} \leqslant 2w_t^2 + cs\theta^{5/4}|\eta p| w^2,$$

and thus the estimate (3.3) can be deduced. $\square$

The estimate (3.3) is obtained for regular initial data, by density we deduce the following result for general initial data.

Proposition 3.2. *For all $T > 0$ and $\beta \in [\alpha, 1)$, there exist two positive constants C and s_0 such that for every $y_0 \in L^2(0,1)$ and all $s \geqslant s_0$, the solution y of (3.2) satisfies*

$$\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s^3\theta^3 x^{2+\alpha-2\beta} y^2 + s\theta x^\alpha y_x^2 + \frac{1}{s\theta} y_t^2 + s\theta^{3/2}|\eta p| y^2\Big) e^{2s\varphi(t,x)}\,dt\,dx \\
&\leqslant C\Big(\int_{Q_{t_0}^T} f^2(t,x) e^{2s\varphi(t,x)}\,dt\,dx + \int_{t_0}^T s\theta(t) y_x^2(t,1) e^{2s\varphi(t,1)} dt\Big).
\end{aligned}$$

Proof. Using the density of $H_\alpha^1(0,1)$ in $L^2(0,1)$, there exists a sequence $(y_0^n)_n \subset H_\alpha^1(0,1)$ converging to y_0. Let y^n be the unique solution in the space $Z_T := L^2(t_0,T;H_\alpha^2) \cap H^1(t_0,T;L^2(0,1))$ of the equation (3.2) associated to the initial data y_0^n. The sequence (y^n) satisfies

$$\begin{aligned}
\|y^m - y^n\|_{Z_T}^2 &:= \int_{t_0}^T \|x^{\alpha/2}(y^m - y^n)_x\|_{L^2(0,1)}^2 + \|(x^\alpha (y^m - y^n)_x)_x\|_{L^2(0,1)}^2 dt \\
&\quad + \int_{t_0}^T \|y^m - y^n\|_{L^2(0,1)}^2 + \|y_t^m - y_t^n\|_{L^2(0,1)}^2 dt \\
&\leqslant C_T \|y_0^m - y_0^n\|_{L^2(0,1)}^2,
\end{aligned} \tag{3.8}$$

hence, it has a limit y in the Banach space Z_T. Using classical argument of semi-group theory, it is easy to show that y is the solution of (3.2) associated to the initial data $y_0 \in L^2(0,1)$. Note that for all $t \in (t_0,T)$ we have

$$s\theta e^{2s\varphi(t,1)} \leqslant L := \max_{y \geqslant 0}(ye^{-2\lambda(d-1)y}).$$

Hence, using the Sobolev trace theorem, $\alpha \in (0,1)$ and (3.8) with $m \to \infty$, one has

$$\begin{aligned}
&\int_{t_0}^T s\theta(t)|(y_x^n - y_x)(t,1)|^2 e^{2s\varphi(t,1)}\,dt \\
&\leqslant C\Big(\int_{t_0}^T \|x^{\alpha/2}(y^n - y)_x\|_{L^2(0,1)}^2\,dt + \int_{t_0}^T \|(x^\alpha (y^n - y)_x)_x\|_{L^2(0,1)}^2\,dt\Big) \\
&\leqslant C_T \|y_0^n - y_0\|_{L^2(0,1)}^2.
\end{aligned}$$

On the other hand since $x^\alpha \leqslant x^{2\alpha-\beta}$ and $x^{2+\alpha-2\beta} \leqslant x^{2+2\alpha-3\beta}$, inequality (3.3) provides

$$\int_{Q_{t_0}^T} \Big(s^3\theta^3 x^{2+\alpha-2\beta}|y^n|^2 + s\theta x^\alpha |y_x^n|^2 + \frac{1}{s\theta}|y_t^n|^2 + s\theta^{3/2}|\eta p||y^n|^2\Big) e^{2s\varphi(t,x)}\,dt\,dx$$

$$\leqslant C\Big(\int_{Q_{t_0}^T} f^2(t,x)e^{2s\varphi(t,x)}\,dt\,dx + \int_{t_0}^T s\theta(t)|y_x^n|^2(t,1)e^{2s\varphi(t,1)}dt\Big).$$

Consequently, since the functions $s^3\theta^3e^{2s\varphi}$, $\frac{1}{s\theta}e^{2s\varphi}$ and x^α are bounded, passing to the limit, we obtain the claim. $\square$

3.2. **ω-Carleman estimates for the system** (1.1)**.** In the present subsection, we shall derive an internal Carleman inequality. As in [18], let

$$\Phi(t,x) = \Psi(x)\theta(t), \quad \Psi(x) := \big(e^{\rho\sigma(x)} - e^{2\rho||\sigma||_\infty}\big), \tag{3.9}$$

with θ defined in (3.1) and σ is a function in $C^2([0,1])$ satisfying $\sigma(x) > 0$ in $(a,1)$, $\sigma(a) = \sigma(1) = 0$ and $|\sigma_x(x)| > 0$ in $[0,1]\backslash\omega_0$ for some open $\omega_0 \subset\subset \omega := (a,b)$. We choose the parameters d, λ and ρ, such that $d \geqslant 5$, $\rho > \frac{4ln2}{||\sigma||_\infty}$ and $\frac{e^{2\rho||\sigma||_\infty}}{d-1} < \lambda < \frac{4}{3d}(e^{2\rho||\sigma||_\infty} - e^{\rho||\sigma||_\infty})$. Thus, one has $\frac{4}{3}\Phi < \varphi < \Phi$.

Let $\xi,\zeta \in C^\infty([0,1])$ such that $\zeta = 1-\xi$, $0 \leqslant \xi(x) \leqslant 1$, $\xi(x) = 1$ for $x \in (0,a'')$ and $\xi(x) = 0$ for $x \in (b'',1)$, where $0 < a < a'' < b'' < b < 1$. Set also $\omega' := (a',b')$ and $\omega'' := (a'',b'')$ where $0 < a < a' < a'' < b'' < b' < b < 1$. To obtain a Carleman estimates for the system (1.1) with internal observations, we will use the following propositions, that provide some local Carleman estimates for the parabolic equation (3.2).

Proposition 3.3. *For all $T > 0$ and $\beta \in [\alpha,1)$, there exist two constants C and s_0 such that, for every $y_0 \in L^2(0,1)$ and all $s \geqslant s_0$, the solution y of* (3.2) *satisfies*

$$\begin{aligned}&\int_{Q_{t_0}^T}\Big(s^3\theta^3x^{2+\alpha-2\beta}\xi^2y^2 + s\theta x^\alpha\xi^2y_x^2 + \frac{1}{s\theta}\xi^2y_t^2 + s\theta^{3/2}|\eta p|\xi^2y^2\Big)e^{2s\varphi(t,x)}\,dt\,dx\\ &\leqslant C\Big(\int_{Q_{t_0}^T}\xi^2f^2(t,x)e^{2s\varphi(t,x)}\,dt\,dx + \int_{t_0}^T\int_{\omega'}(f^2+s^2\theta^2y^2)e^{2s\varphi}\,dx\,dt\Big).\end{aligned}$$

Proof. The function $z := \xi y$ satisfies the parabolic equation

$$\begin{gathered} z_t - (x^\alpha z_x)_x + bz = \xi f - \xi_x x^\alpha y_x - (x^\alpha\xi_x y)_x, \quad x \in (0,1),\ t \in (0,T),\\ z(t,0) = z(t,1) = 0, \quad t \in (0,T),\\ z(0,x) = \xi(x)y_0(x), \quad x \in (0,1).\end{gathered}$$

Using Proposition 3.2, z satisfies the estimate

$$\begin{aligned}&\int_{Q_{t_0}^T}\Big(s^3\theta^3x^{2+\alpha-2\beta}z^2 + s\theta x^\alpha z_x^2 + \frac{1}{s\theta}z_t^2 + s\theta^{3/2}|\eta p|z^2\Big)e^{2s\varphi(t,x)}\,dt\,dx\\ &\leqslant C\int_{Q_{t_0}^T}(\xi^2f^2 + (\xi_xx^\alpha y_x + (x^\alpha\xi_xy)_x)^2)e^{2s\varphi(t,x)}\,dt\,dx.\end{aligned} \tag{3.10}$$

So using $\mathrm{supp}(\xi_x) = \omega''$ and the Caccioppoli inequality (5.2) applying for $\mu_1 = \mu_2 = p$, we obtain

$$\begin{aligned}\int_{Q_{t_0}^T}(\xi_xx^\alpha y_x + (x^\alpha\xi_xy)_x)^2e^{2s\varphi}\,dt\,dx &\leqslant C\int_{t_0}^T\int_{\omega''}[y^2+y_x^2]e^{2s\varphi}\,dx\,dt\\ &\leqslant C\int_{t_0}^T\int_{\omega'}(f^2+s^2\theta^2y^2)e^{2s\varphi}\,dx\,dt.\end{aligned} \tag{3.11}$$

By the definition of z and ξ, we get

$$\int_{Q_{t_0}^T} s\theta x^\alpha \xi^2 y_x^2 e^{2s\varphi}\,dt\,dx \leqslant 2\int_{Q_{t_0}^T} s\theta x^\alpha z_x^2 e^{2s\varphi}\,dt\,dx + 2\int_{t_0}^T\int_{\omega'} s\theta y^2 e^{2s\varphi}\,dt\,dx. \tag{3.12}$$

Thus, from (3.10)-(3.12) and the definition of ξ we deduce the desired estimate. $\square$

Proposition 3.3 gave a Carleman estimate in $(0,a')$. For the rest of the interval, we have the following Proposition. Its proof is similar to the previous result using [18, Lemma 1.2] and Cacciopoli inequality (5.2) applying for $\mu_1 = p$, and $\mu_2 = \Psi$.

Proposition 3.4. *For all $T>0$ and $\beta \in [\alpha,1)$, there exist two constants C and s_0 such that, for every $y_0 \in L^2(0,1)$ and all $s \geqslant s_0$, the solution y of (3.2) satisfies*

$$\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s^3\theta^3 x^{2+\alpha-2\beta}y^2 + s\theta x^\alpha y_x^2 + \frac{1}{s\theta}y_t^2 + s\theta^{3/2}|\eta p|y^2\Big)\zeta^2 e^{2s\Phi(t,x)}\,dt\,dx\\
&\leqslant C\Big(\int_{Q_{t_0}^T} \zeta^2 f^2(t,x)e^{2s\Phi(t,x)}\,dt\,dx + \int_{t_0}^T\int_{\omega'}\Big(f^2e^{2s\varphi} + s^3\theta^3y^2e^{2s(2\Phi-\varphi)}\Big)\,dx\,dt\Big).
\end{aligned}$$

And if $f=0$,

$$\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s^3\theta^3 x^{2+\alpha-2\beta}y^2 + s\theta x^\alpha y_x^2 + \frac{1}{s\theta}y_t^2 + s\theta^{3/2}|\eta p|y^2\Big)\zeta^2 e^{2s\Phi(t,x)}\,dt\,dx\\
&\leqslant C\int_{t_0}^T\int_{\omega'} s^3\theta^3y^2e^{2s\Phi}\,dx\,dt.
\end{aligned}$$

Using the above propositions, we show a Carleman estimate for our coupled system with locally distributed measurements.

Theorem 3.5. *Let $T>0$ and $\beta = \max(\alpha_1,\alpha_2)$. There exist two constants C, $s_0>0$ such that, for every $(u_0,v_0)\in (L^2(0,1))^2$ and all $s\geqslant s_0$, the solution (u,v) of (1.1) satisfies*

$$\begin{aligned}
I(\xi,u,v) &:= \int_{Q_{t_0}^T}\Big(s^3\theta^3(x^{2+\alpha_1-2\beta}u^2 + x^{2+\alpha_2-2\beta}v^2) + s\theta(x^{\alpha_1}u_x^2 + x^{\alpha_2}v_x^2)\\
&\quad + \frac{1}{s\theta}(u_t^2+v_t^2) + s\theta^{3/2}|\eta p|(u^2+v^2)\Big)\xi^2e^{2s\varphi(t,x)}\,dt\,dx\\
&\quad \leqslant C\Big(\int_{Q_{t_0}^T}\xi^2F^2e^{2s\varphi(t,x)}\,dt\,dx + \int_{t_0}^T\int_{\omega'}(F^2+s^2\theta^2(u^2+v^2))e^{2s\varphi}\,dx\,dt\Big),
\end{aligned}$$

and

$$\begin{aligned}
&I(\zeta,u,v)\\
&:= \int_{Q_{t_0}^T}\Big(s^3\theta^3(x^{2+\alpha_1-2\beta}u^2 + x^{2+\alpha_2-2\beta}v^2) + s\theta(x^{\alpha_1}u_x^2 + x^{\alpha_2}v_x^2)\\
&\quad + \frac{1}{s\theta}(u_t^2+v_t^2) + s\theta^{3/2}|\eta p|(u^2+v^2)\Big)\zeta^2e^{2s\Phi(t,x)}\,dt\,dx\\
&\leqslant C\Big(\int_{Q_{t_0}^T}\zeta^2F^2e^{2s\Phi(t,x)}\,dt\,dx + \int_{t_0}^T\int_{\omega'}(s^3\theta^3v^2e^{2s\Phi} + F^2e^{2s\varphi} + u^2)\,dx\,dt\Big).
\end{aligned} \tag{3.13}$$

Proof. The first component u is the solution of the parabolic equation (3.2). Applying Proposition 3.3, for s large enough, we obtain

$$\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s^3\theta^3 x^{2+\alpha_1-2\beta}u^2 + s\theta x^{\alpha_1}u_x^2 + \frac{1}{s\theta}u_t^2 + s\theta^{3/2}|\eta p|u^2\Big)\xi^2 e^{2s\varphi(t,x)}\,dt\,dx \\
&\leqslant C_2\Big(\int_{Q_{t_0}^T} (\xi^2F^2 + \xi^2 b_{12}^2 v^2)e^{2s\varphi(t,x)}\,dt\,dx + \int_{t_0}^T\int_{\omega'} (F^2 + v^2 + s^2\theta^2u^2)e^{2s\varphi}\,dx\,dt\Big).
\end{aligned} \tag{3.14}$$

Proceeding as in (3.6), for s large enough, we have

$$\begin{aligned}
C_2\int_{Q_{t_0}^T} \xi^2 b_{12}^2 v^2 e^{2s\varphi}\,dt\,dx &\leqslant C\int_{Q_{t_0}^T} (x^{\frac{\alpha_2}{2}-1}\xi v e^{s\varphi})(x^{1-\frac{\alpha_2}{2}}\xi v e^{s\varphi})\,dt\,dx \\
&\leqslant \frac{1}{2}\int_{Q_{t_0}^T} (s\theta x^{\alpha_2}\xi^2 v_x^2 + s^3\theta^3 x^{2+\alpha_2-2\beta}\xi^2 v^2)e^{2s\varphi}\,dt\,dx \\
&\quad + C\int_{t_0}^T\int_{\omega'} s^2\theta^2 v^2 e^{2s\varphi}\,dx\,dt.
\end{aligned} \tag{3.15}$$

Therefore, by (3.14) and (3.15) we deduce

$$\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s^3\theta^3 x^{2+\alpha_1-2\beta}u^2 + s\theta x^{\alpha_1}u_x^2 + \frac{1}{s\theta}u_t^2 + s\theta^{3/2}|\eta p|u^2\Big)\xi^2 e^{2s\varphi(t,x)}\,dt\,dx \\
&\leqslant C\Big(\int_{Q_{t_0}^T} \xi^2F^2 e^{2s\varphi(t,x)}\,dt\,dx + \int_{t_0}^T\int_{\omega'} (F^2 + s^2\theta^2(v^2+u^2))e^{2s\varphi}\,dx\,dt\Big) \\
&+ \frac{1}{2}\int_{Q_{t_0}^T} (s^3\theta^3 x^{2+\alpha_2-2\beta}v^2 + s\theta x^{\alpha_2}v_x^2)\xi^2 e^{2s\varphi}\,dt\,dx.
\end{aligned} \tag{3.16}$$

The same can be done for v and we obtain

$$\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s^3\theta^3 x^{2+\alpha_2-2\beta}v^2 + s\theta x^{\alpha_2}v_x^2 + \frac{1}{s\theta}v_t^2 + s\theta^{3/2}|\eta p|v^2\Big)\xi^2 e^{2s\varphi(t,x)}\,dt\,dx \\
&\leqslant C\Big(\int_{t_0}^T\int_{\omega'} s^2\theta^2(u^2+v^2)e^{2s\varphi}\,dx\,dt\Big) \\
&\quad + \frac{1}{2}\int_{Q_{t_0}^T} (s^3\theta^3 x^{2+\alpha_1-2\beta}u^2 + s\theta x^{\alpha_1}u_x^2)\xi^2 e^{2s\varphi}\,dt\,dx.
\end{aligned} \tag{3.17}$$

Therefore, summing (3.16) and (3.17) we obtain

$$\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s^3\theta^3(x^{2+\alpha_1-2\beta}u^2 + x^{2+\alpha_2-2\beta}v^2) + s\theta(x^{\alpha_1}u_x^2 + x^{\alpha_2}v_x^2) \\
&+ \frac{1}{s\theta}(u_t^2+v_t^2) + s\theta^{3/2}|\eta p|(u^2+v^2)\Big)\xi^2 e^{2s\varphi(t,x)}\,dt\,dx \\
&\leqslant C\Big(\int_{Q_{t_0}^T} \xi^2F^2 e^{2s\varphi(t,x)}\,dt\,dx + \int_{t_0}^T\int_{\omega'} (F^2 + s^2\theta^2(u^2+v^2))e^{2s\varphi}\,dx\,dt\Big).
\end{aligned}$$

Similarly, applying Proposition 3.4 to each equation of (1.1), we obtain the estimate (3.13). $\square$

4. Inverse problem

In this section we establish a Lipschitz stability for the term F. More precisely, we show some inequalities estimating F with an upper bound given by some measurements of the component u only. For this aim, we start by giving adequate Carleman estimates for a system (1.1).

The following result play a crucial role to absorb the observations on the component v. For the proof one can adapt a similar technique used in [2, Lemma 3.4] for the adjoint of degenerate parabolic systems.

Lemma 4.1. *Let $\omega_2 \subset\subset \omega_1$. Moreover, assume that $b_{12} \geqslant \mu > 0 \quad$ on ω_1. There is $C > 0$ such that the solution (u, v) of (1.1) satisfies*

$$\int_{t_0}^{T} \int_{\omega_2} s^3\theta^3 v^2 e^{2s\Phi}\, dx\, dt \le \varepsilon J(v) + C \int_{Q_{t_0}^T} F^2 e^{2s\varphi}\, dx\, dt + C \int_{t_0}^{T} \int_{\omega} u^2\, dx\, dt,$$

where $\varepsilon > 0$ is small enough, s is large enough and

$$J(v) = \int_{Q_{t_0}^T} \Big(s\theta x^{\alpha_2} v_x^2 + s^3\theta^3 x^{2+\alpha_2-2\beta} v^2\Big) e^{2s\varphi}\, dx\, dt.$$

The following theorem is a consequence of Theorem 3.5, Lemma 4.1 and the fact that

$$\int_{\omega} F^2 e^{2s\varphi}\, dx \leqslant 2 \int_{\omega} F^2(\xi^2 + \zeta^2) e^{2s\varphi}\, dx \leqslant 2 \int_0^1 F^2(\xi^2 e^{2s\varphi} + \zeta^2 e^{2s\Phi})\, dx.$$

Theorem 4.2. *Let $T > 0$ and $\beta = \max\{\alpha_1, \alpha_2\}$. Moreover, assume that*

$$b_{12} \geqslant \mu > 0 \quad \text{on } \omega' \Subset \omega. \tag{4.1}$$

There exist two positive constants C and s_0 such that, for every $(u_0, v_0) \in (L^2(0,1))^2$ and for all $s \geqslant s_0$, the solution (u, v) of (1.1) satisfies

$$\begin{aligned} J_0(u,v) &:= I(\xi, u, v) + I(\zeta, u, v) \\ &\leqslant C\Big\{ \int_{Q_{t_0}^T} F^2(\zeta^2 e^{2s\Phi(t,x)} + \xi^2 e^{2s\varphi(t,x)})\, dt\, dx + \int_{t_0}^{T}\int_{\omega} u^2\, dt\, dx \Big\} \\ &=: J_1(F, u). \end{aligned} \tag{4.2}$$

The main result of this article is as follows.

Theorem 4.3. *Let $\alpha_1, \alpha_2 \in (0,1)$ and $C_0 > 0$. There exists a positive constant $C = C(T, t_0, s_0, C_0, \alpha_1, \alpha_2)$ such that, for all $F \in S(C_0)$ and $(u_0, v_0) \in (L^2(0,1))^2$, we have*

$$\begin{aligned} &\|F\|^2_{L^2(Q)} \\ &\leqslant C\Big(\|u\|^2_{H^1(t_0,T;L^2(\omega))} + \|(x^{\alpha_1} u_x)_x(T', \cdot)\|^2_{L^2(0,1)} + \|u(T', \cdot)\|^2_{L^2(0,1)}\Big). \end{aligned} \tag{4.3}$$

Proof. The functions $y = u_t$ and $z = v_t$, where (u, v) is the solution of (1.1), are solutions of the system

$$\begin{gathered} y_t - (x^{\alpha_1} y_x)_x + b_{11} y + b_{12} z = F_t, \quad (t,x) \in Q, \\ z_t - (x^{\alpha_2} z_x)_x + b_{21} y + b_{22} z = 0, \quad (t,x) \in Q, \\ y(t,0) = y(t,1) = z(t,0) = z(t,1) = 0, \quad t \in (0,T). \end{gathered}$$

When we apply Carleman estimate (4.2) to (y,z), we obtain

$$\begin{aligned} J_0(y,z) &:= I(\xi,y,z) + I(\zeta,y,z) \\ &\leqslant C\Big\{ \int_{Q_{t_0}^T} F_t^2 (\zeta^2 e^{2s\Phi(t,x)} + \xi^2 e^{2s\varphi(t,x)})\,dt\,dx + \int_{t_0}^T\!\!\int_\omega y^2\,dt\,dx \Big\} \\ &=: J_1(F_t,y). \end{aligned} \tag{4.4}$$

The terms appearing in (4.3) are well defined, indeed, by Proposition 2.1, we have $y \in L^2\big(t_0,T;H^2_{\alpha_1}\big) \cap H^1\big(t_0,T;L^2(0,1)\big)$. As in [7], we divide the proof into three steps.

Step 1. We show first that there exists a constant $C>0$ such that

$$\begin{aligned} J_1(F,u) + J_1(F_t,y) \leqslant C\Big(&\frac{1}{\sqrt{s}} \int_0^1 F^2(T',x)(\zeta^2 e^{2s\Phi(T',x)} + \xi^2 e^{2s\varphi(T',x)})dx \\ &+ \|u\|^2_{L^2(\omega_{t_0}^T)} + \|u_t\|^2_{L^2(\omega_{t_0}^T)} \Big), \end{aligned} \tag{4.5}$$

where, we used $T' = \frac{T+t_0}{2}$.

To obtain (4.5), it remains to prove that

$$\begin{aligned} &\int_{Q_{t_0}^T} (F^2+F_t^2)(\zeta^2 e^{2s\Phi(t,x)} + \xi^2 e^{2s\varphi(t,x)})\ dx\,dt \\ &\leqslant \frac{C}{\sqrt{s}} \int_0^1 F^2(T',x)(\zeta^2 e^{2s\Phi(T',x)} + \xi^2 e^{2s\varphi(T',x)})dx \end{aligned}$$

Since $\Phi_t(T') = \varphi_t(T') = 0$, $\Phi_{tt}(t) \leqslant -\nu_0 < 0$ and $\varphi_{tt}(t) \leqslant -\nu_1 < 0$, then Taylor's formula provides

$$\Phi(t,x) \leqslant \Phi(T',x) - \frac{\nu_0}{2}(t-T')^2, \quad \varphi(t,x) \leqslant \varphi(T',x) - \frac{\nu_1}{2}(t-T')^2$$

and then

$$\begin{aligned} &\int_{t_0}^T e^{2s\Phi(t,x)}dt \leqslant \frac{1}{\sqrt{\nu_0 s}} e^{2s\Phi(T',x)} \int_{-\infty}^{\infty} e^{-l^2}dl \leqslant \frac{C}{\sqrt{s}} e^{2s\Phi(T',x)}, \\ &\int_{t_0}^T e^{2s\varphi(t,x)}dt \leqslant \frac{1}{\sqrt{\nu_1 s}} e^{2s\varphi(T',x)} \int_{-\infty}^{\infty} e^{-l^2}dl \leqslant \frac{C}{\sqrt{s}} e^{2s\varphi(T',x)}. \end{aligned}$$

So,

$$\begin{aligned} &\int_{Q_{t_0}^T} F^2(T',x)(\zeta^2 e^{2s\Phi(t,x)} + \xi^2 e^{2s\varphi(t,x)})\,dx\,dt \\ &\leqslant \frac{C}{\sqrt{s}} \int_0^1 F^2(T',x)(\zeta^2 e^{2s\Phi(T',x)} + \xi^2 e^{2s\varphi(T',x)})dx. \end{aligned}$$

For $F \in S(C_0)$, one has

$$|F(t,x)| \leqslant |F(T',x)| + \int_{T'}^t |F_t(s,x)|ds \leqslant C|F(T',x)|. \tag{4.6}$$

Thus

$$\begin{aligned} &\int_{Q_{t_0}^T} (F^2+F_t^2)(\zeta^2 e^{2s\Phi(t,x)} + \xi^2 e^{2s\varphi(t,x)})\,dx\,dt \\ &\leqslant \frac{C}{\sqrt{s}} \int_0^1 F^2(T',x)(\zeta^2 e^{2s\Phi(T',x)} + \xi^2 e^{2s\varphi(T',x)})dx. \end{aligned}$$

The purpose of the first step is then achieved.

Step 2. Now, let us show that there exists a constant $C > 0$ such that

$$\int_0^1 (y(T',x)+b_{12}v(T',x))^2(\zeta^2 e^{2s\Phi(T',x)}+\xi^2 e^{2s\varphi(T',x)})dx \leqslant C(J_1(F,u)+J_1(F_t,y)). \tag{4.7}$$

Since, for a.e. $x \in (0,1)$,

$$\lim_{t\to t_0}(y(t,x)+b_{12}v(t,x))^2(\zeta^2 e^{2s\Phi(t,x)}+\xi^2 e^{2s\varphi(t,x)}) = 0.$$

Hence

$$\begin{aligned}
&\int_0^1 (y(T',x)+b_{12}v(T',x))^2(\zeta^2 e^{2s\Phi(T',x)}+\xi^2 e^{2s\varphi(T',x)})dx \\
&= \int_0^1\int_{t_0}^{T'} \frac{\partial}{\partial t}\Big((y+b_{12}v)^2(\zeta^2 e^{2s\Phi(t,x)}+\xi^2 e^{2s\varphi(t,x)})\Big)\,dt\,dx \\
&= \int_{t_0}^{T'}\int_0^1 2(y+b_{12}v)(y_t+b_{12}z)(\zeta^2 e^{2s\Phi(t,x)}+\xi^2 e^{2s\varphi(t,x)})\,dx\,dt \\
&\quad + \int_{t_0}^{T'}\int_0^1 (y+b_{12}v)^2(2s\Phi_t\zeta^2 e^{2s\Phi(t,x)}+2s\varphi_t\xi^2 e^{2s\varphi(t,x)})\,dx\,dt.
\end{aligned} \tag{4.8}$$

Using the Young inequality, for s large enough, we obtain

$$\begin{aligned}
&\Big|\int_{t_0}^{T'}\int_0^1 2(y+b_{12}v)(y_t+b_{12}z)(\zeta^2 e^{2s\Phi(t,x)}+\xi^2 e^{2s\varphi(t,x)})\,dx\,dt\Big| \\
&\leqslant \int_{Q_{t_0}^T}\Big(s\theta y^2+s\theta z^2+s\theta v^2+\frac{1}{s\theta}y_t^2\Big)\Big(\zeta^2 e^{2s\Phi(t,x)}+\xi^2 e^{2s\varphi(t,x)}\Big)\,dx\,dt.
\end{aligned} \tag{4.9}$$

On one hand, using Young and Hardy-Poincaré inequalities, we obtain

$$\begin{aligned}
&\int_{Q_{t_0}^T} s\theta(y^2+z^2)\xi^2 e^{2s\varphi(t,x)}\,dt\,dx \leqslant \int_{Q_{t_0}^T} s\theta(x^{\alpha_1-2}y^2+x^{\alpha_2-2}z^2)\xi^2 e^{2s\varphi}\,dt\,dx \\
&\leqslant \int_{Q_{t_0}^T} s\theta\Big\{x^{\alpha_1}[\xi y_x+\xi_x y+s\varphi_x\xi y]^2+x^{\alpha_2}[\xi z_x+\xi_x z+s\varphi_x\xi z]^2\Big\}e^{2s\varphi}\,dt\,dx \\
&\leqslant C\int_{Q_{t_0}^T}\big(s\theta x^{\alpha_1}y_x^2+s^3\theta^3x^{2+\alpha_1-2\beta}y^2+s\theta x^{\alpha_2}z_x^2+s^3\theta^3x^{2+\alpha_2-2\beta}z^2\big)\xi^2 e^{2s\varphi}\,dt\,dx \\
&\quad + C\int_{t_0}^T\int_{\omega''} s\theta(y^2+z^2)e^{2s\varphi}\,dx\,dt.
\end{aligned} \tag{4.10}$$

By the definition of ζ, we obtain

$$\begin{aligned}
&\int_{Q_{t_0}^T} s\theta(y^2+z^2)\zeta^2 e^{2s\Phi(t,x)}\,dt\,dx \\
&\leqslant C\int_{Q_{t_0}^T}\big(s^3\theta^3x^{2+\alpha_1-2\beta}y^2+s^3\theta^3x^{2+\alpha_2-2\beta}z^2\big)\zeta^2 e^{2s\Phi}\,dt\,dx.
\end{aligned} \tag{4.11}$$

Similarly

$$\int_{Q_{t_0}^T} s\theta v^2(\xi^2 e^{2s\varphi(t,x)}+\zeta^2 e^{2s\Phi(t,x)})\,dt\,dx$$

$$\begin{aligned}&\leqslant C\int_{Q_{t_0}^T}\big(s\theta x^{\alpha_2}v_x^2+s^3\theta^3x^{2+\alpha_2-2\beta}v^2\big)(\xi^2e^{2s\varphi}+\zeta^2e^{2s\Phi})\,dt\,dx\\&\quad+C\int_{t_0}^T\int_{\omega''}s\theta v^2e^{2s\varphi}\,dx\,dt.\end{aligned}$$

On the other hand, since $|\theta_t|=|-4\eta\theta^{\frac{5}{4}}|\leqslant C|\eta|\theta^{3/2}$ and $|\Psi|\leqslant|p|$, we have

$$\begin{aligned}&\int_{t_0}^{T'}\int_0^1(y+b_{12}v)^2(2s\Phi_t\zeta^2e^{2s\Phi(t,x)}+2s\varphi_t\xi^2e^{2s\varphi(t,x)})\,dx\,dt\\&\leqslant C\int_{Q_{t_0}^T}s\theta^{3/2}|\eta p|(y^2+v^2)(\xi^2e^{2s\varphi}+\zeta^2e^{2s\Phi})\,dt\,dx.\end{aligned}\tag{4.12}$$

Thus, Lemma 4.1, (4.2), (4.4) and (4.8)-(4.12) yield the estimate (4.7).

Step 3. Combining (4.5) and (4.7), we deduce

$$\begin{aligned}&\int_0^1(y(T',x)+b_{12}v(T',x))^2(\zeta^2e^{2s\Phi(T',x)}+\xi^2e^{2s\varphi(T',x)})dx\\&\leqslant C\Big(\frac{1}{\sqrt{s}}\int_0^1F^2(T',x)(\zeta^2e^{2s\Phi(T',x)}+\xi^2e^{2s\varphi(T',x)})dx+\|u\|^2_{L^2(\omega_{t_0}^T)}+\|u_t\|^2_{L^2(\omega_{t_0}^T)}\Big),\end{aligned}\tag{4.13}$$

Since $y+b_{12}v$ satisfies

$$y(T',x)+b_{12}v(T',x)=(x^{\alpha_1}u_x)_x(T',x)-b_{11}u(T',x)+F(T',x),$$

it follows that

$$\begin{aligned}&\int_0^1F^2(T',x)(\zeta^2e^{2s\Phi(T',x)}+\xi^2e^{2s\varphi(T',x)})dx\\&\leqslant C\Big(\|(x^{\alpha_1}u_x)_x(T')\|^2_{L^2(0,1)}+\|u(T')\|^2_{L^2(0,1)}\\&\quad+\int_0^1(y(T',x)+b_{12}v(T',x))^2(\zeta^2e^{2s\Phi(T',x)}+\xi^2e^{2s\varphi(T',x)})dx\Big).\end{aligned}\tag{4.14}$$

Hence, by (4.13)-(4.14), for s large enough, we obtain

$$\begin{aligned}&\int_0^1F^2(T',x)(\xi^2e^{2s\varphi(T',x)}+\zeta^2e^{2s\Phi(T',x)})dx\\&\leqslant C\Big(\|u\|^2_{H^1(t_0,T;L^2(\omega)}+\|u(T')\|^2_{L^2(0,1)}+\|(x^{\alpha_1}u_x)_x(T')\|^2_{L^2(0,1)}\Big).\end{aligned}\tag{4.15}$$

Moreover, by $\frac{1}{2}\leqslant\xi^2+\zeta^2$ and $e^{2s\varphi(T',x)}\leqslant e^{2s\Phi(T',x)}$, then

$$\begin{aligned}&\gamma\int_0^1F^2(T',x)\,dx\\&\leqslant C\Big(\|u\|^2_{H^1(t_0,T;L^2(\omega))}+\|(x^{\alpha_1}u_x)_x(T',\cdot)\|^2_{L^2(0,1)}+\|u(T',\cdot)\|^2_{L^2(0,1)}\Big),\end{aligned}\tag{4.16}$$

where $\gamma=\min_{x\in[0,1]}\{e^{2s\varphi(T',x)}\}$. Thus (4.6) and (4.16) gives the thesis. $\square$

5. Appendix: Cacciopoli inequality

As in [1, 2, 10], we adapt the proof of the Caccioppoli inequality for nonhomogenuous degenerate parabolic equations. Let ω_1 and ω_2 two arbitrary non empty open subsets of $(0,1)$ such that $\overline{\omega}_2 \subset \omega_1$. Consider the equation

$$U_t - (x^\alpha U_x)_x + b(x)U = F_1(t,x), \quad (t,x) \in \omega_1 \times (t_0,T) := Q_{\omega_1}, \tag{5.1}$$

where $F_1 \in L^2(Q_{\omega_1})$ and $b \in L^\infty(Q_{\omega_1})$.

Lemma 5.1. *Let $\mu_1, \mu_2 \in C^2(\overline{\omega}_1, \mathbb{R}^-)$ such that $\frac{4}{3}\mu_2 \leqslant \mu_1 \leqslant \mu_2$. Then, there exists a constant $C > 0$ such that for any solution U of (5.1), one has*

$$\int_{t_0}^{T}\int_{\omega_2} U_x^2 e^{2s\theta(t)\mu_2(x)}\,dt\,dx \leqslant C\Big(\int_{t_0}^{T}\int_{\omega_1} U^2\,dt\,dx + \int_{t_0}^{T}\int_{\omega_1} F_1^2 e^{2s\theta(t)\mu_1(x)}\,dt\,dx\Big), \tag{5.2}$$

where $\theta(t) = \frac{1}{t^k(T-t)^k}$, $k \geqslant 1$.

Proof. Let $\chi \in \mathcal{C}^\infty(0,1)$ such that $\operatorname{supp}\chi \subset \omega_1$ and $\chi \equiv 1$ on ω_2. We have

$$\begin{aligned}
0 &= \int_{t_0}^{T} \frac{d}{dt}\Big[\int_0^1 \chi^2 U^2 e^{2s\theta(t)\mu_2(x)} dx\Big] dt \\
&= -2\int_{Q_{t_0}^T} \chi^2 x^\alpha U_x^2 e^{2s\theta(t)\mu_2(x)}\,dx\,dt + \int_{Q_{t_0}^T} (x^\alpha(\chi^2 e^{2s\theta(t)\mu_2(x)})_x)_x U^2\,dx\,dt \\
&\quad - 2\int_{Q_{t_0}^T} b\chi^2 U^2 e^{2s\theta(t)\mu_2(x)}\,dx\,dt + 2\int_{Q_{t_0}^T} s\varphi_t \chi^2 U^2 e^{2s\theta(t)\mu_2(x)}\,dx\,dt \\
&\quad + 2\int_{Q_{t_0}^T} \chi^2 U F_1 e^{2s\theta(t)\mu_2(x)}\,dx\,dt.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{Q_{t_0}^T} \chi^2 x^\alpha U_x^2 e^{2s\theta(t)\mu_2(x)}\,dx\,dt \\
&= \int_{Q_{t_0}^T} (x^\alpha(\chi^2 e^{2s\theta(t)\mu_2(x)})_x)_x U^2 dx - \int_{Q_{t_0}^T} b\chi^2 U^2 e^{2s\theta(t)\mu_2(x)}\,dx\,dt \\
&\quad + \int_{Q_{t_0}^T} s\varphi_t \chi^2 U^2 e^{2s\theta(t)\mu_2(x)}\,dx\,dt + \int_{Q_{t_0}^T} \chi^2 U F_1 e^{2s\theta(t)\mu_2(x)}\,dx\,dt \\
&\leqslant \int_{Q_{t_0}^T} \Big\{(x^\alpha(\chi^2 e^{2s\theta(t)\mu_2(x)})_x)_x U^2 + \chi^2 U^2 (b e^{2s\theta(t)\mu_2(x)} + s\varphi_t e^{2s\theta(t)\mu_2(x)} \\
&\quad + \frac{1}{2} e^{2s\theta(t)(2\mu_2(x)-\mu_1(x))})\Big\}\,dx\,dt + \frac{1}{2}\int_{Q_{t_0}^T} F_1 e^{2s\theta(t)\mu_1(x)}\,dx\,dt.
\end{aligned}$$

Therefore, since χ is supported in ω_1, $\chi \equiv 1$ in ω_2 and $\frac{4}{3}\mu_2 \leqslant \mu_1 \leqslant \mu_2$. Then, one obtains

$$\begin{aligned}
&\int_{t_0}^{T}\int_{\omega_2} U_x^2 e^{2s\theta(t)\mu_2(x)}\,dx\,dt \\
&\leqslant C\int_{Q_{t_0}^T} \chi^2 x^\alpha U_x^2 e^{2s\theta(t)\mu_2(x)}\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\Big(\int_{t_0}^{T}\!\!\int_{\omega_1} s^2\theta^2U^2e^{2s\theta(t)(2\mu_2(x)-\mu_1(x))}\,dx\,dt+\int_{t_0}^{T}\!\!\int_{\omega_1} F_1^2e^{2s\theta(t)\mu_1(x)}\,dx\,dt\Big)\\
&\leqslant C\Big(\int_{t_0}^{T}\!\!\int_{\omega_1} U^2\,dx\,dt+\int_{t_0}^{T}\!\!\int_{\omega_1} F_1^2e^{2s\theta(t)\mu_1(x)}\,dx\,dt\Big).
\end{aligned}
$$

This completes the proof. □

References

[1] E. M. Ait Ben Hassi, F. Ammar Khodja, A. Hajjaj, L. Maniar; *Null controllability of degenerate parabolic cascade systems Portugaliae Mathematica*, **68** (2011), 345-367.

[2] E. M. Ait Ben Hassi, F. Ammar Khodja, A. Hajjaj, L. Maniar; *Carleman estimate and null controllability of degenerate parabolic systems*, Evol. Equ. Control Theory, **2** (2013), 441-454.

[3] F. Ammar Khodja, A. Benabdellah, C. Dupaix; *Null-controllability for some reaction-diffusion systems with one control force*, J. Math. Anal. Appl., **320** (2006), 928-943.

[4] F. Ammar Khodja, A. Benabdallah, C. Dupaix, I. Kostin; *Null-controllability of some systems of parabolic type by one control force*, ESAIM Control Optim. Calc. Var., **11** (2005), 426-448.

[5] A. Benabdallah, M. Cristofol, P. Gaitan, M. Yamamoto; *Inverse problem for a parabolic system with two components by measurements of one component*, App. Ana., **88** (2009), 683-709.

[6] A. Benabdallah, M. Cristofol, P. Gaitan, L. De Teresa; *A new Carleman inequality for parabolic systems with a single observation and applications*, C. R. Math. Acad. Sci. Paris, **348** (2010), 25-29.

[7] A. L. Bukhgeim, M. V. Klibanov; *Uniqueness in the large of a class of multidimensional inverse problems*, Soviet Math. Dokl., **17** (1981), 244-247.

[8] M. Campiti, G. Metafune, D. Pallara; *Degenerate self-adjoint evolution equations on the unit interval*, Semigroup Forum, **57** (1998), 1-36.

[9] P. Cannarsa, J. Tort, M. Yamamoto; *Determination of source terms in a degenerate parabolic equation*, Inv. Prob., **26** (2010), 105003 (20pp).

[10] P. Cannarsa, L. De Teresa; *Controllability of 1-d coupled degenerate parabolic equations*, Electron. J. Dif. Equ., **73** (2009), 1-21.

[11] P. Cannarsa, P. Martinez, J. Vancostenoble; *Null controllability of degenerate heat equations*, Adv. Dif. Equ., **10** (2005), 153-190.

[12] P. Cannarsa, G. Fragnelli; *Null controllability of semilinear degenerate parabolic equations in bounded domains*, Electron. J. Dif. Equ., **136** (2006), 1-20.

[13] P. Cannarsa, P. Martinez, J. Vancostenoble; *Carleman estimates for a class of degenerate parabolic operators*, SIAM, J. Control Optim., **47** (2008), 1-19.

[14] M. Cristofol, P. Gaitan, H. Ramoul; *Inverse problems for a 2×2 reaction-diffusion system using a Carleman estimate with one observation*, Inv Prob., **22** (2006), 1561-1573.

[15] M. Cristofol, P. Gaitan, H. Ramoul, M. Yamamoto; *Identification of two independent coefficients with one observation for a nonlinear parabolic system App. Ana.*, **91**, (2012) 2073-2081.

[16] E. Fernández-Cara, S. Guerrero, *Global Carleman inequalities for parabolic systems and applications to controllability*, SIAM J. Control Optim., **45** (2006), 1399-1446.

[17] E. Fernández-Cara, M. González-Burgos, S. Guerrero, J. P. Puel; *Null controllability of the heat equation with boundary Fourier conditions: the linear case*, ESAIM Control Optim. Calc. Var., **12** (2006), 442-465.

[18] A. V. Fursikov, O. Y. Imanuvilov; *Controllability of evolution equations*, (Lectures notes series), **34**, (1996) (Seoul: National University Research Center, Korea)

[19] S. Guerrero, *Null controllability of some systems of two parabolic equations with one control force*, SIAM J. Control Optim., **46** (2007), 379-394.

[20] O. Y. Imanuvilov, M. Yamamoto; *Lipschitz stability in inverse parabolic problem by Carleman estimates*, Inv Prob., **14** (1998), 1229-1245.

[21] X. Liu, H. Gao, P. Lin; *Null controllability of a cascade system of degenerate parabolic equations*, Acta Math. Sci. Ser. A Chin. Ed., **28**, (2008) 985-996.

[22] G. Lebeau, L. Robbiano; *Contrôle exact de l'équation de la chaleur*, Comm. in PDE., **20** (1995), 335-356.

[23] P. Martinez, J. Vancostenoble; *Carleman estimates for one-dimensional degenerate heat equations*, J. Evol. Equ., **6**, (2006) 325-362.
[24] L. Roques, M. Cristofol; *On the determination of the nonlinearity from localized measurements in a reaction-diffusion equation Nonlinearity*, **23** (2010), 675-686.
[25] L. Roques, M. Cristofol; *The inverse problem of determining several coefficients in a nonlinear Lotka-Volterra system*, Inv Prob. 28 (2012), Issue 7, Article number075007, doi:10.1088/0266-5611/28/7/075007.
[26] D. L. Russell; *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations*, Studies in Applied Mathematics, **52** (1973), 189-221.
[27] J. Tort, J. Vancostenoble; *Determination of the insolation function in the nonlinear Sellers climate model*, Ann. I. H. Poincaré-AN., **29** (2012), 638-713.
[28] J. Tort; *Determination of source terms in a degenerate parabolic equation from a locally distributed observation*, C. R. Math. Acad. Sci. Paris, **348**, (2010) 1287-1291.
[29] J. Tort, *An inverse diffusion problem in a degenerate parabolic equation*, Monografias de la Real Academia de Ciencias de Zaragoza, **38**, (2012) 137-145.

Idriss Boutaayamou
Département de Mathématiques, Faculté des Sciences Semlalia, LMDP, UMMISCO (IRD-UPMC), Université Cadi Ayyad, Marrakech 40000, B.P. 2390, Morocco
E-mail address: dsboutaayamou@gmail.com

Abdelkarim Hajjaj
Faculté des Sciences et Techniques, Université Hassan 1er, Laboratoire MISI, B.P. 577, Settat 26000, Morocco
E-mail address: abdelkarim.hajjaj@uhp.ac.ma

Lahcen Maniar
Département de Mathématiques, Faculté des Sciences Semlalia, LMDP, UMMISCO (IRD-UPMC), Université Cadi Ayyad, Marrakech 40000, B.P. 2390, Morocco
E-mail address: maniar@uca.ma