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SOLVABILITY OF NONLOCAL BOUNDARY-VALUE PROBLEMS FOR THE LAPLACE EQUATION IN THE BALL

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Abstract. In this article, we consider a class of nonlocal problems for the Laplace equation with boundary operators of fractional order. We prove the existence, uniqueness and a representation of the solutions. Also it is shown that the smoothness of solutions in Holder classes depends on the order of the boundary operators.

1. Introduction

Let $\Omega = \{x \in R^n : |x| < 1\}$ be the unit ball, and $\partial\Omega = \{x \in R^n : |x| = 1\}$ be a unit sphere. Further let, $u(x)$ be a harmonic function in the ball Ω, $r = |x|$, $\theta = x/|x|$,

$$\frac{d}{dr} = \sum_{j=1}^{n} \frac{x_j}{|x|} \frac{\partial}{\partial x_j}.$$

For an arbitrary positive number $\alpha > 0$, the operator of fractional integration in the Riemann-Liouville sense of order α is the expression [11]:

$$I^{\alpha}[u](x) = \frac{1}{\Gamma(\alpha)} \int_0^r (r-\tau)^{\alpha-1} u(\tau,\varphi)\, d\tau, \quad r > 0.$$

Since $I^{\alpha}[u](x) \to u(x)$ almost everywhere as $\alpha \to 0$, by definition we suppose $I^0[u](x) = u(x)$.

A fractional differentiation operator is naturally defined as the product of a fractional integration operator and differentiation operator of integer order. Thus, depending on the sequence of multiplication of operators their properties are changed. The most famous operator of the fractional differentiation is the Riemann-Liouville operator [11],

$${}_{RL}D^{\alpha}[u](x) = \frac{d}{dr} I^{1-\alpha}[u](x), \quad 0 < \alpha \le 1.$$

In another order of multiplication we obtain fractional differentiation operator in the sense of Caputo [11],

$${}_{C}D^{\alpha}[u](x) = I^{1-\alpha}[u'](x), \quad 0 < \alpha \le 1.$$

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Denote

$$B^{\alpha}[u](x) = r^{\alpha}{}_{RL}D^{\alpha}[u](x),$$
$$B_*^{\alpha}[u](x) = r^{\alpha}{}_{C}D^{\alpha}[u](x),$$
$$B^{-\alpha}[u](x) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}s^{-\alpha}u(sx)\,ds.$$

Note, that properties and some applications of the operators B^{α}, B_*^{α} and $B^{-\alpha}$ to solvability questions of the local and nonlocal boundary-value problems were studied in [10, 20].

The organization of this article is as follows. In Section 2 we give the formulation of the basic problems and some historical information about boundary-value problems with boundary operators of fractional order. In Section 3 we provide auxiliary statements. These statements are related with properties of the solutions of the Dirichlet problem and boundary-value problem with the boundary operators of fractional order. In Section 4 we prove theorems on the uniqueness of solution of the studied problems. Finally, Section 5 is devoted to study of the main problem, where we formulate and prove theorems on existence and smoothness of the solution.

2. Formulation of the problem

Denote

$$\partial\Omega_+ = \partial\Omega \cap \{x \in R^n : x_1 \ge 0\},$$
$$\partial\Omega_- = \partial\Omega \cap \{x \in R^n : x_1 \le 0\},$$
$$I = \partial\Omega \cap \{x \in R^n : x_1 = 0\}.$$

We associate each point $x = (x_1, x_2, \dots, x_n) \in \Omega$ with its "opposite" point

$$x^* = (a_1x_1, a_2x_2, \dots, a_nx_n) \in \Omega,$$

where $a_1 = -1$, and $a_j, j = 2, \dots, n$ take one of values ± 1. Obviously, that if $x \in \partial\Omega_+$, then $x^* \in \partial\Omega_-$. In the domain Ω we consider the following boundary-value problems.

Problem 2.1. Find a function $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $B^{\alpha}[u](x)$ is a continuous function in the domain $\bar{\Omega}$, and satisfies the following conditions:

$$\Delta u(x) = 0, \quad x \in \Omega, \tag{2.1}$$
$$u(x) - (-1)^k u(x^*) = f(x), \quad x \in \partial\Omega_+, \tag{2.2}$$
$${}_{RL}D^{\alpha}[u](x) + (-1)^k{}_{RL}D^{\alpha}[u](x^*) = g(x), \quad x \in \partial\Omega_+. \tag{2.3}$$

Problem 2.2. Find a function $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $B_*^{\alpha}[u](x)$ is a continuous function in the domain $\bar{\Omega}$, and satisfies equation (2.1), equation (2.2) and

$${}_{C}D^{\alpha}[u](x) + (-1)^k{}_{C}D^{\alpha}[u](x^*) = g(x), \quad x \in \partial\Omega_+,$$

where $k = 1, 2$, $0 < \alpha \le 1$, $f(x) \in C^{\lambda+\alpha}(\partial\Omega_+), g(x) \in C^{\lambda}(\partial\Omega_+), 0 < \lambda < 1$, $\lambda + \alpha$ - non-integer.

When $k = 1$, problems 2.1 and 2.2 are called antiperiodical boundary-value problems, and when $k = 2$ - periodical boundary-value problems.

Necessary condition for existence of a solution of the problem 2.1 (problem 2.1) with smoothness $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$, $B^\alpha[u](x) \in C(\bar{\Omega})$ ($B^\alpha_*[u](x) \in C(\bar{\Omega})$) is fulfillment of the matching conditions:

$$f(0, x_2, \dots, x_n) + (-1)^k f(0, a_2x_2, \dots, a_nx_n) = 0, \quad (0, x_2, \dots, x_n) \in I, \tag{2.4}$$

$$g(0, x_2, \dots, x_n) - (-1)^k g(0, a_2x_2, \dots, a_nx_n) = 0, \quad (0, x_2, \dots, x_n) \in I, \tag{2.5}$$

$$\frac{\partial f(0, x_2, \dots, x_n)}{\partial x_j} + (-1)^k \frac{\partial f(0, a_2x_2, \dots, a_nx_n)}{\partial x_j} = 0, \quad (0, x_2, \dots, x_n) \in I, \tag{2.6}$$

when $\lambda + \alpha > 1$. Furthermore we assume that these conditions are satisfied.

Note that numerous publications were devoted to the questions of solvability of boundary-value problems for elliptic equations with boundary operators of fractional order, see [5, 6, 10, 12, 13, 15, 20, 21, 22, 23, 24, 25, 26].

In [5, 10, 12, 13, 15], the Laplace equation nonlocal boundary-value problems with boundary operators of fractional order were investigated. It should also be noted that some questions of solvability of nonlocal problems for fractional order equations in the one-dimensional case were studied in [1, 2, 9, 14, 16, 19, 27]. In these papers the natural generalizations of the Samarskii - Bisadze problem were studied, when nonlocal conditions were given in the relation form of the boundary values with values of the desired function within the domain. In this paper we consider the problems when non-local conditions are given in the form of periodic or antiperiodic conditions.

Since

$${}_{RL}D^1[u](x) = {}_{C}D^1[u](x) = \frac{du(x)}{dr},$$

then when $\alpha = 1$ derivatives ${}_{C}D^\alpha, {}_{RL}D^\alpha$ coincides with derivative in the direction of the vector $r = |x|$. Note, that this case, i.e. when $\alpha = 1$, was studied in [17, 18]. In particular, the following propositions were proved.

Theorem 2.3. *Let $k = 1$, $f(x) \in C^{1+\lambda}(\partial\Omega_+)$, $g(x) \in C^\lambda(\partial\Omega_+)$, $0 < \lambda < 1$ and the matching conditions* (2.4), (2.5), (2.6) *hold. Then a solution of problem 2.1 (problem 2.2) exists, it is unique and represented in the form:*

$$u(x) = -\int_{\partial\Omega_+} \frac{\partial G_1(x, y)}{\partial n_y} f(y)\, ds_y + \int_{\partial\Omega_+} G_1(x, y) g(y)\, ds_y,$$

where $G_1(x, y)$ is Green function of the anti-periodical problem 2.1 (problem 2.2):

$$G_1(x, y) = \frac{1}{2}[G_D(x, y) + G_D(x, y^*) + G_N(x, y) - G_N(x, y^*)],$$

$G_D(x, y)$ - Green function of the Dirichlet problem, $G_N(x, y)$ - Green function of the Neumann problem.

Theorem 2.4. *Let $k = 2$, $f(x) \in C^{1+\lambda}(\partial\Omega_+)$, $g(x) \in C^\lambda(\partial\Omega_+)$, $0 < \lambda < 1$ and the matching conditions* (2.4), (2.5), (2.6) *hold. Then for solvability of problem 2.1 (problem 2.2) it is necessary and sufficient fulfillment of the condition*

$$\int_{\partial\Omega_+} g(y)\, ds_y = 0.$$

If a solution exists, then it is unique with up to a constant and can be represented as

$$u(x) = -\int_{\partial\Omega_+} \frac{\partial G_2(x,y)}{\partial n_y} f(y)\, ds_y + \int_{\partial\Omega_+} G_2(x,y) g(y)\, ds_y + \text{const},$$

where $G_2(x,y)$ is Green function of the periodical problem, that is defined by the equality:

$$G_2(x,y) = \frac{1}{2}[G_D(x,y) - G_D(x,y^*) + G_N(x,y) + G_N(x,y^*)] + \text{const}.$$

Later we will conduct a full investigation of the questions of existence, uniqueness and smoothness of solutions of problems 2.1 and 2.2, depending on the order of the boundary operators within $0 < \alpha \le 1$. Moreover, for completeness of investigation, we give proofs of some statements in the case $\alpha = 1$.

3. Auxiliary statements

To investigate questions on solvability of the problems 2.1 and 2.2 we have to provide some properties of the operators B^α, B_*^α and $B^{-\alpha}$. The following propositions have been proved for the case $0 < \alpha < 1$ in [10], and for $\alpha = 1$ in [4].

Lemma 3.1. *Let function $u(x)$ be harmonic in the domain Ω. Then*
(1) for any $\alpha \in (0,1]$ functions $B^\alpha[u](x)$, $B_*^\alpha[u](x)$ are also harmonic in Ω;
(2) if $\alpha \in (0,1)$, then the function $B^{-\alpha}[u](x)$ is harmonic in Ω;
(3) if $\alpha = 1$, then for $u(0) = 0$ the function $B^{-1}[u](x)$ is harmonic Ω.

Lemma 3.2. *Let $0 < \alpha \le 1$, function $u(x)$ be harmonic in the domain and continuous in $\bar{\Omega}$. Then, if $B^\alpha[u](x)$, $B_*^\alpha[u](x)$ are continuous in the domain $\bar{\Omega}$, Then the following equalities are true:*
(1) for any $\alpha \in (0,1)$,

$$B^{-\alpha}[B^\alpha[u]](x) = B^\alpha[B^{-\alpha}[u]](x) = u(x), \quad x \in \overline{\Omega}, \tag{3.1}$$

(2) for any $\alpha \in (0,1)$,

$$B_*^\alpha[u](x) = B[u](x) - \frac{u(0)}{\Gamma(1-\alpha)}, \quad x \in \overline{\Omega}.$$

(3) if $\alpha = 1$, then

$$B^{-1}[B^1[u]](x) = u(x) - u(0), \quad x \in \overline{\Omega} \tag{3.2}$$

(4) if $\alpha = 1$ and $u(0) = 0$, then

$$B^1[B^{-1}[u]](x) = u(x), x \in \overline{\Omega}.$$

Let $v(x)$ and $w(x)$ be solutions of the following problems:

$$\begin{gathered} \Delta v(x) = 0, \quad x \in \Omega, \\ v(x) = \tau(x), \quad x \in \partial\Omega, \end{gathered} \tag{3.3}$$

and

$$\begin{gathered} \Delta w(x) = 0, x \in \Omega, \\ B^\alpha[w](x) = \mu(x), x \in \partial\Omega. \end{gathered} \tag{3.4}$$

The following propositions refer to the smoothness of the solution of the Dirichlet problem (3.3) (see[3]).

Lemma 3.3. *Let $\lambda > 0$, λ - non-integer and $\tau(x) \in C^{\lambda}(\partial\Omega)$. Then a solution of problem (3.3) exists, belongs to the class $C^{\lambda}(\overline{\Omega})$ and for any multi-index $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ with $|\beta| > \lambda$ the following estimate is true:*

$$|\partial^{\beta} v(x)| \le C(1 - |x|)^{\lambda - |\beta|}, \tag{3.5}$$

where $\partial^{\beta} v(x) = \frac{\partial^{|\beta|} v(x)}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}$.

And the converse is also true.

Lemma 3.4. *Let $\lambda > 0$, $v(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ and for any multi-index $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ with $|\beta| > \lambda$ the inequality (3.5) holds. Then $v(x) \in C^{\lambda}(\overline{\Omega})$.*

The following statement defines smoothness of a solution of probelem (3.4) (see [24]).

Lemma 3.5. *Let $\lambda > 0$, $0 < \alpha < 1$, $\mu(x) \in C^{\lambda}(\partial\Omega)$, λ and $\lambda + \alpha$ - non-integer. Then a solution of the problem (3.4) exists, it is unique, belongs to the class $C^{\lambda+\alpha}(\overline{\Omega})$ and can be represented as*

$$w(x) = \int_{\partial\Omega} P_{\alpha}(x, y)\mu(y)\, ds_y,$$

where

$$P_{\alpha}(x, y) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} s^{-\alpha} P(sx, y) ds,$$

$P(x, y) = \frac{1}{\omega_n} \frac{1 - |x|^2}{|x - y|^n}$ - Poisson kernel of the Dirichlet problem (3.3).

Let us give a proposition about smoothness of the fractional derivative of the Dirichlet problem.

Lemma 3.6. *Let $\lambda > \alpha$, $0 < \alpha \le 1$, λ and $\lambda - \alpha$ non-integer. Further, let $\tau(x) \in C^{\lambda}(\partial\Omega)$, $v(x)$ be a solution of the Dirichlet problem (3.3). Then $B^{\alpha}[v](x) \in C^{\lambda - \alpha}(\overline{\Omega})$.*

Proof. Let $v(x)$ be a solution of the problem (3.3). Introduce the function $B^{\alpha}[v](x)$ in the form

$$\begin{aligned}
B^{\alpha}[v](x) &= \frac{r^{\alpha}}{\Gamma(1 - \alpha)} \frac{d}{dr} \int_0^r (r - \tau)^{-\alpha} v(\tau\theta) d\tau \underset{\tau = r\xi}{=} \\
&= \frac{r^{\alpha}}{\Gamma(1 - \alpha)} \frac{d}{dr} r^{1 - \alpha} \int_0^1 (1 - \xi)^{-\alpha} v(\xi x) d\xi \\
&= \frac{1}{\Gamma(1 - \alpha)} r^{\alpha} [(1 - \alpha) r^{-\alpha} + r^{1 - \alpha} \frac{d}{dr}] \int_0^1 (1 - \xi)^{-\alpha} v(\xi x) d\xi \\
&= \frac{1}{\Gamma(1 - \alpha)} (r \frac{d}{dr} + 1 - \alpha) \int_0^1 (1 - \xi)^{-\alpha} v(\xi x) d\xi.
\end{aligned}$$

Denote

$$v_1(x) = \int_0^1 (1 - \xi)^{-\alpha} v(\xi x) d\xi.$$

Let β be a multi-index $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ with $|\beta| > \lambda + 1 - \alpha$. Since $\tau(x) \in C^{\lambda}(\partial\Omega)$, due to the lemma 3.3 $v(x) \in C^{\lambda}(\overline{\Omega})$ and

$$|\partial^{\beta} v(x)| \le C(1 - |x|)^{\lambda - |\beta|}.$$

Then
$$|\partial^\beta v_1(x)| \le C\int_0^1 (1-\xi)^{-\alpha}\xi^{|\beta|}(1-\xi|x|)^{\lambda-|\beta|}d\xi.$$
Represent the last integral in the form
$$\int_0^1 = \int_0^{|x|} + \int_{|x|}^1 = I_1 + I_2$$
and estimate I_1.

We consider two cases:

(a) Let $1/2 \le |x| \le 1$. Since for any $\xi \in [0,|x|]$ inequalities $1-\xi|x| \ge 1-\xi$ and $|\xi|^{|\beta|} \le 1$ are true, it follows that
$$\begin{aligned} I_1 &\le C\int_0^{|x|}(1-\xi)^{\lambda-\alpha-|\beta|}d\xi = \frac{(1-\xi)^{\lambda+1-\alpha-|\beta|}}{|\beta|-\lambda-1+\alpha}\Big|_0^{|x|} \\ &= \frac{1}{|\beta|-\lambda-1+\alpha}[(1-|x|)^{\lambda+1-\alpha-|\beta|}-1] \le C(1-|x|)^{\lambda+1-\alpha-|\beta|}. \end{aligned}$$

(b) Let $|x| \le 1/2$. In this case $1-\xi|x| \ge 1-|x|^2 \ge 1-\frac{1}{4} = \frac{3}{4}$. Consequently, $I_1 \le C$, i.e. I_1 is bounded. Thus, in general case
$$I_1 \le C(1-|x|)^{\lambda+1-\alpha-|\beta|}.$$

Next we estimate integral I_2. In this case for all $\xi \in [|x|,1]$ inequality $1-\xi|x| \ge 1-|x|$ holds, and, thus
$$(1-\xi|x|)^{\lambda-|\beta|} \le (1-|x|)^{\lambda-|\beta|}.$$
Then
$$\begin{aligned} I_2 &\le (1-|x|)^{\lambda-|\beta|}\int_0^1 (1-\xi)^{-\alpha}d\xi \\ &= (1-|x|)^{\lambda-|\beta|}\frac{(1-\xi)^{1-\alpha}}{1-\alpha}\Big|_{|x|}^1 \\ &= \frac{(1-|x|)^{\lambda+1-\alpha-|\beta|}}{1-\alpha}. \end{aligned}$$
Hence, for any multi-index $\beta = (\beta_1,\beta_2,\dots,\beta_n)$ with $|\beta| > \lambda+1-\alpha$ the inequality
$$|\partial^\beta v_1(x)| \le C(1-|x|)^{\lambda+1-\alpha-|\beta|}$$
holds. The by lemma 3.4 $v_1(x) \in C^{\lambda+1-\alpha}(\overline{\Omega})$. Since
$$B^\alpha[v](x) = \frac{1}{\Gamma(1-\alpha)}\Big(r\frac{d}{dr}+1-\alpha\Big)v_1(x),$$
then obviously, $B^\alpha[v](x) \in C^{\lambda-\alpha}(\overline{\Omega})$. The proof is complete. □

Lemma 3.7. *Let $\tau(x) \in C(\partial\Omega)$ and $v(x)$ be a solution of (3.3). If $\tau(x)$ has the property*
$$\tau(x) = \pm\tau(x^*), \quad x \in \partial\Omega_+,$$
then for any $x \in \overline{\Omega}$, we have $v(x) = \pm v(x^*)$.

Proof. If $\tau(x) \in C(\partial\Omega)$, then a solution of (3.3) exists and can be represented as a Poisson integral:
$$v(x) = \int_{\partial\Omega} P(x,y)\tau(y)\,ds_y. \tag{3.6}$$
Using property of the function $\tau(x)$, the function (3.6) can be represented as follows:
$$\begin{aligned}
v(x) &= \frac{1}{\omega_n}\int_{\partial\Omega_+} \frac{1-|x|^2}{|x-y|^n}\tau(y)\,ds_y + \frac{1}{\omega_n}\int_{\partial\Omega_-} \frac{1-|x|^2}{|x-y|^n}\tau(y)\,ds_y \\
&= \frac{1}{\omega_n}\int_{\partial\Omega_+} \frac{1-|x|^2}{|x-y|^n}\tau(y)\,ds_y + \frac{1}{\omega_n}\int_{\partial\Omega_+} \frac{1-|x|^2}{|x-y^*|^n}\tau(y^*)\,ds_y \\
&= \frac{1}{\omega_n}\int_{\partial\Omega_+} \frac{1-|x|^2}{|x-y|^n}\tau(y)\,ds_y \pm \frac{1}{\omega_n}\int_{\partial\Omega_+} \frac{1-|x|^2}{|x-y^*|^n}\tau(y)\,ds_y \\
&= \frac{1}{\omega_n}\int_{\partial\Omega_+} [\frac{1-|x|^2}{|x-y|^n} \pm \frac{1-|x|^2}{|x-y^*|^n}]\tau(y)\,ds_y.
\end{aligned}$$
Then for $v(x^*)$ we have
$$v(x^*) = \frac{1}{\omega_n}\int_{\partial\Omega_+} [\frac{1-|x^*|^2}{|x^*-y|^n} \pm \frac{1-|x^*|^2}{|x^*-y^*|^n}]\tau(y)\,ds_y.$$
Further, since $|x| = |x^*|$ and
$$\begin{aligned}
|x^*-y|^2 &= \sum_{j=1}^n (\alpha_j x_j - y_j)^2 = \sum_{j=1}^n (\alpha_j^2 x_j^2 - 2\alpha_j x_j y_j + y_j^2)^2 \\
&= \sum_{j=1}^n (x_j^2 - 2\alpha_j x_j y_j + \alpha_j^2 y_j^2)^2 \\
&= \sum_{j=1}^n (x_j - \alpha_j y_j)^2 \\
&= |x-y^*|^2
\end{aligned}$$
then
$$\begin{aligned}
v(x^*) &= \frac{1}{\omega_n}\int_{\partial\Omega_+} [\frac{1-|x^*|^2}{|x^*-y|^n} \pm \frac{1-|x^*|^2}{|x^*-y^*|^n}]\tau(y)\,ds_y \\
&= \pm\frac{1}{\omega_n}\int_{\partial\Omega_+} [\frac{1-|x|^2}{|x-y|^n} \pm \frac{1-|x|^2}{|x-y^*|^n}]\tau(y)\,ds_y = v(x).
\end{aligned}$$
The proof is complete. □

4. Uniqueness of a solutions to problems 2.1 and 2.2

Theorem 4.1. *If a solution of problem 2.1 exists, then*
(1) when $k = 1, 2$ for all $\alpha \in (0, 1)$ the solution is unique;
(2) in the case $\alpha = 1$ when $k = 1$ the solution is unique, and when $k = 2$ it is unique up to a constant value.

Proof. Suppose that $u(x)$ is a solution of the homogenous problem. Then due to (2.2) we obtain
$$u(x) = (-1)^k u(x^*), \quad x \in \partial\Omega_+ \tag{4.1}$$

Let $\alpha \in (0,1)$. Apply the operator B^α to the function $u(x)$. Then by the lemma 3.1 function $B^\alpha[u](x)$ is harmonic in the domain Ω, and since

$$B^\alpha[u](x)\big|_{\partial\Omega} = {}_{RL}D^\alpha[u](x)\big|_{\partial\Omega},$$

and due to the boundary condition (2.3), it follows that

$$B^\alpha[u](x) = -(-1)^k B^\alpha[u](x^*), \quad x \in \partial\Omega_+.$$

Further, since $B^\alpha[u](x) \in C(\overline{\Omega})$, then from lemma 3.7 for any $x \in \overline{\Omega}$ it follows that

$$B^\alpha[u](x) = -(-1)^k B^\alpha[u](x^*) \tag{4.2}$$

Applying the operator $B^{-\alpha}$ to the equality (4.2), and taking account equality (3.1), we obtain

$$\begin{aligned} u(x) &= B^{-\alpha}[B^\alpha[u]](x) \\ &= -(-1)^k B^{-\alpha}[B^\alpha[u]](x^*) \\ &= -(-1)^k u(x^*), \quad x \in \overline{\Omega}. \end{aligned}$$

i.e. for all $x \in \overline{\Omega}$, we have $u(x) = -(-1)^k u(x^*)$.

In particular, we get

$$u(x) = -(-1)^k u(x^*), \quad x \in \partial\Omega. \tag{4.3}$$

Comparing equalities (4.1) and (4.3) we have $u(x) = 0$ for $x \in \partial\Omega_+$; thus

$$u(x) = 0, \quad x \in \partial\Omega.$$

Then due to maximum principle for harmonic functions:

$$u(x) \equiv 0, \quad x \in \overline{\Omega}.$$

Now let $\alpha = 1$ and $k = 1$. Then from the boundary condition (2.2) we have $u(x) = -u(x^*)$, and from (3.2) and condition (2.3),

$$u(x) - u(0) = u(x^*) - u(0).$$

Consequently, $u(x) = u(x^*)$; thus, $u(x) = 0, \quad x \in \partial\Omega$. Then

$$u(x) \equiv 0, \quad x \in \overline{\Omega}.$$

If $k = 2$, then from condition (2.2) we obtain $u(x) = u(x^*)$, and from the conditions (3.2) and (2.3):

$$u(x) - u(0) = -[u(x^*) - u(0)].$$

Then

$$u(x) = 2u(0) \equiv \text{const}, \quad x \in \partial\Omega,$$

hence $u(x) \equiv C, x \in \overline{\Omega}$. The proof is complete. □

The following result can be proved analogously, as the above theorem.

Theorem 4.2. *If a solution of problem 2.2 exists, then*
(1) when $k = 1$ for all $\alpha \in (0,1]$ the solution is unique;
(2) when $k = 2$ for all $\alpha \in (0,1]$ the solution is unique up to constant item.

5. Existence of a solution

Let a function $P_\alpha(x,y)$ be defined by

$$P_\alpha(x,y)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}s^{-\alpha}P(sx,y)ds, & 0<\alpha<1\\ \int_0^1[P(sx,y)-1]\frac{ds}{s}, & \alpha=1\end{cases} \tag{5.1}$$

Theorem 5.1. *Assume that in problem 2.1: $f(x)\in C^{\lambda+\alpha}(\partial\Omega_+)$ and $g(x)\in C^{\lambda}(\partial\Omega_+)$, where $0<\lambda<1$, $0<\alpha\le 1$, λ and $\lambda+\alpha$ - non-integer. Then*

(1) if $\alpha\in(0,1)$ and $k=1,2$, then a solution of the problem exists and is unique;

(2) if $\alpha=1$, then for $k=1$ a solution of the problem exists and is unique, and for $k=2$, for existence of a solution of the problem it is necessary any sufficient the fulfillment of the condition:

$$\int_{\partial\Omega_+} g(y)\,ds_y=0, \tag{5.2}$$

If a solution of the problem exists, then it is unique up to constant term;

(3) if a solution of the problem exists, then it belongs to the class $C^{\lambda+\alpha}(\bar\Omega)$, and can be represented as follows:

$$\begin{aligned}u(x)&=\frac12\int_{\partial\Omega_+}[P(x,y)-(-1)^kP(x,y^*)]f(y)\,ds_y\\&\quad+\frac12\int_{\partial\Omega_+}[P_\alpha(x,y)+(-1)^kP_\alpha(x,y^*)]g(y)\,ds_y.\end{aligned} \tag{5.3}$$

Proof. We introduce the auxiliary functions:

$$v(x)=\frac12(u(x)-(-1)^ku(x^*)),\quad w(x)=\frac12(u(x)+(-1)^ku(x^*)).$$

It is obvious, that $u(x)=v(x)+w(x)$. Assuming, that $u(x)$ is a solution of 2.1, we find two problems, satisfied by $v(x)$ and $w(x)$. The function $v(x)$ is a solution of Dirichlet problem (3.3), and the function $w(x)$ is a solution of the problem (3.4), where

$$\tau(x)=\begin{cases}\frac12 f(x), & \text{if } x\in\partial\Omega_+\\ -\frac{(-1)^k}{2}f(x^*), & \text{if } x\in\partial\Omega_-\end{cases} \tag{5.4}$$

and

$$\mu(x)=\begin{cases}\frac12 g(x), & \text{if } x\in\partial\Omega_+\\ \frac{(-1)^k}{2}f(x^*), & \text{if } x\in\partial\Omega_-\end{cases} \tag{5.5}$$

Indeed, if $x\in\partial\Omega_+$, then

$$\tau(x)\equiv v(x)\big|_{\partial\Omega_+}=\frac12[u(x)-(-1)^ku(x^*)]\big|_{\partial\Omega_+}=\frac{f(x)}{2},$$

And if $x\in\partial\Omega_-$, then in this case $x^*\in\partial\Omega_+$ and

$$\begin{aligned}\tau(x)&\equiv v(x)|_{\partial\Omega_-}=[u(x)-(-1)^ku(x^*)]\\&=-\frac{(-1)^k}{2}[u(x^*)-(-1)^ku(x)]=-(-1)^k\frac{f(x^*)}{2}.\end{aligned}$$

Thus, a function $\tau(x)$ is defined by equality (5.4).

Further,
$$B^\alpha[w](x)=\frac{1}{2}[B^\alpha[u](x)+(-1)^kB^\alpha[u](x^*)],$$
and, hence
$$\begin{gathered}B^\alpha[w](x)|_{\partial\Omega_+}=\frac{1}{2}g(x),\\ B^\alpha[w](x)|_{\partial\Omega_-}=(-1)^k\frac{g(x^*)}{2}.\end{gathered}$$
i.e. for the function $\mu(x)$ we obtain equality (5.5).

If $\tau(x)\in C^{\lambda+\alpha}(\partial\Omega)$, then for any $\alpha\in(0,1]$ a solution of the Dirichlet problem (3.3) exists, belongs to the class $v(x)\in C^{\lambda+\alpha}(\overline{\Omega})$ and is represented as (3.6). Further, since for $\tau(y)$ equality (5.4) holds, then
$$\begin{aligned}v(x)&=\frac{1}{\omega_n}\int_{\partial\Omega_+}\frac{1-|x|^2}{|x-y|^n}\tau(y)\,ds_y+\frac{1}{\omega_n}\int_{\partial\Omega_-}\frac{1-|x|^2}{|x-y|^n}\tau(y)\,ds_y\\ &=\frac{1}{2\omega_n}\int_{\partial\Omega_+}\frac{1-|x|^2}{|x-y|^n}f(y)\,ds_y-\frac{(-1)^k}{2\omega_n}\int_{\partial\Omega_-}\frac{1-|x|^2}{|x-y|^n}f(y^*)\,ds_y\\ &=\frac{1}{2\omega_n}\int_{\partial\Omega_+}\frac{1-|x|^2}{|x-y|^n}f(y)\,ds_y-\frac{(-1)^k}{2\omega_n}\int_{\partial\Omega_+}\frac{1-|x|^2}{|x-y^*|^n}f(y)\,ds_y\\ &=\frac{1}{2}\int_{\partial\Omega_+}[P(x,y)-(-1)^kP(x,y^*)]f(y)\,ds_y.\end{aligned}$$
Let $0<\alpha<1$. By lemma 3.5 when $\mu(y)\in C^\lambda(\partial\Omega)$ a solution of problem (3.4) exists, belongs to the class $C^{\lambda+\alpha}(\overline{\Omega})$ and is represented in the form
$$w(x)=\int_{\partial\Omega}P_\alpha(x,y)\mu(y)\,ds_y.$$
Then, using the representation of the function $\mu(y)$, we have
$$\begin{aligned}w(x)&=\int_{\partial\Omega}P_\alpha(x,y)\mu(y)\,ds_y\\ &=\frac{1}{2}\int_{\partial\Omega_+}P_\alpha(x,y)g(y)\,ds_y+\frac{(-1)^k}{2}\int_{\partial\Omega_-}P_\alpha(x,y)g(y^*)\,ds_y\\ &=\frac{1}{2}\int_{\partial\Omega_+}[P_\alpha(x,y)+(-1)^kP_\alpha(x,y^*)]g(y)\,ds_y.\end{aligned}$$
Thus, in the case $0<\alpha<1$, $k=1,2$ for a solution of the problem 2.1 representation (5.3) holds. Now let $\alpha=1$. In this case the problem (3.4) is the Neumann problem and for the existence of a solution of this problem it is necessary and sufficient fulfillment of the condition:
$$\int_{\partial\Omega}\mu(y)\,ds_y=0. \tag{5.6}$$
If $k=1$, then due to the equality (5.5),
$$\begin{aligned}\int_{\partial\Omega}\mu(y)\,ds_y&=\frac{1}{2}\int_{\partial\Omega_+}g(y)\,ds_y-\frac{1}{2}\int_{\partial\Omega_-}g(y^*)\,ds_y\\ &=\frac{1}{2}\int_{\partial\Omega_+}g(y)\,ds_y-\frac{1}{2}\int_{\partial\Omega_+}g(y)\,ds_y=0;\end{aligned}$$

i.e. in this case condition of solvability (5.6) always holds, hence a solution of problem (3.4) exists. If $k=2$, then

$$\int_{\partial\Omega}\mu(y)\,ds_y=\frac{1}{2}\int_{\partial\Omega_+}g(y)\,ds_y+\frac{1}{2}\int_{\partial\Omega_-}g(y^*)\,ds_y=\int_{\partial\Omega_+}g(y)\,ds_y,$$

and then condition on solvability of Neumann problem (5.6) can be rewritten in the form (5.3). It is known [7, 8], that a solution of the Neumann problem is represented as follows:

$$w(x)=\int_{\partial\Omega}P_1(x,y)\mu(y)\,ds_y+C \tag{5.7}$$

where

$$P_1(x,y)=\int_0^1[P(sx,y)-1]\frac{ds}{s}.$$

Further, using representation of the function $\mu(x)$, function (5.7) is easy reduced to the form

$$w(x)=\int_{\partial\Omega_+}[P_1(x,y)-P_1(x^*,y)]g(y)\,ds_y+C. \tag{5.8}$$

Note, that if $x^*=(-x_1,\alpha_2x_2,\dots,\alpha_nx_n)$, then

$$(x^*)^*=(x_1,x_2,\dots,x_n)=x.$$

Then

$$\begin{aligned}w(x^*)&=\frac{1}{2}(u(x^*)+(-1)^ku(x^{**}))\\&=\frac{1}{2}((-1)^ku(x)+u(x^*))\\&=\frac{(-1)^k}{2}(u(x)+(-1)^ku(x^*))=(-1)^kw(x).\end{aligned}$$

Thus, when $k=1$ the function $w(x)$ has the symmetric property

$$w(x)=-w(x^*),x\in\Omega.$$

For the function (5.8) this is possible, only when $C=0$.

Hence, when $k=1$ for a solution of problem 2.1 we obtain representation (5.3). If $k=2$, then $w(x)=w(x^*)$, $x\in\Omega$. In this case the solution of Problem 2.1 is unique up to a constant term and the representation (5.3) holds. The theorem is proved. □

Let a function $P^*_\alpha(x,y)$ be defined by

$$P^*_\alpha(x,y)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}s^{-\alpha}[P(sx,y)-1]ds, & 0<\alpha<1\\ \int_0^1 s^{-1}[P(sx,y)-1]ds, & \alpha=1\end{cases} \tag{5.9}$$

The following proposition can be proved analogously to the above theorem.

Theorem 5.2. *In problem 2.2 let* $0<\alpha\le 1$, $f(x)\in C^{\lambda+\alpha}(\partial\Omega_+)$, $g(x)\in C^\lambda(\partial\Omega_+)$, $0<\lambda<1$, λ *and* $\lambda+\alpha$ *- non-integer. Then*

(1) if $k=1$ *a solution of the problem exists and unique;*

(2) if $k=2$ *then for solvability of the problem it is necessary any sufficient fulfillment of the condition* (5.2). *If a solution exists, then it is unique up to constant term;*

(3) if a solution of the problem exists, then it belongs to the class $C^{\lambda+\alpha}(\bar{\Omega})$, and can be represented as:

$$\begin{aligned} u(x) &= \frac{1}{2}\int_{\partial\Omega_+}[P(x,y)-(-1)^k P(x,y^*)]f(y)\,ds_y \\ &\quad + \frac{1}{2}\int_{\partial\Omega_+}[P^*_\alpha(x,y)+(-1)^k P^*_\alpha(x,y^*)]g(y)\,ds_y. \end{aligned} \tag{5.10}$$

6. Examples

Example 6.1. Let $n=2$, $a_2=-1$, $k=1$. Then in problem 2.1 we obtain the boundary value conditions:

$$\begin{gathered} u(1,\varphi)+u(1,\varphi+\pi)=f(\varphi), \quad 0\le\varphi\le\pi, \\ B^\alpha[u](1,\varphi)-B^\alpha[u](1,\varphi+\pi)=g(\varphi), \quad 0\le\varphi\le\pi. \end{gathered}$$

By the theorem 5.1, problem 2.1 has a unique solution, which can be represented as follows

$$\begin{aligned} u(x) &= \frac{1}{4\pi}\int_0^\pi [P(r,\varphi-\theta)+P(-r,\varphi-\theta)]f(\theta)\,d\theta \\ &\quad + \frac{1}{4\pi}\int_0^\pi [P_\alpha(r,\varphi-\theta)-P_\alpha(-r,\varphi-\theta)]g(\theta)\,d\theta. \end{aligned}$$

In [21], an explicit form of the function (5.1) was obtained:

$$P_\alpha(r,\gamma)=2\Gamma(1-\alpha)\Big(\frac{\cos[(1-\alpha)\arctan\frac{r\sin\gamma}{1-r\cos\gamma}]}{(1-2r\cos\gamma+r^2)^{\frac{1-\alpha}{2}}}-\frac{1}{2}\Big).$$

Then a solution of the problem has the form:

$$\begin{aligned} u(x) &= \frac{1}{2\pi}\int_0^\pi \frac{1-r^4}{1-2r^2\cos 2(\varphi-\theta)+r^4}f(\theta)\,d\theta \\ &\quad + \frac{\Gamma(1-\alpha)}{2\pi}\int_0^\pi \frac{\cos[(1-\alpha)\arctan\frac{r\sin(\varphi-\theta)}{1-r\cos(\varphi-\theta)}]}{(1-2r\cos(\varphi-\theta)+r^2)^{\frac{1-\alpha}{2}}}g(\theta)\,d\theta \\ &\quad - \frac{\Gamma(1-\alpha)}{2\pi}\int_0^\pi \frac{\cos[(1-\alpha)\arctan\frac{r\sin(\varphi-\theta)}{1+r\cos(\varphi-\theta)}]}{(1+2r\cos(\varphi-\theta)+r^2)^{\frac{1-\alpha}{2}}}g(\theta)\,d\theta. \end{aligned}$$

Example 6.2. Let $n=2$, $a_2=1$, $k=2$. In this case, the boundary conditions of the problem 2.2 have the form

$$\begin{gathered} u(1,\varphi)-u(1,2\pi-\varphi)=f(\varphi), \quad 0\le\varphi\le\pi, \\ B^\alpha_*[u](1,\varphi)-B^\alpha_*[u](1,2\pi-\varphi)=g(\varphi), 0\le\varphi\le\pi. \end{gathered}$$

By Theorem 5.2 for the solvability of the considered problem it is necessary and sufficient fulfillment of the condition $\int_0^\pi g(\theta)\,d\theta=0$. The problem has a unique solution up to a constant, which is represented in the form (5.10). As in Example 6.1 one can construct the explicit form of the function (5.9), and then the solution has the form:

$$u(x)=\frac{1}{2\pi}\int_0^\pi \Big[\frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}-\frac{1-r^2}{1-2r\cos(\varphi+\theta)+r^2}\Big]f(\theta)\,d\theta$$

$$+\frac{\Gamma(1-\alpha)}{2\pi}\int_0^\pi \frac{\cos[(1-\alpha)\arctan\frac{r\sin(\varphi-\theta)}{1-r\cos(\varphi-\theta)}]}{(1-2r\cos(\varphi-\theta)+r^2)^{\frac{1-\alpha}{2}}}g(\theta)\,d\theta$$
$$+\frac{\Gamma(1-\alpha)}{2\pi}\int_0^\pi \frac{\cos[(1-\alpha)\arctan\frac{r\sin(\varphi+\theta)}{1-r\cos(\varphi+\theta)}]}{(1-2r\cos(\varphi+\theta)+r^2)^{\frac{1-\alpha}{2}}}g(\theta)\,d\theta.$$

Conclusion. In this paper questions about solvability of some nonlocal boundary-value problems for the Laplace equation are studied. Boundary conditions are given in the form of periodic or anti-periodic conditions, i.e. values of the function and values of the fractional derivative in the upper part of the boundary are associated with the values of these functions in the bottom part of the boundary. Theorems on existence and uniqueness of solutions are proved, and conditions for solvability of the investigated problems are established. Moreover, in the Holder class the order of smoothness of the solution are studied depending on the order of the boundary operator.

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References

[1] B. Ahmad, J. J. Nieto; *Anti-Periodic Fractional Boundary Value Problems With Nonlinear Term Depending On Lower Order Derivative*, Fractional Calculus and Applied Analysis. 15 No.3 (2012), 451–462.

[2] B. Ahmad, J. J. Nieto; *Anti-periodic fractional boundary value problems*, Computational Mathematics and Applications 62 (2011), 1050–1056.

[3] Sh. A. Alimov; *One Problem with an Oblique Derivative*, Differential Equations. 17 No. 10 (1981), 1738–1751.

[4] I. I. Bavrin; *Operators for harmonic functions and their applications*, Differential Equations, 21 No.1 (1985), 6–10.

[5] A. S. Berdyshev, B. J. Kadirkulov, J. J. Nieto; *Solvability of an elliptic partial differential equation with boundary condition involving fractional derivatives*, Complex Variables and Elliptic Equations, (2013) DOI:10.1080/17476933.2013.777711

[6] A. S. Berdyshev, B. Kh. Turmetov, B. J. Kadirkulov; *Some properties and applications of the integrodifferential operators of Hadamard-Marchaud type in the class of harmonic functions*, Siberian Mathematical Journal 53 No.4 (2012) 600-610.

[7] A. V. Bitsadze; *The Neumann problem for harmonic functions*, Doklady akademii nauk SSSR 990 No.1 (1990), 11–13.

[8] A. V. Bitsadze; *Singular integral equations of the first kind with Neumann kernels*, Differential Equations 22 No.5 (1986), 823–828.

[9] Guoqing Chai; *Existence results for anti-periodic boundary value problems of fractional differential equations*, Advances in Difference Equations 2013 No.53 (2013), doi:10.1186/1687-1847-2013-53

[10] V. V. Karachik, B. Kh. Turmetov, B. T. Torebek; *On some integro-differential operators in the class of harmonic functions and their applications*, Matematicheskiye trudy 14 No.1 (2011), 99–125 (transl.) Siberian Advances in Mathematics 22 No.2 (2012), 115-134.

[11] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*, Elsevier. North-Holland. Mathematics Studies. (2006) -539p.

[12] M. Kirane, N. -e. Tatar; *Nonexistence for the Laplace equation with a dynamical boundary condition of fractional type*, Siberian Mathematical Journal 48 No.5 (2007), 1056–1064.

[13] M. Kirane, N. -e. Tatar; *Absence of local and global solutions to an elliptic system with time-fractional dynamical boundary conditions*, Siberian Mathematical Journal 48 No.3 (2007), 593–605.

[14] Xiaoyou Liu, Yiliang Liu; *Fractional Differential Equations With Fractional Non-Separated Boundary Conditions*, Electronic Journal of Differential Equations 2013, No.25 1–13.
[15] M. A. Muratbekova, K. M. Shinaliyev, B. Kh. Turmetov; *On solvability of a nonlocal problem for the Laplace equation with the fractional-order boundary operator*, Boundary Value Problems (2014), doi:10.1186/1687-2770-2014-29.
[16] N. Nyamoradi, M. Javidi; *Existence Of Multiple Positive Solutions For Fractional Differential Inclusions With M-Point Boundary Conditions And Two Fractional Orders*, Electronic Journal of Differential Equations. 2012 No.187 (2012) 1–26.
[17] M. A. Sadybekov, B. Kh. Turmetov; *On analogues of periodic boundary value problems for the Laplace operator in ball*, Eurasian Mathematical Journal 3 No.1 (2012) 143-146.
[18] M. A. Sadybekov, B. Kh. Turmetov; *On an Analog of Periodic Boundary Value Problems for the Poisson Equation in the Disk*, Differential Equations 50 No.2 (2014) 264-268.
[19] N. -e. Tatar; *Existence Of Mild Solutions For A Neutral Fractional Equation With Fractional Nonlocal Conditions*, Electronic Journal of Differential Equations. 2012 No.153 (2012) 1–12.
[20] B. T. Torebek, B. Kh. Turmetov; *On solvability of a boundary value problem for the Poisson equation with the boundary operator of a fractional order*, Boundary Value Problems 2013 No.93 (2013) doi:10.1186/1687-2770-2013-93.
[21] B. T. Torebek, B. Kh. Turmetov; *On the solvability of some inverse problems for the Laplace equation with the boundary operator in the Riemann-Liouville*, Vestnik KarGU, seriya Mathematika 69 No.1 (2013) 113–121. (In Russian)
[22] B. Kh. Turmetov; *On a boundary value problem for a harmonic equation*, Differential Equations 32 No.8 (1996) 1093-1096.
[23] B. Kh. Turmetov; *On Smoothness of a Solution to a Boundary Value Problem with Fractional Order Boundary Operator*, Matematicheskiye trudy 7 No.1 (2004), 189–199 (transl.) Siberian Advances in Mathematics 15 No.2 (2005), 115-125.
[24] B. Kh. Turmetov, B. T. Torebek; *On smoothness of solutions to some boundary value problems for the Laplace equation in Holder classes*, Vestnik KazNU. Series Mathematics, Mechanics, Informatics. No.1 (2011) 79–86. (In Russian)
[25] S. R. Umarov; *On some boundary value problems for elliptic equations with a boundary operator of fractional order*, Doklady Russian Academy of Science 333 No.6 (1993), 708-710.
[26] S. R. Umarov, Yu. F. Luchko, R. Gorenflo; *On boundary value problems for elliptic equations with boundary operators of fractional order*, Fractional Calculus and Applied Analysis 2 No.4 (2000), 454-468.
[27] Fang Wang, Zhenhai Liu; *Anti-periodic fractional boundary value problems for nonlinear differential equations of fractional order*, Advances in Difference Equations 2012 No.116 (2012), doi:10.1186/1687-1847-2012-116.

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