Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 153, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

STABILITY OF SOLITARY WAVES FOR A THREE-WAVE INTERACTION MODEL

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Abstract. In this article we consider the normalized one-dimensional three-wave interaction model

$$\begin{aligned} i\frac{\partial z_1}{\partial t} &= -\frac{d^2 z_1}{dx^2} - z_3\bar{z}_2 \\ i\frac{\partial z_2}{\partial t} &= -\frac{d^2 z_2}{dx^2} - z_3\bar{z}_1 \\ i\frac{\partial z_3}{\partial t} &= -\frac{d^2 z_3}{dx^2} - z_1 z_2. \end{aligned}$$

Solitary waves for this model are solutions of the form

$$z_1(t,x) = e^{i\omega_1 t}u_1(x) \quad z_2(t,x) = e^{i\omega_2 t}u_2(x) \quad z_3(t,x) = e^{i(\omega_1+\omega_2)t}u_3(x),$$

where ω_1 and ω_2 are positive frequencies, and $u_i(x)$, $i = 1,2,3$ are real-valued functions that satisfy the ODE system

$$\begin{aligned} -\frac{d^2 u_1}{dx^2} - u_2u_3 + \omega_1 u_1 &= 0 \\ -\frac{d^2 u_2}{dx^2} - u_1u_3 + \omega_2 u_2 &= 0 \\ -\frac{d^2 u_3}{dx^2} - u_1u_2 + (\omega_1+\omega_2)u_3 &= 0. \end{aligned}$$

For the case $\omega_1 = \omega_2 = \omega$, we prove existence, uniqueness and stability of solitary waves corresponding to positive solutions $u_i(x)$ that tend to zero as x tends to infinity.

The full model has more parameters, and the case we consider corresponds to the exact phase matching. However, as we will see, even in the simpler case, a formal proof of stability depends on a nontrivial spectral analysis of the linearized operator. This is so because the spectral analysis depends on some calculations on a full neighborhood of the parameter (ω,ω) and the solution is not known explicitly.

1. Introduction and statement of results

In this article we consider the normalized one-dimensional three-wave interaction model presented in [1]:

2000 *Mathematics Subject Classification.* 34A34.
Key words and phrases. Dispersive equations; variational methods; stability.

Submitted January 10, 2013. Published June 30, 2014.

$$
\begin{aligned}
i\frac{\partial z_1}{\partial t} &= -\frac{d^2 z_1}{dx^2} - z_3\bar{z}_2\\
i\frac{\partial z_2}{\partial t} &= -\frac{d^2 z_2}{dx^2} - z_3\bar{z}_1\\
i\frac{\partial z_3}{\partial t} &= -\frac{d^2 z_3}{dx^2} - z_1 z_2.
\end{aligned} \tag{1.1}
$$

Here, $x \in \mathbb{R}$, $z_i(t,x)$, $i = 1, 2, 3$ are complex values and $\bar{z}_i$ denotes the conjugate of z_i.

In [2] a model with more nonlinear terms and more space variables is analyzed. In that case, the authors are able to study the stability/instability of solitary waves with only one nonzero component. In [3], the stability of a semitrivial standing wave for a system of two Schrodinger equations in several space variables is discussed. The model considered here is this paper is one dimensional in the space variable and the components of the solitary waves are nonzero.

System (1.1) has the following conserved quantities:

$$
E^0(z_1, z_2, z_3) = \frac{1}{2}\sum_{i=1}^{i=3}\int_{-\infty}^{+\infty} |\frac{dz_i(x)}{dx}|^2\,dx - \mathrm{Re}\Big(\int_{-\infty}^{+\infty} z_1(x)z_2(x)\bar{z}_3(x)\,dx\Big) \tag{1.2}
$$

$$
Q_1^0(z_1, z_3) = \frac{1}{2}\int_{-\infty}^{+\infty} (|z_1(x)|^2 + |z_3(x)|^2)\,dx \tag{1.3}
$$

$$
Q_2^0(z_2, z_3) = \frac{1}{2}\int_{-\infty}^{+\infty} (|z_2(x)|^2 + |z_3(x)|^2)\,dx \tag{1.4}
$$

Solitary waves of (1.1) are solutions of the form

$$
z_1(t,x) = e^{i\omega_1 t}u_1(x) \quad z_2(t,x) = e^{i\omega_2 t}u_2(x) \quad z_3(t,x) = e^{i(\omega_1+\omega_2)t}u_3(x)\,, \tag{1.5}
$$

where the frequencies ω_I are positive values, and $u_i(x)$ are real-value functions for $i = 1, 2, 3$. Therefore, the $u_i(x)$s have to satisfy the ODE system

$$
\begin{aligned}
-\frac{d^2u_1}{dx^2} - u_2u_3 + \omega_1 u_1 = 0\\
-\frac{d^2u_2}{dx^2} - u_1u_3 + \omega_2 u_2 = 0\\
-\frac{d^2u_3}{dx^2} - u_1u_2 + (\omega_1+\omega_2)u_3 = 0\,.
\end{aligned} \tag{1.6}
$$

Defining

$$
E(u_1, u_2, u_3) = \frac{1}{2}\sum_{i=1}^{i=3}\int_{-\infty}^{+\infty} \big(\frac{du_i(x)}{dx}\big)^2\,dx - \int_{-\infty}^{+\infty} u_1(x)u_2(x)u_3(x)\,dx \tag{1.7}
$$

$$
Q_1(u_1, u_3) = \frac{1}{2}\int_{-\infty}^{+\infty} (u_1^2(x) + u_3^2(x))\,dx \tag{1.8}
$$

$$
Q_2(u_2, u_3) = \frac{1}{2}\int_{-\infty}^{+\infty} (u_2^2(x) + u_3^2(x))\,dx \tag{1.9}
$$

we see that solutions of (1.6) are critical points of

$$
E(u_1, u_2, u_3) + \omega_1 Q_1(u_1, u_3) + \omega_2 Q_2(u_2, u_3).
$$

By a *positive solution* of system (1.6) we mean a solution $(u_1(x), u_2(x), u_3(x))$ defined for all $x \in \mathbb{R}$ such that $u_i(x) > 0$ for all x, and $u_i(x)$ tends to zero exponentially as $|x|$ approaches infinity, $i = 1, 2, 3$ (this implies that the derivatives also tend to zero).

Let $H = H^1(\mathbb{R}, \mathbb{C}) \times H^1(\mathbb{R}, \mathbb{C}) \times H^1(\mathbb{R}, \mathbb{C})$ be the space of the complex valued functions $z(x) = (z_1(x), z_2(x), z_3(x))$ defined for $x \in \mathbb{R}$ with norm

$$\|z\|^2 = \sum_{i=1}^{3} \int_{-\infty}^{+\infty} |\frac{dz_i(x)}{dx}|^2 \, dx + \sum_{i=1}^{3} \int_{-\infty}^{+\infty} |z_i(x)|^2 \, dx.$$

We denote by $u(x) = (u_1(x), u_2(x), u_3(x))$ a solution of (1.6) in the space H.

Definition 1.1. The solitary wave (1.5) is orbitally stable with respect to system (1.1) if for each $\epsilon > 0$ there is a $\delta > 0$ such that if $z_0 \in H$ and $\|z_0 - u\| < \delta$ then the solution $z(t)$ of (1.1) with $z(0) = z_0$ satisfies

$$\sup_{-\infty<t<+\infty} \inf\{\|z(t) - (e^{i\theta_1} u_1(\cdot + c), e^{i\theta_2} u_2(\cdot + c), e^{i(\theta_1+\theta_2)} u_3(\cdot + c)\|, \theta_1, \theta_2, c \in \mathbb{R}\}) < \epsilon.$$

In the definition of orbital stability, the supremum is taken over $-\infty < t < +\infty$ because we are dealing with conservative systems and the Cauchy problem is well posed for all values of t. Next we state our main results.

Theorem 1.2. *The following assertions hold:*

(1) For any $\omega_1, \omega_2 > 0$ system (1.6) *has a positive solution that tends to zero exponentially.*

(2) Except for a translation in the x variable (the same translation for all components), any positive solution of (1.6) *is symmetric and decreasing.*

(3) If $\omega_1 = \omega_2$ then the solution (u_1, u_2, u_3) given in part one satisfies $u_1 = u_2$, it is unique and the linearized operator $L = (L_1, L_2, L_3)$ where

$$\begin{aligned} L_1(h_1, h_2, h_3) &= -\frac{d^2 h_1}{dx^2} - u_3 h_2 - u_2 h_3 + \omega_1 h_1 \\ L_2(h_1, h_2, h_3) &= -\frac{d^2 h_2}{dx^2} - u_3 h_1 - u_1 h_3 + \omega_2 h_2 \\ L_3(h_1, h_2, h_3) &= -\frac{d^2 h_3}{dx^2} - u_2 h_1 - u_1 h_2 + \omega_3 h_3 \end{aligned} \tag{1.10}$$

has zero as a simple eigenvalue corresponding to $(u_1'(x), u_2'(x), u_3'(x))$, as eigenfunction, and it has exactly one negative eigenvalue. Moreover, such a solution gives rise to an orbitally stable solitary wave of the evolution system (1.1).

Remark 1.3. In the case $\omega_1 = \omega_2 = \omega$, $u_1 = u_2 = u$, $u_3 = v$ system (1.6) becomes

$$\begin{aligned} -\frac{d^2 u}{dx^2} - uv + \omega u = 0 \\ -\frac{d^2 v}{dx^2} - u^2 + 2\omega v = 0. \end{aligned} \tag{1.11}$$

System (1.11) possesses no explicit solutions (u, v) of the form $(\text{sech}^2, \text{sech}^2)$, $(\text{sech}^2, \text{sech})$, $(\text{sech}, \text{sech}^2)$, $(\text{sech}, \text{sech})$. In [5] a model with more parameters is considered. In that case, explicit solutions are given. However, in the case we are considering here, those solutions become that trivial one.

2. Proof of main results

The proof of Throem 1.2 will be broken in several lemmas the first of which deals with the existence of positive solution.

Lemma 2.1. *System* (1.6) *has a C^∞ positive solution that tends to zero exponentially as x tends to infinity.*

Proof. For $u_i \in H^1(\mathbb{R})$, $i = 1, 2, 3$, we minimize $E(u_1, u_2, u_3)$ under

$$\omega_1 Q_1(u_1, u_2, u_3) + \omega_2 Q_2(u_1, u_2, u_3) = 1,$$

where $E(u_1, u_2, u_3), Q_1(u_1, u_2, u_3)$ and $Q_2(u_1, u_2, u_3)$ are defined by (1.7), (1.8) and (1.9). The existence of a minimizer follows from the method of concentration compactness ([7]). The corresponding Euler-Lagrange equation has a multiplier that can be absorbed by a scaling argument. Since $E(u_1, u_2, u_3)$ does not increase if we replace (u_1, u_2, u_3) by $(|u_1|, |u_2|, |u_3|)$ we can assume that the components are nonnegative. The maximum principle implies that each component is actually strictly positive. The exponential decay follows from linearization at $(0, 0, 0)$.

The assertion concerning the symmetry is a Gidas-Ni-Nirenberg-Troy-type result and its proof in the one dimensional case has been given in [6]. This completes the proof. $\square$

Lemma 2.2. *If $\omega_1 = \omega_2 = \omega$ then the solution (u_1, u_2, u_3) given by the previous lemma satisfies $u_1 = u_2$ and it is unique. Moreover the linearized operator $L = (L_1, L_2, L_3)$ given by* (1.10) *at that solution has zero as a simple eigenvalue corresponding to the eigenfunction $(u_1'(x), u_2'(x), u_3'(x))$ and it has exactly one negative eigenvalue.*

Proof. If $\omega_1 = \omega_2 = \omega$, systems (1.6) becomes

$$\begin{aligned} -\frac{d^2u_1}{dx^2} - u_2u_3 + \omega u_1 = 0 \\ -\frac{d^2u_2}{dx^2} - u_1u_3 + \omega u_2 = 0 \\ -\frac{d^2u_3}{dx^2} - u_1u_2 + 2\omega u_3 = 0 \end{aligned} \tag{2.1}$$

and then

$$-\frac{d^2(u_1 - u_2)}{dx^2} + u_3(u_1 - u_2) + \omega(u_1 - u_2) = 0.$$

If we multiply this last equality by $(u_1 - u_2)$ and integrate we see that we must have $u_1 = u_2$. Setting $u_1 = u_2 = u$ and $u_3 = v$, we get the system

$$\begin{aligned} -\frac{d^2u}{dx^2} - uv + \omega u = 0 \\ -\frac{d^2v}{dx^2} - u^2 + 2\omega v = 0. \end{aligned} \tag{2.2}$$

Notice that system (1.6) is variational but (2.2) is not. We fix that defining $U = \sqrt{2}u, V = v$. Then (2.2) takes the variational form

$$\begin{aligned} -\frac{d^2U}{dx^2} - UV + \omega U = 0 \\ -\frac{d^2V}{dx^2} - \frac{U^2}{2} + 2\omega V = 0. \end{aligned} \tag{2.3}$$

Next we define the linearized operator $M(h,k) = (M_1(h,k), M_2(h,k)$ of (2.3) where

$$\begin{aligned} M_1(h,k) &= -\frac{d^2h}{dx^2} - Uk - Vh + \omega h \\ M_2(h,k) &= -\frac{d^2k}{dx^2} - Uh + 2\omega k. \end{aligned} \tag{2.4}$$

According to [8] and [9], the positive solution of (2.3) is unique, the linearized operator $M = (M_1, M_2)$ has zero as a simple eigenvalue corresponding to the eigenfunction (U', V') and it has exactly one negative eigenvalue.

Now let $\lambda \le 0$ be an eigenvalue of $L = (L_1, L_2, L_3)$ defined by (1.10), with eigenfunction (h_1, h_2, h_3). Then

$$\begin{aligned} -\frac{d^2h_1}{dx^2} - vh_2 - uh_3 + \omega h_1 - \lambda h_1 &= 0 \\ -\frac{d^2h_2}{dx^2} - vh_1 - uh_3 + \omega h_2 - \lambda h_2 &= 0. \\ -\frac{d^2h_3}{dx^2} - uh_1 - uh_2 + 2\omega h_3 - \lambda h_3 &= 0. \end{aligned} \tag{2.5}$$

Defining $p = h_1 - h_2$ and using the first two equations of (2.5) we get

$$-\frac{d^2p}{dx^2} + vp + \omega p - \lambda p = 0.$$

Multiplying this last equation by p and integrating we get $p = 0$ (because $\lambda \le 0$). In other words, if $\lambda \le 0$ is an eigenvalue of L with eigenfunction (h_1, h_2, h_3) then we must have $h_1 = h_2$. Setting $h_1 = h_2 = h$ and $h_3 = k$, system (2.5) becomes

$$\begin{aligned} -\frac{d^2h}{dx^2} - vh - uk + \omega h - \lambda h &= 0 \\ -\frac{d^2k}{dx^2} - 2uh + 2\omega k - \lambda k &= 0. \end{aligned} \tag{2.6}$$

As before, (2.6) is not selfadjoint and then we define $H = \sqrt{2}h$ and $K = k$, and (2.6) becomes

$$\begin{aligned} -\frac{d^2H}{dx^2} - UK - VH + \omega H - \lambda H &= 0 \\ -\frac{d^2K}{dx^2} - UH + 2\omega K - \lambda K &= 0. \end{aligned} \tag{2.7}$$

Notice that (2.7) is precisely the equation for the eigenvalues of the linearized operator $M = (M_1, M_2)$ defined by (2.4). The conclusion is: if $\omega_1 = \omega_2$ and $\lambda \le 0$ is an eigenvalue of $L = (L_1, L_2, L_3)$ defined by (1.10) with eigenfunctions (h_1, h_2, h_3), then $h_1 = h_2$ and λ is an eigenvalue of $M = (M_1, M_2)$ defined by (2.4) with eigenfunction $(h_1/\sqrt{2}, h_3)$. Therefore, the spectral properties of L claimed in lemma follow from the spectral properties of M stated above. The proof is complete. □

Next we discuss the stability of the solitary wave in the sense of Definition 1.1. If we fix an $\omega > 0$, then according to Lemma 2.2, zero is a simple eigenvalue of the operator L and the corresponding eigenfuntion is odd. Since the coefficients of L are even, the set of even functions is invariant under L. Consequently, L is invertible in the class of even functions because the only eigenfunction of L corresponding to the zero eigenvalue is odd. Therefore, from the implicit function theorem, there is a smooth family $u_i(\omega_1,\omega_2), i = 1,2,3$ of positive symmetric solution of (1.6) for (ω_1,ω_2) in a neighborhood of (ω,ω). We define

$$Q_1(\omega_1,\omega_2) = Q_1(u_1(\omega_1,\omega_2), u_3(\omega_1,\omega_2))$$

and

$$Q_2(\omega_1,\omega_2) = Q_2(u_2(\omega_1,\omega_2), u_3(\omega_1,\omega_2)),$$

where $Q_1(u_1,u_3)$ and $Q_2(u_2,u_3)$ are defined by (1.8) and (1.9), respectively. According to [4] and due to the spectral properties of the operator L, the solitary wave (1.5) is orbitally stable provided the matrix

$$A(\omega_1,\omega_2) = \begin{pmatrix} \frac{\partial Q_1(\omega_1,\omega_2)}{\partial\omega_1} & \frac{\partial Q_1(\omega_1,\omega_2)}{\partial\omega_2} \\ \frac{\partial Q_2(\omega_1,\omega_1)}{\partial\omega_1} & \frac{\partial Q_2(\omega_1,\omega_2)}{\partial\omega_2} \end{pmatrix} \tag{2.8}$$

has exactly one negative eigenvalue; that is, if

$$\det A(\omega_1,\omega_2) < 0. \tag{2.9}$$

As we have seen, for $\omega_1 = \omega_2 = \omega$, the positive symmetric solution of (1.6) is (u_1,u_2,u_3) with $u_1 = u_2 = u$, $u_3 = v$ and

$$\begin{gathered} -\frac{d^2u}{dx^2} - uv + \omega u = 0 \\ -\frac{d^2v}{dx^2} - u^2 + 2\omega v = 0. \end{gathered} \tag{2.10}$$

If we denote by (ϕ,ψ) the solution of (2.10) corresponding to $\omega = 1$, then the unique positive symmetric solution of (2.10) is

$$u(x) = \omega\phi(\sqrt{\omega}x), , v(x) = \omega\psi(\sqrt{\omega}x).$$

If we set

$$I = \int_{-\infty}^{+\infty} (\phi(x)^2 + \psi(x)^2)\,dx$$

then

$$Q_1(\omega,\omega) = Q_2(\omega,\omega) = \omega^{3/2} I. \tag{2.11}$$

Differentiating (2.11) with respect to ω we get

$$\begin{gathered} \frac{\partial Q_1(\omega,\omega)}{\partial\omega_1} + \frac{\partial Q_1(\omega,\omega)}{\partial\omega_2} = \frac{3}{2}\omega^{1/2} I \\ \frac{\partial Q_2(\omega,\omega)}{\partial\omega_1} + \frac{\partial Q_2(\omega,\omega)}{\partial\omega_2} = \frac{3}{2}\omega^{1/2} I. \end{gathered} \tag{2.12}$$

Remark 2.3. Notice that even in the case $\omega_1 = \omega_2$, if the quantities $Q_i(\beta_1,\beta_2), i = 1,2$ were known explicitly in terms of β_1 and β_2 in a full neighborhood of (ω,ω), then the verification of condition (2.9) would be easy. As we will see, the matrix $A(\omega_1,\omega_2)$ is symmetric. Therefore, the scaling invariance gives us two equations (2.12) involving three quantities. Due to that, the verification of (2.9) requires further analysis that will be carried out next.

Define

$$U_{11}(x) = \frac{\partial u_1(x,\omega,\omega)}{\partial\omega_1}, \quad U_{12}(x) = \frac{\partial u_1(x,\omega,\omega)}{\partial\omega_2},$$
$$U_{21}(x) = \frac{\partial u_2(x,\omega,\omega)}{\partial\omega_1}, \quad U_{22}(x) = \frac{\partial u_2(x,\omega,\omega)}{\partial\omega_2},$$

$$U_{31}(x) = \frac{\partial u_3}{\partial \omega_1}, \quad U_{32} = \frac{\partial u_3}{\partial \omega_2},$$

and differentiate with respect to ω_1 and ω_2 in a neighborhood of (ω, ω). We obtain

$$\begin{gathered}
-\frac{d^2U_{11}}{dx^2} - u_3U_{21} - u_2U_{31} + \omega_1U_{11} = -u_1 \\
-\frac{d^2U_{21}}{dx^2} - u_3U_{11} - u_1U_{31} + \omega_2U_{21} = 0 \\
-\frac{d^2U_{31}}{dx^2} - u_1U_{21} - u_2U_{11} + (\omega_1 + \omega_2)U_{31} = -u_3
\end{gathered} \tag{2.13}$$

and

$$\begin{gathered}
-\frac{d^2U_{12}}{dx^2} - u_3U_{22} - u_2U_{32} + \omega_1U_{12} = 0 \\
-\frac{d^2U_{22}}{dx^2} - u_3U_{12} - u_1U_{32} + \omega_2U_{22} = -u_2 \\
-\frac{d^2U_{32}}{dx^2} - u_1U_{22} - u_2U_{12} + (\omega_1 + \omega_2)U_{32} = -u_3
\end{gathered} \tag{2.14}$$

Setting $\omega_1 = \omega_2 = \omega$, $u_1 = u_2 = u$ and $u_3 = v$, Equations (2.13) and (2.14) become

$$\begin{gathered}
-\frac{d^2U_{11}}{dx^2} - vU_{21} - uU_{31} + \omega U_{11} = -u \\
-\frac{d^2U_{21}}{dx^2} - vU_{11} - uU_{31} + \omega U_{21} = 0 \\
-\frac{d^2U_{31}}{dx^2} - uU_{21} - uU_{11} + 2\omega U_{31} = -v
\end{gathered} \tag{2.15}$$

and

$$\begin{gathered}
-\frac{d^2U_{12}}{dx^2} - vU_{22} - uU_{32} + \omega U_{12} = 0 \\
-\frac{d^2U_{22}}{dx^2} - vU_{12} - uU_{32} + \omega U_{22} = -u \\
-\frac{d^2U_{32}}{dx^2} - uU_{22} - uU_{12} + 2\omega U_{32} = -v.
\end{gathered} \tag{2.16}$$

Interchanging the first two equations of (2.16) we obtain

$$\begin{gathered}
-\frac{d^2U_{22}}{dx^2} - vU_{12} - uU_{32} + \omega U_{22} = -u \\
-\frac{d^2U_{12}}{dx^2} - vU_{22} - uU_{32} + \omega U_{12} = 0 \\
-\frac{d^2U_{32}}{dx^2} - uU_{22} - uU_{12} + 2\omega U_{32} = -v.
\end{gathered} \tag{2.17}$$

Comparing (2.17) and (2.15), we see that we must have

$$U_{11} = U_{22}, \quad U_{21} = U_{12}, \quad U_{31} = U_{32}, \tag{2.18}$$

because, as we have seen, the operator L is invertible in the space of even functions.

Furthermore, from (1.8) and (1.9) we have

$$\frac{\partial Q_1(\omega_1, \omega_2)}{\partial \omega_1} = 2\int_{-\infty}^{\infty} (u_1U_{11} + u_3U_{31})\, dx, \quad \frac{\partial Q_1(\omega_1, \omega_2)}{\partial \omega_2} = 2\int_{-\infty}^{\infty} (u_1U_{12} + u_3U_{32}),$$

$$\frac{\partial Q_2(\omega_1,\omega_2)}{\partial \omega_1} = 2\int_{-\infty}^{\infty}(u_2U_{21}+u_3U_{31})\,dx \quad \frac{\partial Q_2(\omega_1,\omega_2)}{\partial \omega_2} = 2\int_{-\infty}^{\infty}(u_2U_{22}+u_3U_{32}).$$

Setting $\omega_1=\omega_2=\omega$, $u_1=u_2=u$ and $u_3=v$ and using (2.18) we have

$$\begin{gathered}
\frac{\partial Q_1(\omega,\omega)}{\partial \omega_1} = 2\int_{-\infty}^{\infty}(uU_{11}+vU_{31})\,dx, \quad \frac{\partial Q_1(\omega,\omega)}{\partial \omega_2} = 2\int_{-\infty}^{\infty}(uU_{12}+vU_{31}),\\
\frac{\partial Q_2(\omega,\omega)}{\partial \omega_1} = 2\int_{-\infty}^{\infty}(uU_{12}+vU_{31})\,dx, \quad \frac{\partial Q_2(\omega,\omega))}{\partial \omega_2} = 2\int_{-\infty}^{\infty}(uU_{11}+vU_{31}).
\end{gathered}$$

We conclude that

$$\begin{gathered}
\frac{\partial Q_1(\omega,\omega)}{\partial \omega_1} = \frac{\partial Q_2(\omega,\omega)}{\partial \omega_2},\\
\frac{\partial Q_1(\omega,\omega)}{\partial \omega_2} = \frac{\partial Q_2(\omega,\omega)}{\partial \omega_1}.
\end{gathered}$$

This second equality we already knew because the matrix $A(\omega_1,\omega_2)$ is symmetric. Then

$$\begin{aligned}
\det(A(\omega,\omega)) &= \Big(\frac{\partial Q_1(\omega,\omega)}{\partial \omega_1}\Big)^2 - \Big(\frac{\partial Q_1(\omega,\omega)}{\partial \omega_2}\Big)^2\\
&= \Big(\frac{\partial Q_1(\omega,\omega)}{\partial \omega_1} - \frac{\partial Q_1(\omega,\omega)}{\partial \omega_2}\Big)\Big(\frac{\partial Q_1(\omega,\omega)}{\partial \omega_1} + \frac{\partial Q_1(\omega,\omega)}{\partial \omega_2}\Big).
\end{aligned}$$

From (2.12) we conclude that

$$\frac{\partial Q_1(\omega,\omega)}{\partial \omega_1} + \frac{\partial Q_1(\omega,\omega)}{\partial \omega_2} > 0.$$

Therefore, to show that $det(A(\omega,\omega)) < 0$ we have to show that

$$\int_{-\infty}^{\infty} u(U_{11}-U_{12})\,dx < 0. \tag{2.19}$$

Defining $W = U_{11}-U_{12}$, from the first two equations (2.15) and taking in account that $U_{21}=U_{12}$ we see that

$$-\frac{d^2W}{dx^2} + vW + \omega W = -u$$

and this implies $W<0$ (because W cannot have a positive maximum). The proof of Theorem 1.2 is complete.

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