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SOLVABILITY OF AN OPTIMAL CONTROL PROBLEM IN COEFFICIENTS FOR ILL-POSED ELLIPTIC BOUNDARY-VALUE PROBLEMS

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Abstract. We study an optimal control problem (OCP) associated to a linear elliptic equation $-\operatorname{div}(A(x)\nabla y + C(x)\nabla y) = f$. The characteristic feature of this control object is the fact that the matrix $C(x)$ is skew-symmetric and belongs to L^2-space (rather than L^∞). We adopt a symmetric positive defined matrix $A(x)$ as control in $L^\infty(\Omega;\mathbb{R}^{N\times N})$. In spite of the fact that the equations of this type can exhibit non-uniqueness of weak solutions, we prove that the corresponding OCP, under rather general assumptions on the class of admissible controls, is well-posed and admits a nonempty set of solutions. The main trick we apply to the proof of the existence result is the approximation of the original OCP by regularized OCPs in perforated domains with fictitious boundary controls on the holes.

1. Introduction

In this article we study the following optimal control problem (OCP) for a linear elliptic equation with unbounded coefficients in the main part of the elliptic operator

$$
\begin{gathered}
\text{Minimize } I(A,y) = \|y - y_d\|^2_{H^1(\Omega)} \\
\text{subject to the constraints} \\
-\operatorname{div}\big(A(x)\nabla y + C(x)\nabla y\big) = f \quad \text{in } \Omega, \\
y = 0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.1}
$$

where a symmetric matrix $A \in L^\infty(\Omega;\mathbb{S}^N_{\rm sym})$ is adopted as a control, $y_d \in H^1(\Omega)$ and $f \in H^{-1}(\Omega)$ are given distributions, and $C \in L^2(\Omega;\mathbb{S}^N_{\rm skew})$ is a given skew-symmetric matrix. We define a class of admissible controls $A_{\rm ad}$ as the set of uniformly coercive and uniformly bounded symmetric matrices with H^{-1}-bounded divergence of their columns. The optimal control problem is to minimize the discrepancy between a given distribution $y_d \in H^1(\Omega)$ and the solution of a Dirichlet problem $(1.1)_2$–$(1.1)_3$ choosing an appropriate matrix of coefficients $A \in L^\infty(\Omega;\mathbb{S}^N_{\rm sym})$.

This kind of problems naturally appears in the optimal design theory for linearized elliptic boundary-value problems. Their characteristic feature is that the matrix $C(x) = [c_{ij}(x)]_{i,j=1,\dots,N}$ is skew-symmetric, $c_{ij}(x) = -c_{ji}(x)$, measurable,

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and belongs to L^2-space (rather than L^∞). As a result, the existence, uniqueness, and variational properties of the weak solution to (1.1) usually are drastically different from the corresponding properties of solutions to the elliptic equations with L^∞-matrices in coefficients. In most cases, the situation can change dramatically for the matrices C with unremovable singularities. Typically, in such cases, the boundary-value problem may admit many or even infinitely many weak solutions [9, 22]. Usually, such solutions may have a character of non-variational solutions [22], singular solutions [2, 8, 13, 14, 23], pathological solutions [18, 21] and others.

The aim of this work is to study the existence of optimal controls to problem (1.1) and propose a scheme of their approximation. Since the range of optimal control problems in coefficients is very wide, including as well optimal shape design problems, some problems originating in mechanics and others, this topic has been widely studied by many authors. (see, for instance, [17], Pironneau [19]). However, most of the existing results and methods rely on linear PDEs with bounded coefficients in the main part of elliptic operators, while only very few articles deal with nonlinear problems with unbounded and degenerate coefficients, see [8, 13, 14, 15, 16].

As was pointed earlier, the principal feature of OCP (1.1) is that the corresponding boundary-value problem $(1.1)_2$–$(1.1)_3$ is ill-posedness and the class of admissible controls $A \in \mathfrak{A}_{\rm ad}$ belongs neither to the Sobolev space $W^{1,\infty}(\Omega;;\mathbb{S}^N_{\rm sym})$ nor to the space of matrices with bounded variation $BV(\Omega;\mathbb{R}^{N\times N})$. In fact, we consider in this paper the OCP (1.1) subject to the control constraints: $A \in \mathfrak{A}_{\rm ad}$ if and only if $\alpha I \le A(x) \le \beta I$ a.e. in Ω and $\operatorname{div} A \in Q$, where Q is a given compact subset of $H^{-1}(\Omega)^N$. We note that these assumptions on the class of admissible controls together with L^2-properties of the skew-symmetric matrix C are essentially weaker than they usually are in the literature. We give the precise definition of such controls in Section 3 and, using the direct method in the Calculus of variations, we show that a set of optimal pairs to the above problem is nonempty provided two conditions holds — the so-called non-triviality condition and the condition of closedness of the set of admissible solutions.

The non-triviality condition is closely related with the existence of weak solutions to the boundary-value problem $(1.1)_2$–$(1.1)_3$. We show in Section 4 that this condition can be satisfied due to the approximation approach. As for the condition of closedness of the set of admissible solutions, we provide its substantiation in Section 5. With that in mind we construct a special sequence of OCPs in perforated domains with fictitious boundary controls on the holes. As a result, we show that the set of admissible solutions to the problem (1.1) is the limit in the sense of Kuratowski of the sequences of admissible solutions to the regularized OCPs. Hence, due to the main properties of Kuratowski convergence, it gives that the limit set is sequentially closed.

2. Notation and Preliminaries

Let Ω be a bounded open connected subset of $\mathbb{R}^N$ $(N \ge 2)$ with Lipschitz boundary $\partial\Omega$. Let $\mathcal{L}^N(E)$ be the N-dimensional Lebesgue measure of $E \subset \mathbb{R}^N$. The spaces $\mathcal{D}'(\Omega)$ of distributions in Ω is the dual of the space $C_0^\infty(\Omega)$. By $H_0^1(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$-functions in the Sobolev space $H^1(\Omega)$, while $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. The norm in $H_0^1(\Omega)$ is defined by $\|y\|_{H_0^1(\Omega)} = \left(\int_\Omega \|\nabla y\|^2_{\mathbb{R}^N}\,dx\right)^{1/2}$. Let Γ be a part of the boundary $\partial\Omega$ with positive $(N-1)$-dimensional measures. Let $C_0^\infty(\mathbb{R}^N;\Gamma) = \{\varphi \in C_0^\infty(\mathbb{R}^N) : \varphi = 0 \text{ on } \Gamma\}$.

We define the Banach space $H_0^1(\Omega;\Gamma)$ as the closure of $C_0^\infty(\mathbb{R}^N;\Gamma)$ with respect to the norm $\|y\| = \big(\int_\Omega \|\nabla y\|^2_{\mathbb{R}^N}\,dx\big)^{1/2}$.

Skew-Symmetric Matrices. Let $\mathbb{M}^N$ be the set of all squared $N\times N$ matrices. We denote by $\mathbb{S}^N_{\rm skew}$ the set of all skew-symmetric matrices $C=[c_{ij}]_{i,j=1}^N$, i.e., C is a square matrix with $c_{ij}=-c_{ji}$ and, hence, $c_{ii}=0$. Therefore, the set $\mathbb{S}^N_{\rm skew}$ can be identified with the Euclidean space $\mathbb{R}^{\frac{N(N-1)}{2}}$.

Let $L^2(\Omega)^{\frac{N(N-1)}{2}} = L^2\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$ be the space of measurable square-integrable functions whose values are skew-symmetric matrices. For each $C\in L^2\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$ we have the bilinear form $\varphi(y,v)_C=\int_\Omega \big(\nabla v, C(x)\nabla y\big)_{\mathbb{R}^N}\,dx$, for all $y,v\in C_0^1(\Omega)$. It is easy to see that, in general, this form is unbounded on $H_0^1(\Omega)$. This motivates the introduction of the following set.

Definition 2.1. Let $C\in L^2\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$ be a given matrix. We say that an element $y\in H_0^1(\Omega)$ belongs to the set $D(C)$ if

$$\Big|\int_\Omega \big(\nabla\varphi, C\nabla y\big)_{\mathbb{R}^N}\,dx\Big| \le c(y)\Big(\int_\Omega |\nabla\varphi|^2_{\mathbb{R}^N}\,dx\Big)^{1/2},\quad \forall\varphi\in C_0^\infty(\Omega) \tag{2.1}$$

with some constant c depending on y.

As a result, having set $[y,\varphi]_C=\int_\Omega \big(\nabla\varphi, C(x)\nabla y\big)_{\mathbb{R}^N}\,dx$, for all $y\in D(C)$, and all $\varphi\in C_0^\infty(\Omega)$, we see that the form $[y,\varphi]_C$ can be defined for all $\varphi\in H_0^1(\Omega)$ using the standard rule

$$[y,\varphi]_C=\lim_{\varepsilon\to 0}\,[y,\varphi_\varepsilon]_C, \tag{2.2}$$

where $\{\varphi_\varepsilon\}_{\varepsilon>0}\subset C_0^\infty(\Omega)$ and $\varphi_\varepsilon\to\varphi$ strongly in $H_0^1(\Omega)$. In this case the value $[v,v]_C$ is finite for every $v\in D(C)$, although the "integrand" $\big(\nabla v(x), C(x)\nabla v(x)\big)_{\mathbb{R}^N}$ need not be integrable, in general.

Skew-symmetric matrices $C\in L^2\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$ of the $\mathfrak{F}$-type. Let ε be a small parameter. Assume that the parameter ε varies within a strictly decreasing sequence of positive real numbers which converge to 0. For a given sequence $\{\varepsilon>0\}$, we define the cut-off operators $\mathbb{T}_\varepsilon:\mathbb{S}^N_{\rm skew}\to\mathbb{S}^N_{\rm skew}$ as follows $\mathbb{T}_\varepsilon(C)=[T_\varepsilon(c_{ij})]_{i,j=1}^N$ for every $\varepsilon>0$ and $C\in L^2\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$, where $T_\varepsilon(s)=\max\{\min\{s,\varepsilon^{-1}\},-\varepsilon^{-1}\}$. The following property of $\mathbb{T}_\varepsilon$ is well known (see [11]):

$$\mathbb{T}_\varepsilon(C)\in L^\infty(\Omega;\mathbb{S}^N_{\rm skew}),\ \forall\varepsilon>0\quad\text{and}\quad \mathbb{T}_\varepsilon(C)\to C \text{ strongly in } L^2(\Omega;\mathbb{S}^N_{\rm skew}). \tag{2.3}$$

In what follows, for every $\varepsilon>0$, we associate with operator $\mathbb{T}_\varepsilon$ the perforated domain

$$\Omega_\varepsilon=\Omega\setminus Q_\varepsilon,\quad \forall\varepsilon>0, \tag{2.4}$$

where

$$Q_\varepsilon=\text{closure}\{x\in\Omega:\|C(x)\|_{\mathbb{S}^N_{\rm skew}}:=\max_{1\le i<j\le N}|c_{ij}(x)|\ge\varepsilon^{-1}\}. \tag{2.5}$$

Definition 2.2. We say that a matrix $C\in L^2\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$ is of the $\mathfrak{F}$-type, if there exists a strictly decreasing sequence of positive real numbers $\{\varepsilon\}$ converging to 0 such that the corresponding collection of sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$, defined by (2.4), possesses the following properties:

(i) Ω_ε are open connected subsets of Ω with Lipschitz boundaries for which there exists a positive value $\delta>0$ such that $\partial\Omega\subset\partial\Omega_\varepsilon$ and $\text{dist}(\Gamma_\varepsilon,\partial\Omega)>\delta$ for all $\varepsilon>0$.

(ii) The surface measure of the boundaries of holes $Q_\varepsilon = \Omega \setminus \Omega_\varepsilon$ is small enough in the following sense:

$$\mathcal{H}^{N-1}(\Gamma_\varepsilon) = o(\varepsilon) \quad \text{and} \quad \frac{\varepsilon \mathcal{H}^{N-1}(\Gamma_\varepsilon)}{|\Omega \setminus \Omega_\varepsilon|} = O(1), \quad \forall \varepsilon > 0. \tag{2.6}$$

(iii) For each element $h \in D(C)$, there is a constant $c = c(h)$ depending on h and independent of ε such that

$$\Big| \int_{\Omega\setminus\Omega_\varepsilon} \big(\nabla\varphi, C\nabla h\big)_{\mathbb{R}^N}\, dx\Big| \le c(h)\sqrt{\frac{|\Omega\setminus\Omega_\varepsilon|}{\varepsilon}}\Big(\int_{\Omega\setminus\Omega_\varepsilon} |\nabla\varphi|^2_{\mathbb{R}^N}\, dx\Big)^{1/2} \tag{2.7}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$.

Remark 2.3. As immediately follows from Definition 2.2, if $C \in L^2\big(\Omega; \mathbb{S}^N_{\rm skew}\big)$ is of the $\mathfrak{F}$-type then each of the sets Ω_ε is locally located on one side of its Lipschitz boundary $\partial\Omega_\varepsilon$ and the boundary $\partial\Omega_\varepsilon$ can be divided into two parts $\partial\Omega_\varepsilon = \partial\Omega \cup \Gamma_\varepsilon$. Besides, the sequence of perforated domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ is monotonically expanding, i.e., $\Omega_{\varepsilon_k} \subset \Omega_{\varepsilon_{k+1}}$ for all $\varepsilon_k > \varepsilon_{k+1}$, and perimeters of Q_ε tend to zero as $\varepsilon \to 0$. Moreover, in view of the structure of subdomains Q_ε (see (2.5)) and L^2-property of the matrix C, we have

$$\frac{|\Omega\setminus\Omega_\varepsilon|}{\varepsilon^2} \le \int_{\Omega\setminus\Omega_\varepsilon} \|C(x)\|^2_{\mathbb{S}^N_{\rm skew}}\, dx, \quad \forall \varepsilon > 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \|C\|_{L^2(\Omega\setminus\Omega_\varepsilon; \mathbb{S}^N_{\rm skew})} = 0.$$

Hence, $|\Omega \setminus \Omega_\varepsilon| = o(\varepsilon^2)$ and, therefore, $\lim_{\varepsilon\to 0} |\Omega_\varepsilon| = |\Omega|$.

It is easy to see also that if $C \in L^2\big(\Omega; \mathbb{S}^N_{\rm skew}\big)$ is of the $\mathfrak{F}$-type, then conditions (1)–(ii) of Definition 2.2 and Sobolev Trace Theorem [1] imply the inequality

$$\|\varphi\|_{L^2(\Gamma_\varepsilon)} \le \frac{M}{\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}}\, \|\varphi\|_{H^1_0(\Omega_\varepsilon;\partial\Omega)}, \quad \forall \varphi \in C_0^\infty(\Omega), \tag{2.8}$$

which holds for ε small enough with a constant $M = M(\Omega)$ independent of ε.

Symmetric Matrices. Let $\mathbb{S}^N_{\rm sym}$ be the set of all $N \times N$ symmetric matrices, which are obviously determined by $N(N+1)/2$ scalars. Let α and β be two fix constants such that $0 < \alpha \le \beta < +\infty$. We define $M_\alpha^\beta(\Omega)$ as a set of all matrices $A = [a_{i\,j}]$ in $L^\infty(\Omega; \mathbb{S}^N_{\rm sym})$ such that

$$\alpha I \le A(x) \le \beta I, \quad \text{a.e. in } \Omega. \tag{2.9}$$

In (2.9) I stands for the identity matrix in $\mathbb{M}^N$, and the above inequalities are in the sense of the quadratic forms defined by $(A\xi, \xi)_{\mathbb{R}^N}$ for $\xi \in \mathbb{R}^N$.

Solenoidal vector fields. For any vector field $v \in L^2(\Omega; \mathbb{R}^N)$, the divergence of v is an element $\operatorname{div} v$ of the space $H^{-1}(\Omega)$ defined by the formula

$$\langle \operatorname{div} v, \varphi\rangle_{H^{-1}(\Omega); H^1_0(\Omega)} = -\int_\Omega (v, \nabla\varphi)_{\mathbb{R}^N}\, dx, \quad \forall \varphi \in C_0^\infty(\Omega), \tag{2.10}$$

where $\langle \cdot, \cdot\rangle_{H^{-1}(\Omega); H^1_0(\Omega)}$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$, and $(\cdot, \cdot)_{\mathbb{R}^N}$ stands for the scalar product in $\mathbb{R}^N$.

We define the divergence $\operatorname{div} A$ of a matrix $A \in L^2\big(\Omega; \mathbb{S}^N_{\rm sym}\big)$ as a vector-valued distribution $d \in H^{-1}(\Omega; \mathbb{R}^N)$ by the following rule

$$\langle d_i, \varphi\rangle_{H^{-1}(\Omega); H^1_0(\Omega)} = -\int_\Omega (a_i, \nabla\varphi)_{\mathbb{R}^N}\, dx, \quad \forall \varphi \in C_0^\infty(\Omega),\ \forall i \in \{1, \dots, N\}, \tag{2.11}$$

where a_i stands for the i-th row of the matrix A.

Variational convergence of optimal control problems. Throughout the paper ε denotes a small parameter which varies within a strictly decreasing sequence of positive numbers converging to 0. When we write $\varepsilon > 0$, we consider only the elements of this sequence, in the case $\varepsilon \geq 0$ we also consider its limit $\varepsilon = 0$.

Let $I_\varepsilon : \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \to \overline{\mathbb{R}}$ be a cost functional, $\mathbb{Y}_\varepsilon$ be a space of states, and $\mathbb{U}_\varepsilon$ be a space of controls. Let $\min\{I_\varepsilon(u, y) : (u, y) \in \Xi_\varepsilon\}$ be a parameterized OCP, where

$$\Xi_\varepsilon \subset \{(u_\varepsilon, y_\varepsilon) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon : u_\varepsilon \in U_\varepsilon,\ F(u_\varepsilon, y_\varepsilon) = 0,\ I_\varepsilon(u_\varepsilon, y_\varepsilon) < +\infty\}$$

is the set of all admissible pairs linked by some state equation $F(u_\varepsilon, y_\varepsilon) = 0$. Hereinafter, we always associate to such OCP the corresponding constrained minimization problem:

$$(\mathrm{CMP}_\varepsilon): \quad \Big\langle \inf_{(u,y)\in\,\Xi_\varepsilon} I_\varepsilon(u, y)\Big\rangle. \tag{2.12}$$

Since the sequence of constrained minimization problems (2.12) lives in variable spaces $\mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon$, we assume that there exists a Banach space $\mathbb{U} \times \mathbb{Y}$ such that a convergence in the scale $\{\mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ is well defined. Following the scheme of the direct variational convergence [12] (see also [4, 5, 6]), we adopt the following definition for the convergence of minimization problems in variable spaces.

Definition 2.4. A problem $\langle \inf_{(u,y)\in\Xi} I(u, y)\rangle$ is the variational limit of the sequence (2.12) as $\varepsilon \to 0$. in symbols,

$$\Big\langle \inf_{(u,y)\in\,\Xi_\varepsilon} I_\varepsilon(u, y)\Big\rangle \xrightarrow[\varepsilon\to 0]{\mathrm{Var}} \Big\langle \inf_{(u,y)\in\Xi} I(u, y)\Big\rangle$$

if and only if the following conditions are satisfied:

(d) The space $\mathbb{U} \times \mathbb{Y}$ possesses the strong approximation property with respect to the scale of spaces $\{\mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$, that is, for every pair $(u, y) \in \mathbb{U} \times \mathbb{Y}$, there exists a sequence $\{(u_\varepsilon, y_\varepsilon) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ such that $(u_\varepsilon, y_\varepsilon) \to (u, y)$.

(dd) If sequences $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and $\{(u_k, y_k)\}_{k\in\mathbb{N}}$ are such that $\varepsilon_k \to 0$ as $k \to \infty$, $(u_k, y_k) \in \Xi_{\varepsilon_k}$ for all $k \in \mathbb{N}$, and $(u_k, y_k) \to (u, y)$, then $(u, y) \in \Xi$ and $I(u, y) \leq \liminf_{k\to\infty} I_{\varepsilon_k}(u_k, y_k)$.

(ddd) For every $(u, y) \in \Xi \subset \mathbb{U} \times \mathbb{Y}$, there are a constant $\varepsilon^0 > 0$ and a sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ (called a Γ-realizing sequence) such that

$$(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon, \quad \forall \varepsilon \leq \varepsilon^0, \quad (u_\varepsilon, y_\varepsilon) \to (u, y), \tag{2.13}$$

$$I(u, y) \geq \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon). \tag{2.14}$$

3. Setting of the Optimal Control Problem

Let $f \in H^{-1}(\Omega)$ be a given distribution. The optimal control problem we consider in this paper is to minimize the discrepancy (tracking error) between a given distribution $y_d \in H^1(\Omega)$ and a solution y of the Dirichlet boundary-value problem for the linear elliptic equation

$$-\operatorname{div}\big(A(x)\nabla y + C(x)\nabla y\big) = f \quad \text{in } \Omega, \tag{3.1}$$

$$y = 0 \quad \text{on } \partial\Omega \tag{3.2}$$

by choosing an appropriate control $A \in L^\infty(\Omega;\mathbb{S}^N_{\rm sym})$. Here, $C \in L^2(\Omega;\mathbb{S}^N_{\rm skew})$ is a given matrix. More precisely, we are concerned with the OCP

$$\text{Minimize } I(A,y) = \|y-y_d\|^2_{H^1(\Omega)} := \int_\Omega (y-y_d)^2\,dx + \int_\Omega \|\nabla y - \nabla y_d\|^2_{\mathbb{R}^N}\,dx \tag{3.3}$$

subject to the constraints

$$(A,y) \text{ are related by (3.1)–(3.2)}, \tag{3.4}$$

$$y \in H^1_0(\Omega), \quad A \in \mathfrak{A}_{\rm ad} \subset L^\infty(\Omega;\mathbb{S}^N_{\rm sym}), \tag{3.5}$$

where, by analogy with [7], we define the class of admissible controls $\mathfrak{A}_{\rm ad}$ as follows:

$$\mathfrak{A}_{\rm ad} = \{A = [a_1,\dots,a_N] \in M^\beta_\alpha(\Omega) |\operatorname{div} a_i \in Q_i,\ \forall i = 1,\dots,N\}. \tag{3.6}$$

Here, $\{Q_1,\dots,Q_N\}$ is a given collection of nonempty compact subsets of $H^{-1}(\Omega)$. In view of the compactness of the embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, we can consider $\{Q_1,\dots,Q_N\}$ as a collection of bounded closed subsets of $L^2(\Omega)$. Hereinafter, we assume that $\mathfrak{A}_{\rm ad} \subset L^\infty(\Omega;\mathbb{S}^N_{\rm sym})$ is a nonempty subset of $L^\infty(\Omega;\mathbb{S}^N_{\rm sym})$.

For our further analysis, we use the following obvious result.

Proposition 3.1. *The set $\mathfrak{A}_{\rm ad}$ is sequentially compact with respect to the weak-$*$ topology of $L^\infty(\Omega;\mathbb{S}^N_{\rm sym})$.*

The distinguishing feature of optimal control problem (3.3)–(3.5) is the fact that the skew-symmetric matrix C is merely measurable and belongs to the space $L^2\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$ (rather than the space of bounded matrices $L^\infty\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$). As a rule this entails a number of pathologies with respect to the standard properties of optimal control problems for the classical elliptic equations, even if f in the right-hand has "good" properties. In particular, the unboundedness of the skew-symmetric matrix C may lead to the non-uniqueness of weak solutions to the corresponding boundary-value problem.

Definition 3.2. We say that a function $y = y(A,C,f)$ is a weak solution to the boundary-value problem (3.1)–(3.2) for given matrices $A \in \mathfrak{A}_{\rm ad}$ and $C \in L^2\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$ and a given distribution $f \in H^{-1}(\Omega)$, if $y \in H^1_0(\Omega)$ and the integral identity

$$\int_\Omega \big(\nabla\varphi, A\nabla y + C\nabla y\big)_{\mathbb{R}^N}\,dx = \langle f,\varphi\rangle_{H^{-1}(\Omega);H^1_0(\Omega)} \tag{3.7}$$

holds for every $\varphi \in C^\infty_0(\Omega)$.

Proposition 3.3. *Let $A \in \mathfrak{A}_{\rm ad}$ be a given control. Let $y \in H^1_0(\Omega)$ be a weak solution to the boundary-value problem* (3.1)–(3.2) *in the sense of Definition 3.2. Then $y \in D(C)$.*

Proof. To verify the validity of this assertion it is enough to rewrite the integral identity (3.7) in the form

$$[y,\varphi]_C = -\int_\Omega \big(A\nabla y,\nabla\varphi\big)_{\mathbb{R}^N}\,dx + \langle f,\varphi\rangle_{H^{-1}(\Omega);H^1_0(\Omega)} \tag{3.8}$$

and apply Hölder's inequality to the right-hand side of (3.8). As a result, we arrive at the relation

$$\Big|[y,\varphi]_C\Big| \le \Big(\|A\|_{L^\infty(\Omega;\mathbb{S}^N_{\rm sym})}\|\nabla y\|_{L^2(\Omega;\mathbb{R}^N)} + \|f\|_{H^{-1}(\Omega)}\Big)\,\|\varphi\|_{H^1_0(\Omega)},$$

which leads us immediately to estimate (2.1). The proof is complete. □

Remark 3.4. Due to Proposition 3.3, Definition 3.2 can be reformulated as follows: y is a weak solution to the problem (3.1)–(3.2) for a given control $A \in \mathfrak{A}_{\rm ad}$, if and only if $y \in D(C)$ and

$$\int_\Omega \big(A\nabla y, \nabla\varphi\big)_{\mathbb{R}^N}\,dx + [y,\varphi]_C = \langle f,\varphi\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}, \quad \forall \varphi \in H_0^1(\Omega). \tag{3.9}$$

Moreover, as immediately follows from (2.2) and (3.9), every weak solution $y \in D(C)$ to the problem (3.1)–(3.2) satisfies the energy equality

$$\int_\Omega \big(A\nabla y, \nabla y\big)_{\mathbb{R}^N}\,dx + [y,y]_C = \langle f,y\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}. \tag{3.10}$$

It is well known that boundary-value problem (3.1)–(3.2) is ill-posed, in general (see, for instance, [9, 18, 20, 21, 22]). It means that there exists a matrix $C \in L^2\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$ such that the corresponding state $y \in H_0^1(\Omega)$ may be not unique.

Definition 3.5. We say that (A,y) is an admissible pair to the OCP (3.3)–(3.4) if $A \in \mathfrak{A}_{\rm ad} \subset L^\infty\big(\Omega;\mathbb{S}^N_{\rm sym}\big)$, $y \in D(C) \subset H_0^1(\Omega)$, and the pair (A,y) is related by the integral identity (3.9).

We denote by Ξ the set of all admissible pairs for the OCP (3.3)–(3.4). Let τ be the topology on the set $\Xi \subset L^\infty\big(\Omega;\mathbb{S}^N_{\rm sym}\big) \times H_0^1(\Omega)$ which we define as the product of the weak-$*$ topology of $L^2\big(\Omega;\mathbb{S}^N_{\rm sym}\big)$ and the weak topology of $H_0^1(\Omega)$. We say that a pair $(A^0,y^0) \in L^\infty\big(\Omega;\mathbb{S}^N_{\rm sym}\big) \times D(C)$ is optimal for problem (3.3)–(3.4) if

$$(A^0,y^0) \in \Xi \quad\text{and}\quad I(A^0,y^0) = \inf_{(A,y)\in\Xi} I(A,y).$$

Remark 3.6. As follows from the definition of the form $[y,\varphi]_C$, the value $[y,y]_C$ is not of constant sign for all $y \in D(C)$. It means that, for a given $C \in L^2\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$, we can admit the existence of elements $y^* \in D(C)$ and $y^\sharp \in D(C)$ such that $y^* \neq y^\sharp$, $[y^*,y^*]_C > 0$, and $[y^\sharp,y^\sharp]_C < 0$, whereas $[y,y]_C = 0$ for all $y \in H_0^1(\Omega)$ provided $C \in L^\infty\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$ (for the details, we refer to Zhikov [22]). Hence, the energy equality (3.10) does not allow us to derive a reasonable a priory estimate for the weak solutions of boundary-value problem (3.1)–(3.2) with respect to H_0^1-norm. Moreover, the monotonicity properties of operator $-\operatorname{div}\big(A\nabla + C\nabla\big) : D(C) \to H^{-1}(\Omega)$ for each admissible $A \in \mathfrak{A}_{\rm ad}$ remain an open problem for the time being. Hence, it is unknown whether the set of admissible pairs Ξ is closed with respect to the τ-topology on $L^\infty\big(\Omega;\mathbb{S}^N_{\rm sym}\big) \times H_0^1(\Omega)$.

In view of these remarks, use the following Hypotheses.

- (H1) OCP (3.3)–(3.5) is regular in the following sense: there exists at least one pair $(A,y) \in L^\infty\big(\Omega;\mathbb{S}^N_{\rm sym}\big) \times H_0^1(\Omega)$ such that $(A,y) \in \Xi$.
- (H2) The set of admissible solutions Ξ to OCP (3.3)–(3.5) is sequentially closed with respect to the τ-topology.

Then, following standard techniques (see, for instance, [10]), it is easy to prove the existence of optimal pairs to OCP (3.3)–(3.5).

Theorem 3.7. *If Hypotheses* (H1) *and* (H2) *are valid, then, for each* $f \in H^{-1}(\Omega)$ *and* $y_d \in H^1(\Omega)$, *optimal control problem* (3.3)–(3.5) *admits at least one solution*

$$(A^{opt},y^{opt}) \in \Xi \subset L^\infty(\Omega;\mathbb{S}^N_{\rm sym}) \times H_0^1(\Omega).$$

4. Substantiation of Hypothesis (H1)

The question we are going to discuss in this section is to show that Hypothesis (H1) on the regularity property of OCP (3.3)–(3.5) in Theorem 3.7 can be eliminated due to the approximation approach. It is clear that condition $C \in L^2(\Omega;\mathbb{S}^N_{\rm skew})$ ensures the existence of the sequence of skew-symmetric matrices $\{C_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{S}^N_{\rm skew})$ such that $C_k \to C$ strongly in $L^2(\Omega;\mathbb{S}^N_{\rm skew})$.

Theorem 4.1. *Let $y_d \in H^1(\Omega)$ and $f \in H^{-1}(\Omega)$ be given distributions. Then the set of admissible solutions Ξ to OCP (3.3)–(3.5) is nonempty. Moreover, for each admissible control $A \in A_{\rm ad}$, there exists a weak solution $y(A) \in H^1_0(\Omega)$ to the problem (3.1)–(3.2) such that*

$$[\widehat{y},\widehat{y}]_C \ge 0. \tag{4.1}$$

Proof. Let $A \in A_{\rm ad}$ be a given admissible control. Let $\{C_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{S}^N_{\rm skew})$ be an arbitrary L^2-approximation of the matrix $C \in L^2(\Omega;\mathbb{S}^N_{\rm skew})$. Then, for each $k \in \mathbb{N}$, the corresponding form $[y,\varphi]_{C_k} = \int_\Omega \big(\nabla\varphi, C_k\nabla y\big)_{\mathbb{R}^N}\,dx$ is bounded on $H^1_0(\Omega)\times H^1_0(\Omega)$ and satisfies the identity

$$\int_\Omega \big(\nabla\varphi, C_k\nabla y\big)_{\mathbb{R}^N}\,dx = -\int_\Omega \big(\nabla y, C_k\nabla \varphi\big)_{\mathbb{R}^N}\,dx.$$

Therefore,

$$\int_\Omega \big(\nabla v, C_k(x)\nabla v\big)_{\mathbb{R}^N}\,dx = 0, \quad \forall v \in H^1_0(\Omega) \tag{4.2}$$

and, hence, by the Lax-Milgram lemma the boundary-value problem

$$-\operatorname{div}\big(A\nabla y + C_k\nabla y\big) = f \quad \text{in } \Omega, \tag{4.3}$$

$$y = 0 \quad \text{on } \partial\Omega \tag{4.4}$$

has a unique solution $y_k \in H^1_0(\Omega)$ for each $C_k \in L^\infty(\Omega;\mathbb{S}^N_{\rm skew})$ such that

$$\int_\Omega \big(\nabla\varphi, A\nabla y_k + C_k\nabla y_k\big)_{\mathbb{R}^N}\,dx = \langle f,\varphi\rangle_{H^{-1}(\Omega);H^1_0(\Omega)}, \quad \forall\varphi \in C^\infty_0(\Omega), \tag{4.5}$$

$$\int_\Omega \big(\nabla y_k, A\nabla y_k\big)_{\mathbb{R}^N}\,dx = \langle f, y_k\rangle_{H^{-1}(\Omega);H^1_0(\Omega)}. \tag{4.6}$$

Thus, $\|y_k\|_{H^1_0(\Omega)} \le \alpha^{-1}\|f\|_{H^{-1}(\Omega)}$ and we can assume that the sequence $\{y_k\}_{k\in\mathbb{N}}$ is weakly convergent: $y_k \rightharpoonup \widehat{y}$ in $H^1_0(\Omega)$. Since, $\nabla y_k \rightharpoonup \nabla\widehat{y}$ in $L^2(\Omega;\mathbb{R}^N)$ and $C_k\nabla\varphi \to C\nabla\varphi$ in $L^2(\Omega;\mathbb{R}^N)$ for all $\varphi \in C^\infty_0(\Omega)$, we can pass to the limit in (4.5). As a result, we have

$$\int_\Omega \big(\nabla\varphi, A\nabla\widehat{y} + C\nabla\widehat{y}\big)_{\mathbb{R}^N}\,dx = \langle f,\varphi\rangle_{H^{-1}(\Omega);H^1_0(\Omega)}, \quad \forall\varphi \in C^\infty_0(\Omega).$$

Thus, $\widehat{y}$ is a weak solution to the original boundary-value problem (3.1)–(3.2), and, hence, $\widehat{y} \in D(C)$ by Proposition 3.3. This yields: $(A,\widehat{y}) \in \Xi$.

To proof property (4.1), it remains to pass to the limit in the energy equality (4.6) using the lower semicontinuity of the norm $\|\cdot\|_{L^2(\Omega;\mathbb{R}^N)}$ with respect to the weak convergence $\nabla y_k \rightharpoonup \nabla\widehat{y}$ in $L^2(\Omega;\mathbb{R}^N)$. We obtain

$$\begin{aligned}\langle f,\widehat{y}\,\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} &= \lim_{k\to\infty}\int_\Omega (\nabla y_k, A\nabla y_k)_{\mathbb{R}^N}\,dx\\ &= \lim_{k\to\infty}\int_\Omega \|A^{1/2}\nabla y_k\|^2_{\mathbb{R}^N}\,dx\\ &\geq \int_\Omega \|A^{1/2}\nabla\widehat{y}\|^2_{\mathbb{R}^N}\,dx = \int_\Omega (\nabla\widehat{y}, A\nabla\widehat{y})_{\mathbb{R}^N}\,dx.\end{aligned}\tag{4.7}$$

Thus, the desired inequality (4.1) obviously follows from (3.10) and (4.7). □

Remark 4.2. As Theorem 4.1 proves, for any approximation $\{C_k\}_{k\in\mathbb{N}}$ of the matrix $C \in L^2(\Omega;\mathbb{S}^N_{\rm skew})$ with properties $\{C_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{S}^N_{\rm skew})$ and $C_k \to C$ strongly in $L^2(\Omega;\mathbb{S}^N_{\rm skew})$, the corresponding solutions to the regularized boundary-value problem (4.3)–(4.4) always lead in the limit to some weak solution $\widehat{y}$ of the original problem (3.1)–(3.2). However, this solution can depend on the choice of the approximative sequence $\{C_k\}_{k\in\mathbb{N}}$.

Definition 4.3. We say that a pair $(\widehat{A},\widehat{y}\,) \in L^\infty(\Omega;\mathbb{S}^N_{\rm sym}) \times H_0^1(\Omega)$ is a variational solution to OCP (3.3)–(3.5) if there exists an L^2-approximation $\{C_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{S}^N_{\rm skew})$ of the matrix $C \in L^2(\Omega;\mathbb{S}^N_{\rm skew})$ such that $(\widehat{A},\widehat{y}\,) \in \Xi$ and

$$A_k^0 \overset{*}{\rightharpoonup} \widehat{A} \quad \text{in } L^\infty(\Omega;\mathbb{S}^N_{\rm sym}),\tag{4.8}$$

$$y_k^0 \rightharpoonup \widehat{y} \quad \text{in } H_0^1(\Omega),\tag{4.9}$$

$$I(A_k^0,y_k^0) \xrightarrow[k\to\infty]{} I(\widehat{A},\widehat{y}) = \inf_{(A,y)\in\Xi} I(A,y),\tag{4.10}$$

where (A_k^0,y_k^0) is an optimal pair to the approximate problem (3.3),(3.5),(4.3),(4.4).

As a direct consequence of Definition 4.3 and Theorem 4.1, we have the following characteristic property of variational solutions.

Proposition 4.4. *Let $(\widehat{A},\widehat{y}\,) \in L^\infty(\Omega;\mathbb{S}^N_{\rm sym}) \times H_0^1(\Omega)$ be a variational solution to OCP (3.3)–(3.5) in the sense of Definition 4.3. Then $[\widehat{y},\widehat{y}]_C = 0$.*

Proof. Let $\{(A_k^0,y_k^0)\}_{k\in\mathbb{N}}$ be a sequence of optimal pairs to approximate problems (3.3), (3.5), (4.3), (4.4). In view of the property (4.10), we have

$$\inf_{(A,y)\in\,\Xi} I(A,y) = I\big(\widehat{A},\widehat{y}\big) := \|\widehat{y}-y_d\|^2_{H^1(\Omega)} = \lim_{k\to\infty} I(A_k^0,y_k^0) = \lim_{k\to\infty}\|y_k^0-y_d\|^2_{H^1(\Omega)}.\tag{4.11}$$

Hence, in addition to (4.9), $y_k^0 \to \widehat{y}$ strongly in $H_0^1(\Omega)$. As a result, we finally get

$$\begin{aligned}0 &\overset{\text{by (4.2)}}{=} \lim_{k\to\infty}[y_k^0,y_k^0]_{C_k}\\ &\overset{\text{by (4.6)}}{=} -\lim_{k\to\infty}\int_\Omega \big(\nabla y_k^0, A_k^0\nabla y_k^0\big)_{\mathbb{R}^N}\,dx + \lim_{k\to\infty}\langle f,y_k^0\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}\\ &\overset{\text{by (4.8), (4.11)}}{=} -\int_\Omega \Big(\nabla\widehat{y},\widehat{A}\nabla\widehat{y}\Big)_{\mathbb{R}^N}\,dx + \langle f,\widehat{y}\,\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \overset{\text{by (3.10)}}{=} [\widehat{y},\widehat{y}]_C.\end{aligned}$$

□

Since, for some matrices $C \in L^2(\Omega;\mathbb{S}^N_{\rm skew})$, the weak solutions to the boundary-value problem (3.1)–(3.2) are not unique in general, it follows that even if the OCP (3.3)–(3.4) has a unique solution (A^0,y^0) and even if this solution possesses the property $[y^0, y^0]_C \geq 0$, it does not ensure that the pair (A^0, y^0) is the variational solution to the above problem in the sense of Definition 4.3.

5. Substantiation of Hypothesis (H2)

Let us consider the following sequence of regularized OCPs associated with perforated domains Ω_ε,

$$\Big\{\Big\langle \inf_{(A,v,y)\in\Xi_\varepsilon} I_\varepsilon(A,v,y)\Big\rangle, \quad \varepsilon \to 0\Big\}, \tag{5.1}$$

where

$$I_\varepsilon(A,v,y) := \|y-y_d\|^2_{H^1(\Omega_\varepsilon)} + \frac{1}{\varepsilon^\sigma}\|v\|^2_{H^{-1/2}(\Gamma_\varepsilon)}, \tag{5.2}$$

$$\begin{aligned}\Xi_\varepsilon = \Big\{&(A,v,y) : -\operatorname{div}\big(A\nabla y + C\nabla y\big) = f_\varepsilon \quad \text{in } \Omega_\varepsilon, y = 0 \text{ on } \partial\Omega,\\ &\partial y/\partial\nu_{A+C} = v \text{ on } \Gamma_\varepsilon,\ v \in H^{-1/2}(\Gamma_\varepsilon),\\ &y \in H^1_0(\Omega_\varepsilon;\partial\Omega), A \in \mathfrak{A}_{\rm ad} \subset L^\infty(\Omega;\mathbb{S}^N_{\rm sym})\Big\}.\end{aligned} \tag{5.3}$$

Here, $y_d \in H^1(\Omega)$ and $f_\varepsilon \in L^2(\Omega)$ are given functions, $\{\varepsilon\}$ is a strictly decreasing sequence of positive real numbers converging to 0, the sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ are defined by (2.4), ν is the outward normal unit vector at Γ_ε to Ω_ε, $v \in H^{-1/2}(\Gamma_\varepsilon)$ is a fictitious control, σ is a positive number, and $\Gamma_\varepsilon = \partial\Omega_\varepsilon \setminus \partial\Omega$.

To begin, we show that each of the regularized OCPs (5.1)–(5.3) is solvable. With that in mind, we use the following version of the Compensated Compactness Lemma.

Lemma 5.1. *Let $\varepsilon > 0$ be a given value. Let $\{f_k\}_{k\in\mathbb{N}} \subset L^2(\Omega_\varepsilon;\mathbb{R}^N)$ and $\{g_k\}_{k\in\mathbb{N}} \subset L^2(\Omega_\varepsilon;\mathbb{R}^N)$ be sequences of vector-valued functions such that $f_k \rightharpoonup f_0$ and $g_k \rightharpoonup g_0$ in $L^2(\Omega_\varepsilon;\mathbb{R}^N)$. Assume that*

$$\{\operatorname{div} f_k\}_{k\in\mathbb{N}} \text{ is compact with respect to the strong topology of } H^{-1}(\Omega_\varepsilon), \tag{5.4}$$

$$\text{and} \quad \operatorname{curl} g_k = 0, \quad \forall k \in \mathbb{N}. \tag{5.5}$$

Then

$$\lim_{k\to\infty}\int_{\Omega_\varepsilon} \phi(f_k, g_k)_{\mathbb{R}^N}\,dx = \int_{\Omega_\varepsilon} \phi\big(f_0, g_0\big)_{\mathbb{R}^N}\,dx, \quad \forall \phi \in C_0^\infty(\Omega_\varepsilon). \tag{5.6}$$

For the proof of this lemma, we refer to [7].

Theorem 5.2. *For every $\varepsilon > 0$ the constrained minimization problem*

$$\Big\langle \inf_{(A,v,y)\in\Xi_\varepsilon} I_\varepsilon(A,v,y)\Big\rangle, \tag{5.7}$$

where I_ε and Ξ_ε are defined by (5.2)–(5.3), admits at least one minimizer $(A^0_\varepsilon, v^0_\varepsilon, y^0_\varepsilon)$ in Ξ_ε.

Proof. By analogy with the proof of Theorem 3.7, we can infer the existence of minimizing sequence $\{(A_{k,\varepsilon}, v_{k,\varepsilon}, y_{k,\varepsilon})\}_{k\in\mathbb{N}} \subset \Xi_\varepsilon$ for problem (5.7) such that $\sup_{k\in\mathbb{N}} I_\varepsilon(A_{k,\varepsilon}, v_{k,\varepsilon}, y_{k,\varepsilon}) \leq \widehat{c}$, where the constant $\widehat{c}$ is independent of k. Using the fact that $I_\varepsilon(A_{k,\varepsilon}, v_{k,\varepsilon}, y_{k,\varepsilon}) \geq 0$, we obtain

$$\sup_{k\in\mathbb{N}}\|y_{k,\varepsilon}\|^2_{H^1_0(\Omega_\varepsilon;\partial\Omega)} \leq 4\|y_d\|^2_{H^1(\Omega)} + 2\sup_{k\in\mathbb{N}} I_\varepsilon(A_{k,\varepsilon}, v_{k,\varepsilon}, y_{k,\varepsilon}) \leq 4\|y_d\|^2_{H^1(\Omega)} + 2\widehat{c},$$

$$\sup_{k\in\mathbb{N}}\|v_{k,\varepsilon}\|^2_{H^{-1/2}(\Gamma_\varepsilon)} \leq \varepsilon^\sigma\widehat{c}.$$

Due to this estimate and Proposition 3.1, we may suppose that, up to a subsequence, there exists a triplet $(A^0_\varepsilon, v^0_\varepsilon, y^0_\varepsilon) \in \mathfrak{A}_{\rm ad} \times H^{-1/2}(\Gamma_\varepsilon) \times H^1_0(\Omega_\varepsilon; \partial\Omega)$ such that

$$A_{k,\varepsilon} \overset{*}{\rightharpoonup} A^0_\varepsilon \quad \text{in } L^\infty(\Omega_\varepsilon; \mathbb{S}^N_{\rm sym}), \tag{5.8}$$

$$y_{k,\varepsilon} \rightharpoonup y^0_\varepsilon \quad \text{in } H^1_0(\Omega_\varepsilon; \partial\Omega), \tag{5.9}$$

$$v_{k,\varepsilon} \rightharpoonup v^0_\varepsilon \quad \text{in } H^{-1/2}(\Gamma_\varepsilon), \tag{5.10}$$

$$I_\varepsilon(A^0_\varepsilon, v^0_\varepsilon, y^0_\varepsilon) < +\infty. \tag{5.11}$$

Our next aim is to prove that $(A^0_\varepsilon, v^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon$. With that in mind we note that $C \in L^\infty(\Omega_\varepsilon; \mathbb{S}^N_{\rm skew})$ for every $\varepsilon > 0$, and $\{\xi_k := A_{k,\varepsilon}\nabla y_{k,\varepsilon}\}_{k\in\mathbb{N}}$ is the bounded sequence in $L^2(\Omega_\varepsilon; \mathbb{R}^N)$. So, passing to a subsequence, we may assume the existence of a vector-function $\xi \in L^2(\Omega_\varepsilon; \mathbb{R}^N)$ such that $\xi_k \rightharpoonup \xi$ and $(C\nabla y_{k,\varepsilon}) \rightharpoonup \big(C\nabla y^0_\varepsilon\big)$ in $L^2(\Omega_\varepsilon; \mathbb{R}^N)$. Since, for each $k \in \mathbb{N}$, the integral identity

$$\int_{\Omega_\varepsilon} \big(\nabla\varphi, A_{k,\varepsilon}\nabla y_{k,\varepsilon} + C\nabla y_{k,\varepsilon}\big)_{\mathbb{R}^N}\, dx = \int_{\Omega_\varepsilon} f_\varepsilon \varphi\, dx + \langle v_{k,\varepsilon}, \varphi\rangle_{H^{-1/2}(\Gamma_\varepsilon); H^{1/2}(\Gamma_\varepsilon)} \tag{5.12}$$

holds for all $\varphi \in C^\infty_0(\Omega_\varepsilon; \partial\Omega)$, we can pass to the limit in (5.12) as $k \to \infty$. As a result, we obtain

$$\int_{\Omega_\varepsilon} \big(\nabla\varphi, \xi + C\nabla y^0_\varepsilon\big)_{\mathbb{R}^N}\, dx = \int_{\Omega_\varepsilon} f_\varepsilon \varphi\, dx + \langle v^0_\varepsilon, \varphi\rangle_{H^{-1/2}(\Gamma_\varepsilon); H^{1/2}(\Gamma_\varepsilon)}, \tag{5.13}$$

for all $\varphi \in C^\infty_0(\Omega_\varepsilon; \partial\Omega)$. It remains to show only that $\xi = A^0_\varepsilon \nabla y^0_\varepsilon$. With this aim, we consider the scalar function $v(x) = (z, x)_{\mathbb{R}^N}$, where z is a fixed element of $\mathbb{R}^N$. Since the operator

$$\mathcal{A} := -\operatorname{div}(A\nabla y + C\nabla y) : H^1_0(\Omega_\varepsilon; \partial\Omega) \to H^{-1}(\Omega_\varepsilon; \partial\Omega)$$

is strictly monotone (because $C \in L^\infty(\Omega_\varepsilon; \mathbb{S}^N_{\rm skew})$ for every $\varepsilon > 0$), it follows that, for every $z \in \mathbb{R}^N$ and every positive function $\phi \in C^\infty_0(\Omega_\varepsilon)$, we have

$$\begin{aligned} &\int_{\Omega_\varepsilon} \phi(x)\left(A_{k,\varepsilon}\nabla(y_{k,\varepsilon} - v) + C\nabla(y_{k,\varepsilon} - v), \nabla(y_{k,\varepsilon} - v)\right)_{\mathbb{R}^N}\, dx \\ &\overset{\text{by } C \in L^\infty(\Omega_\varepsilon; \mathbb{S}^N_{\rm skew})}{=} \int_{\Omega_\varepsilon} \phi(x)\left(A\nabla(y_{k,\varepsilon} - v), \nabla(y_{k,\varepsilon} - v)\right)_{\mathbb{R}^N}\, dx \geq 0, \end{aligned}$$

or, taking into account the definition of function $v = v(x)$, this inequality can be rewritten as

$$\int_{\Omega_\varepsilon} \phi(x)\left(\left(A_{k,\varepsilon} + C\right)\left(\nabla y_{k,\varepsilon} - z\right), \nabla y_{k,\varepsilon} - z\right)_{\mathbb{R}^N}\, dx \geq 0. \tag{5.14}$$

Our next intention is to pass to the limit in (5.14) as $k \to \infty$ using Lemma 5.1. Since

$$\begin{gathered} -\operatorname{div}\left(\left(A_{k,\varepsilon} + C\right)\nabla y_{k,\varepsilon}\right) \to f \quad \text{strongly in } H^{-1}(\Omega_\varepsilon), \\ \operatorname{curv}\left(\nabla y_{k,\varepsilon} - z\right) = \operatorname{curv}\nabla y_{k,\varepsilon} = 0, \quad \forall k \in \mathbb{N}, \end{gathered} \tag{5.15}$$

it remains to show that the sequence $\{\operatorname{div}\left(A_{k,\varepsilon} z + Cz\right)\}_{k\in\mathbb{N}}$ is compact with respect to the strong topology of $H^{-1}(\Omega_\varepsilon)$.

Indeed, for every $\phi \in C_0^\infty(\Omega_\varepsilon)$, we have

$$\begin{aligned} &\langle -\operatorname{div}\left(A_{k,\varepsilon} z + Cz\right), \phi\rangle_{H^{-1}(\Omega_\varepsilon);H_0^1(\Omega_\varepsilon)} \\ &= \int_{\Omega_\varepsilon} \left(A_{k,\varepsilon} z, \nabla\phi\right)_{\mathbb{R}^N}\,dx + \int_{\Omega_\varepsilon} \left(Cz, \nabla\phi\right)_{\mathbb{R}^N}\,dx, \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} &\int_{\Omega_\varepsilon} \big(A_{k,\varepsilon}(x) z, \nabla\phi\big)_{\mathbb{R}^N}\,dx \\ &= \int_{\Omega_\varepsilon} \left(A_{k,\varepsilon}(x) z, \nabla\phi\right)\,dx = \int_{\Omega_\varepsilon} \left(\begin{bmatrix} (a^1_{k,\varepsilon}(x), z)_{\mathbb{R}^N} \\ \dots \\ (a^N_{k,\varepsilon}(x), z)_{\mathbb{R}^N} \end{bmatrix}, \nabla\phi \right)_{\mathbb{R}^N} dx \\ &= \int_{\Omega_\varepsilon} \sum_{i=1}^N \big(a^i_{k,\varepsilon}(x), z\big)_{\mathbb{R}^N} \frac{\partial\phi}{\partial x_i}\,dx = \int_{\Omega_\varepsilon} \sum_{i=1}^N \sum_{j=1}^N a^{(i,j)}_{k,\varepsilon}(x) \frac{\partial\phi}{\partial x_i} z_j\,dx \\ &= \sum_{j=1}^N z_j \int_{\Omega_\varepsilon} (a^j_{k,\varepsilon}(x), \nabla\phi)_{\mathbb{R}^N}\,dx = \sum_{j=1}^N z_j \int_{\Omega} (a^j_{k,\varepsilon}(x), \nabla\widetilde{\phi})_{\mathbb{R}^N}\,dx \\ &= -\sum_{j=1}^N z_j \langle \operatorname{div} a^j_{k,\varepsilon}, \widetilde{\phi}\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} = J_k. \end{aligned} \tag{5.17}$$

Here, by $\widetilde{\phi}$ we denote the zero-extension of ψ to $\mathbb{R}^N$. Then, in view of the definition of the class of admissible controls, we obtain

$$\begin{aligned} \lim_{k\to\infty} J_k &= \sum_{j=1}^N z_j \lim_{k\to\infty} \langle -\operatorname{div} a^j_{k,\varepsilon}, \widetilde{\phi}\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \\ &= \sum_{j=1}^N z_j \langle -\operatorname{div} a^j_{0,\varepsilon}, \widetilde{\phi}\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}, \end{aligned} \tag{5.18}$$

where $a^j_{0,\varepsilon}$ denotes the j-th column of matrix $A^0_\varepsilon \in \mathfrak{A}_{\rm ad}$. Making the converse transformations with (5.18), as we did it in (5.17), we come to the relation

$$\begin{aligned} &\lim_{k\to\infty} \big\langle -\operatorname{div}\big(A_{k,\varepsilon} z + Cz\big), \phi\big\rangle >_{H^{-1}(\Omega_\varepsilon);H_0^1(\Omega_\varepsilon)} \\ &= \int_{\Omega_\varepsilon} \left(Cz, \nabla\phi\right)_{\mathbb{R}^N}\,dx + \lim_{k\to\infty} \sum_{j=1}^N z_j \langle -\operatorname{div} a^j_{0,\varepsilon}, \widetilde{\phi}\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \\ &= \int_{\Omega_\varepsilon} \left(Cz, \nabla\phi\right)_{\mathbb{R}^N}\,dx + \int_{\Omega_\varepsilon} \big(A^0_\varepsilon(x) z, \nabla\phi\big)_{\mathbb{R}^N}\,dx \\ &= \langle -\operatorname{div}\big(A^0_\varepsilon z + Cz\big), \phi\rangle_{H^{-1}(\Omega_\varepsilon);H_0^1(\Omega_\varepsilon)}. \end{aligned} \tag{5.19}$$

Since for every $i = 1, \dots, N$ the sequences $\{\operatorname{div} a^j_{k,\varepsilon}\}_{k\in\mathbb{N}}$ are strongly convergent in $H^{-1}(\Omega)$, from (5.17)–(5.19) it follows that

$$\begin{aligned} &\lim_{k\to\infty} \langle -\operatorname{div}\big(A_{k,\varepsilon} z + Cz\big), \phi_k\rangle_{H^{-1}(\Omega_\varepsilon);H_0^1(\Omega_\varepsilon)} \\ &= \langle -\operatorname{div}\big(A^0_\varepsilon z + Cz\big), \phi\rangle_{H^{-1}(\Omega_\varepsilon);H_0^1(\Omega_\varepsilon)} \end{aligned} \tag{5.20}$$

for each sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega_\varepsilon)$ such that $\phi_k \rightharpoonup \phi$ in $H^{-1}(\Omega_\varepsilon)$. Thus, summing up the above results, we obtain

$$\operatorname{div}\big(A_{k,\varepsilon} z + Cz\big) \to \operatorname{div}\big(A_\varepsilon^0 z + Cz\big) \quad \text{strongly in } H^{-1}(\Omega_\varepsilon). \tag{5.21}$$

As a result, combining properties (5.15) and (5.21), it has been shown that all suppositions of Lemma 5.1 are fulfilled. So, taking into account (5.15), (5.21), and passing to the limit in inequality (5.14) as $k \to \infty$, we obtain

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \phi(x)\big(\xi + C\nabla y_\varepsilon^0 - A_\varepsilon^0 z - Cz, \nabla y_\varepsilon^0 - z\big)_{\mathbb{R}^N}\,dx \\
&= \int_{\Omega_\varepsilon} \phi(x)\big(\xi - A_\varepsilon^0 z, \nabla y_\varepsilon^0 - z\big)_{\mathbb{R}^N}\,dx + \int_{\Omega_\varepsilon} \phi(x)\big(C\nabla y_\varepsilon^0 - Cz, \nabla y_\varepsilon^0 - z\big)_{\mathbb{R}^N}\,dx \\
&\qquad \text{(by the skew-symmetry property of } C\text{)} \\
&= \int_{\Omega_\varepsilon} \phi(x)\big(\xi - A_\varepsilon^0 z, \nabla y_\varepsilon^0 - z\big)_{\mathbb{R}^N}\,dx \geq 0, \quad \forall z \in \mathbb{R}^N
\end{aligned}$$

for all positive $\phi \in C_0^\infty(\Omega)$. After localization, we have $(\xi - A_\varepsilon^0 z, \nabla y_\varepsilon^0 - z)_{\mathbb{R}^N} \geq 0$, $\forall z \in \mathbb{R}^N$. Since the operator $-\operatorname{div}(A_\varepsilon^0 \nabla y) : H_0^1(\Omega_\varepsilon) \to H^{-1}(\Omega_\varepsilon)$ is strictly monotone, it follows that $\xi = A_\varepsilon^0 \nabla y_\varepsilon^0$. Therefore, integral identity (5.13) takes the form

$$\int_{\Omega_\varepsilon} \big(\nabla\varphi, A_\varepsilon^0 \nabla y_\varepsilon^0 + C\nabla y_\varepsilon^0\big)_{\mathbb{R}^N}\,dx = \int_{\Omega_\varepsilon} f_\varepsilon \varphi\,dx + \langle v_\varepsilon^0, \varphi\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)},$$

for all $\varphi \in C_0^\infty(\Omega_\varepsilon; \partial\Omega)$. Hence, the triplet $(A_\varepsilon^0, v_\varepsilon^0, y_\varepsilon^0)$ belongs to the set Ξ_ε and, therefore, it is admissible to optimal control problem (5.1)–(5.3).

To show that $(A_\varepsilon^0, v_\varepsilon^0, y_\varepsilon^0)$ is an optimal solution, it is enough to use the lower semicontinuity of the cost functional I_ε with respect to the convergence (5.8)–(5.10):

$$I_\varepsilon(A_\varepsilon^0, v_\varepsilon^0, y_\varepsilon^0) \leq \liminf_{k\to\infty} \Big[\|y_{k,\varepsilon} - y_d\|^2_{H^1(\Omega_\varepsilon)} + \frac{\|v_{k,\varepsilon}\|^2_{H^{-1/2}(\Gamma_\varepsilon)}}{\varepsilon^\sigma}\Big] = \inf_{(A,v,y)\in\Xi_\varepsilon} I_\varepsilon(A,v,y).$$

The proof is complete. □

The main goal of this section is to show that Hypothesis (H2) is valid. With that in mind we show that if the skew-symmetric matrices $C \in L^2\big(\Omega; \mathbb{S}^N_{\rm skew}\big)$ is of the $\mathfrak{F}$-type, then the set of admissible pairs Ξ to OCP (3.3)–(3.5) is a limit in the sense of Kuratowski of the sequence $\{\Xi_\varepsilon \subset \mathfrak{A}_{\rm ad} \times H^{-1/2}(\Gamma_\varepsilon) \times H_0^1(\Omega_\varepsilon; \partial\Omega)\}_{\varepsilon>0}$ with respect to an appropriate convergence in variable spaces $L^\infty(\Omega; \mathbb{S}^N_{\rm sym}) \times H^{-1/2}(\Gamma_\varepsilon) \times H_0^1(\Omega_\varepsilon; \partial\Omega)$. As a result, due to the main properties of Kuratowski convergence [12], it is easy to conclude that the limit set is sequentially closed.

It is easy to see that monotonicity property of $\{\chi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ leads us to the following obvious conclusion.

Proposition 5.3. *Assume that $C \in L^2\big(\Omega; \mathbb{S}^N_{\rm skew}\big)$ is of the $\mathfrak{F}$-type. Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a sequence of perforated domains of Ω given by (2.5), and let $\{\chi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ be the corresponding sequence of characteristic functions. Then*

$$\chi_{\Omega_\varepsilon} \to \chi_\Omega \quad \text{strongly in } L^2(\Omega). \tag{5.22}$$

For our further analysis, we need to formalize the convergence concept in the scale of variable spaces $\{L^\infty(\Omega; \mathbb{S}^N_{\rm sym}) \times H^{-1/2}(\Gamma_\varepsilon) \times H_0^1(\Omega_\varepsilon; \partial\Omega)\}_{\varepsilon>0}$.

Definition 5.4. We say that a sequence $\{y_\varepsilon \in H_0^1(\Omega_\varepsilon;\partial\Omega)\}_{\varepsilon>0}$ weakly converges to an element $y \in H_0^1(\Omega)$ in variable space $H_0^1(\Omega_\varepsilon;\partial\Omega)$ (in symbols, $y_\varepsilon \rightharpoonup y$ in $H_0^1(\Omega_\varepsilon;\partial\Omega)$), if

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} (\nabla y_\varepsilon, \nabla\varphi)_{\mathbb{R}^N}\, dx = \int_\Omega (\nabla y, \nabla\varphi)_{\mathbb{R}^N}\, dx, \quad \forall\varphi \in C_0^\infty(\Omega).$$

Remark 5.5. Since $\chi_{\Omega_\varepsilon} \to \chi_\Omega$ strongly in $L^1(\Omega)$, it follows that $\mathfrak{F}$-property of the skew-symmetric matrix C implies the so-called strong connectedness of the sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ (see [3]). Hence, there exist extension operators P_ε from $H_0^1(\Omega_\varepsilon;\partial\Omega)$ to $H_0^1(\Omega)$ such that, for some positive constant M independent of ε,

$$\|\nabla\left(P_\varepsilon y\right)\|_{L^2(\Omega;\mathbb{R}^N)} \le M\|\nabla y\|_{L^2(\Omega_\varepsilon;\mathbb{R}^N)}, \quad \forall y \in H_0^1(\Omega_\varepsilon;\partial\Omega). \tag{5.23}$$

Let $y^* \in H_0^1(\Omega)$ be a weak limit in $H_0^1(\Omega)$ of the sequence $\{P_\varepsilon y_\varepsilon \in H_0^1(\Omega)\}_{\varepsilon>0}$. Since $\int_{\Omega_\varepsilon} (\nabla y_\varepsilon, \nabla\varphi)_{\mathbb{R}^N}\, dx = \int_\Omega \left(\nabla\left(P_\varepsilon y_\varepsilon\right), \nabla\varphi\right)_{\mathbb{R}^N} \chi_{\Omega_\varepsilon}\, dx$ for all $\varepsilon > 0$, it follows that

$$\begin{aligned}\int_\Omega (\nabla y, \nabla\varphi)_{\mathbb{R}^N}\, dx &= \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} (\nabla y_\varepsilon, \nabla\varphi)_{\mathbb{R}^N}\, dx = \lim_{\varepsilon\to 0}\int_\Omega \left(\nabla\left(P_\varepsilon y_\varepsilon\right), \nabla\varphi\right)_{\mathbb{R}^N} \chi_{\Omega_\varepsilon}\, dx \\ &\overset{\text{by (5.22), (5.23)}}{=} \int_\Omega (\nabla y^*, \nabla\varphi)_{\mathbb{R}^N}\, dx, \quad \forall\varphi \in C_0^\infty(\Omega).\end{aligned}$$

Hence, the weak limit in the sense of Definition 5.4 does not depend on the choice of extension operators $P_\varepsilon : H_0^1(\Omega_\varepsilon;\partial\Omega) \to H_0^1(\Omega)$ with properties (5.23). Therefore, hereinafter, we suppose that functions y_ε of $H_0^1(\Omega_\varepsilon, \partial\Omega)$, by default, are extended by operators P_ε outside of Ω_ε.

Definition 5.6. We say that a sequence $\{(A_\varepsilon, v_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ converges weakly to a pair $(A, y) \in \mathfrak{A}_{\rm ad} \times H_0^1(\Omega)$ in the scale of spaces

$$\{L^\infty(\Omega;\mathbb{S}^N_{\rm sym}) \times H^{-1/2}(\Gamma_\varepsilon) \times H_0^1(\Omega_\varepsilon;\partial\Omega)\}_{\varepsilon>0}, \tag{5.24}$$

(in symbols, $(A_\varepsilon, v_\varepsilon, y_\varepsilon) \overset{w}{\to} (A, y)$), if

$$A_\varepsilon \overset{*}{\rightharpoonup} A \quad \text{in } L^\infty(\Omega;\mathbb{S}^N_{\rm sym}), \tag{5.25}$$

$$y_\varepsilon \rightharpoonup y \quad \text{in variable } H_0^1(\Omega_\varepsilon;\partial\Omega), \tag{5.26}$$

$$\sup_{\varepsilon>0} \frac{1}{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}\|v_\varepsilon\|^2_{H^{-1/2}(\Gamma_\varepsilon)} < +\infty. \tag{5.27}$$

We are now in a position to state the main result of this section.

Theorem 5.7. *Assume that the matrix $C \in L^2\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$ is of the $\mathfrak{F}$-type. Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a sequence of perforated subdomains of Ω associated with matrix C. Let $f \in H^{-1}(\Omega)$ and $y_d \in H^1(\Omega)$ be given distributions. Let $\{f_\varepsilon \in L^2(\Omega)\}_{\varepsilon>0}$ be a sequence such that $\chi_{\Omega_\varepsilon} f_\varepsilon \to f$ strongly in $H^{-1}(\Omega)$. Assume that the parameter σ in* (5.1) *satisfies condition*

$$\varepsilon^{-\sigma}\mathcal{H}^{N-1}(\Gamma_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0 \quad (\text{see (2.6)}). \tag{5.28}$$

Then the optimal control problem $\langle \inf_{(A,y)\in\Xi} I(A, y)\rangle$ is the variational limit of the sequence (5.1)–(5.3) *(in the sense of Definition 2.4) as the parameter ε tends to zero.*

Proof. Since each of the optimization problems $\langle \inf_{(A,v,y)\in\Xi_\varepsilon} I_\varepsilon(A,v,y)\rangle$ lives in the corresponding space $\mathfrak{A}_{\rm ad}\times H^{-1/2}(\Gamma_\varepsilon)\times H_0^1(\Omega_\varepsilon;\partial\Omega)$, we have to show that in this case all conditions of Definition 2.1 hold. To do so, we divide this proof into three steps.

Step 1. We show that the set $\mathfrak{A}_{\rm ad}\times H_0^1(\Omega)$ possesses the strong approximation property with respect to the w-convergence in the scale of spaces (5.24). Indeed, let $(A,y)\in\mathfrak{A}_{\rm ad}\times H_0^1(\Omega)$ be an arbitrary pair. We define h as an element of $C_0^\infty(\Omega)$ such that

$$\operatorname{div}(A\nabla h+C\nabla h)\in L^2(\Omega). \tag{5.29}$$

Hence, $h\in D(A)$. As a result, we construct the sequence

$$\{(A_\varepsilon,v_\varepsilon,y_\varepsilon)\in L^\infty(\Omega;\mathbb{S}^N_{\rm sym})\times H^{-1/2}(\Gamma_\varepsilon)\times H_0^1(\Omega_\varepsilon;\partial\Omega)\}_{\varepsilon>0}$$

as follows $A_\varepsilon=A$, $v_\varepsilon=\frac{\partial h}{\partial\nu_{A+C}}$ on Γ_ε, and $y_\varepsilon=y$, $\forall\varepsilon>0$. Here,

$$\frac{\partial h}{\partial\nu_{A+C}}=\sum_{i,j=1}^N\Big(a_{ij}(x)+c_{ij}(x)\Big)\frac{\partial h}{\partial x_j}\cos(\nu,x_i),$$

$\cos(n,x_i)$ is the i-th directing cosine of ν, and ν is the outward unit normal vector at Γ_ε to Ω_ε. By Proposition 5.3, we have

$$\lim_{\varepsilon\to0}\int_{\Omega_\varepsilon}(\nabla\varphi,\nabla y_\varepsilon)_{\mathbb{R}^N}\,dx=\lim_{\varepsilon\to0}\int_\Omega(\nabla\varphi,\nabla y)_{\mathbb{R}^N}\,\chi_{\Omega_\varepsilon}\,dx=\int_\Omega(\nabla\varphi,\nabla y)_{\mathbb{R}^N}\,dx$$

for every $\varphi\in C_0^\infty(\Omega)$. Hence, $y_\varepsilon\rightharpoonup y$ in variable $H_0^1(\Omega_\varepsilon;\partial\Omega)$ as $\varepsilon\to0$.

It remains to show that the sequence $\{v_\varepsilon\in H^{-1/2}(\Gamma_\varepsilon)\}_{\varepsilon>0}$ is bounded in the sense of Definition 5.6. Following Green's identity, for an arbitrary $\varphi\in C_0^\infty(\Omega)$, we obtain

$$\begin{aligned}
&\big|\big\langle\frac{\partial h}{\partial\nu_{A+C}},\varphi\big\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}\big|\\
&\le\big|\int_{Q_\varepsilon}\operatorname{div}(A(x)\nabla h+C(x)\nabla h)\,\varphi\,dx\big|+\big|\int_{Q_\varepsilon}(\nabla\varphi,A(x)\nabla h+C(x)\nabla h)_{\mathbb{R}^N}\,dx\big|\\
&\le\Big(\int_{Q_\varepsilon}|\operatorname{div}(A(x)\nabla h+C(x)\nabla h)|^2\,dx\Big)^{1/2}\|\varphi\|_{L^2(Q_\varepsilon)}\\
&\quad+\beta\|\nabla h\|_{L^2(Q_\varepsilon;\mathbb{R}^N)}\|\nabla\varphi\|_{L^2(Q_\varepsilon;\mathbb{R}^N)}\overset{\text{by (2.7)}}{+}c(h)\sqrt{\frac{|\Omega\setminus\Omega_\varepsilon|}{\varepsilon}}\Big(\int_{\Omega\setminus\Omega_\varepsilon}|\nabla\varphi|^2_{\mathbb{R}^N}\,dx\Big)^{1/2}\\
&\le(I_1+I_2+I_3)\,\|\varphi\|_{H^1(\Omega\setminus\Omega_\varepsilon)}.
\end{aligned}$$

Since $|\Omega\setminus\Omega_\varepsilon|=o(\varepsilon^2)$ by the $\mathfrak{F}$-type properties of C, it follows that there exist a suitable change of variables and a constant $M>0$ independent of ε such that

$$\begin{aligned}
I_2&=\beta\|\nabla h\|_{L^2(Q_\varepsilon;\mathbb{R}^N)}\\
&=\beta\Big(M\frac{|\Omega\setminus\Omega_\varepsilon|}{\varepsilon}\int_{\Omega\setminus\Omega_1}\|\nabla h(y)\|^2_{\mathbb{R}^N}\,dy\Big)^{1/2}\\
&\overset{\text{by }(2.6)_2}{\le}M_1\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}\|h\|_{H^1(\Omega)}.
\end{aligned}\tag{5.30}$$

Following similar arguments, in view of (5.29), we obtain

$$I_1=\|\operatorname{div}(A(x)\nabla h+C(x)\nabla h)\|_{L^2(Q_\varepsilon)}\le M_2(h)\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}.$$

As a result, summing up the previous inequalities, we come to the following conclusion: there exists a constant $M = M(h)$ independent of ε such that

$$\frac{1}{\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}}\Big\langle\frac{\partial h}{\partial\nu_{A+C}},\varphi\Big\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)} \le M(h)\|\varphi\|_{H^1(\Omega\setminus\Omega_\varepsilon)}, \quad \forall\varphi\in C_0^\infty(\Omega).$$

Hence,

$$\sup_{\varepsilon>0}\Big(\frac{1}{\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}}\Big\|\frac{\partial h}{\partial\nu_{A+C}}\Big\|_{H^{-1/2}(\Gamma_\varepsilon)}\Big) \le M. \tag{5.31}$$

Thus, the weak approximation property is proved.

Step 2. We show that condition (ddd) of Definition 2.1 holds. Let $(A, y) \in \Xi$ be an arbitrary admissible pair to the original OCP (3.3)–(3.4) and let $L(C)$ be a subspace of $H_0^1(\Omega)$ such that

$$L(C) = \{h \in D(C) : \int_\Omega \big(\nabla\varphi, A\nabla h + C\nabla h\big)_{\mathbb{R}^N}\,dx = 0 \ \forall\varphi\in C_0^\infty(\mathbb{R}^N)\}, \tag{5.32}$$

i.e., $L(C)$ is the set of all weak solutions of the homogeneous problem

$$-\operatorname{div}\big(A\nabla y + C\nabla y\big) = 0 \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega. \tag{5.33}$$

We distinguish two cases.

Case 1. The set $L(C)$ is a singleton. It means that $h \equiv 0$ is a unique solution of homogeneous problem (5.33);

Case 2. The set $L(C)$ is not a singleton. So, we suppose that the set $L(C)$ is a linear subspace of $H_0^1(\Omega)$ and it contains at least one non-trivial element of $D(C) \subset H_0^1(\Omega)$.

We start with the Case 2. Let $h \in D(C)$ be an element of the set $L(C)$ such that h is a non-trivial solution of homogeneous problem (5.33). In the sequel, the choice of element $h \in L(C)$ will be specified (see (5.50)). Then we construct a Γ-realizing sequence $\{(A_\varepsilon, v_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ in the following way:

(j) $A_\varepsilon = A$ for all $\varepsilon > 0$. In view of definition of the set $\mathfrak{A}_{\rm ad}$, we obviously have that $\{A_\varepsilon \in \mathfrak{A}_{\rm ad} \subset L^2(\Omega; \mathbb{S}^N_{\rm sym})\}_{\varepsilon>0}$ is a sequence of admissible controls to the problems (5.1). Note that in this case the property (5.25) is obviously true for the sequence $\{A_\varepsilon\}_{\varepsilon>0}$.

(jj) Fictitious controls $\{v_\varepsilon \in H^{-1/2}(\Gamma_\varepsilon)\}_{\varepsilon>0}$ are defined as follows

$$v_\varepsilon := w_\varepsilon + \frac{\partial h}{\partial\nu_{A_\varepsilon+C}}, \quad \forall\varepsilon>0, \tag{5.34}$$

where distributions w_ε are such that

$$\sup_{\varepsilon>0}\Big(\frac{1}{\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}}\|w_\varepsilon\|_{H^{-1/2}(\Gamma_\varepsilon)}\Big) \le M \tag{5.35}$$

with some constant M independent of ε.

(jjj) $\{y_\varepsilon \in H_0^1(\Omega_\varepsilon; \partial\Omega)\}_{\varepsilon>0}$ is the sequence of weak solutions to the corresponding boundary-value problems

$$-\operatorname{div}\big(A\nabla y_\varepsilon + C\nabla y_\varepsilon\big) = f_\varepsilon \quad \text{in } \Omega_\varepsilon, \tag{5.36}$$

$$y_\varepsilon = 0 \text{ on } \partial\Omega, \quad \partial y_\varepsilon/\partial\nu_{A+C} = v_\varepsilon \quad \text{on } \Gamma_\varepsilon. \tag{5.37}$$

Since $C = T_\varepsilon(C)$ whenever $x \in \Omega_\varepsilon$ for every $\varepsilon > 0$, it means that $C \in L^\infty(\Omega_\varepsilon; \mathbb{M}^N)$. Hence, due to the Lax-Milgram lemma, the sequence $\{y_\varepsilon \in H_0^1(\Omega_\varepsilon; \partial\Omega)\}_{\varepsilon>0}$ is defined in a unique way and for every $\varepsilon > 0$ we have the following decomposition $y_\varepsilon = y_{\varepsilon,1} + y_{\varepsilon,2}$, where $y_{\varepsilon,1}$ and $y_{\varepsilon,2}$ are elements of $H_0^1(\Omega_\varepsilon)$ such that

$$\int_\Omega \big(\nabla\varphi, A\nabla y_{\varepsilon,1} + C\nabla y_{\varepsilon,1}\big)_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx = \int_\Omega f_\varepsilon\chi_{\Omega_\varepsilon}\varphi\,dx + \langle w_\varepsilon,\varphi\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}, \tag{5.38}$$

$$\begin{gathered}\int_\Omega \big(\nabla\varphi, A\nabla y_{\varepsilon,2} + C\nabla y_{\varepsilon,2}\big)_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx = \langle \frac{\partial h}{\partial\nu_{A+C}},\varphi\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)},\\ \forall\varphi\in C_0^\infty(\Omega;\partial\Omega).\end{gathered} \tag{5.39}$$

By the skew-symmetry property of $C \in L^\infty(\Omega_\varepsilon;\mathbb{S}^N_{\rm skew})$, we have

$$\int_{\Omega_\varepsilon} \big(\nabla y_{\varepsilon,i}, C\nabla y_{\varepsilon,i}\big)_{\mathbb{R}^N}\,dx = 0, \quad \text{for } i=1,2.$$

Then (5.38)–(5.39) lead us to the energy equalities

$$\int_\Omega \big(\nabla y_{\varepsilon,1}, A\nabla y_{\varepsilon,1}\big)_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx = \int_\Omega f_\varepsilon\chi_{\Omega_\varepsilon} y_{\varepsilon,1}\,dx + \langle w_\varepsilon, y_{\varepsilon,1}\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}, \tag{5.40}$$

$$\int_\Omega \big(\nabla y_{\varepsilon,2}, A\nabla y_{\varepsilon,2}\big)_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx = \langle \frac{\partial h}{\partial\nu_{A+C}}, y_{\varepsilon,2}\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}. \tag{5.41}$$

By the initial assumptions, we have $h \in L(C)$. Then the condition (iii) of Definition 2.2 implies that (for the details we refer to (5.30))

$$\begin{aligned}|\langle \frac{\partial h}{\partial\nu_{A+C}},\varphi\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}| &= |\int_{\Omega\setminus\Omega_\varepsilon}\big(\nabla\varphi, A\nabla h + C\nabla h\big)_{\mathbb{R}^N}\,dx|\\ &\le \sqrt{\frac{|\Omega\setminus\Omega_\varepsilon|}{\varepsilon}}\,(M_1(h)+M_2(h))\,\|\varphi\|_{H^1(\Omega\setminus\Omega_\varepsilon)}\\ &\overset{\text{by (2.6)}}{\le} M(h)\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}\|\varphi\|_{H^1(\Omega\setminus\Omega_\varepsilon)},\end{aligned}$$

for all $\varphi \in H_0^1(\Omega)$, with some constant $M(h)$ independent of ε. Hence,

$$\sup_{\varepsilon>0}\big(\mathcal{H}^{N-1}(\Gamma_\varepsilon)\big)^{-1}\|\frac{\partial h}{\partial\nu_{A+C}}\|^2_{H^{-1/2}(\Gamma_\varepsilon)} < M(h) < +\infty. \tag{5.42}$$

Thus, using the continuity of the embedding $H^{1/2}(\Gamma_\varepsilon)\hookrightarrow L^2(\Gamma_\varepsilon)$ and Sobolev Trace Theorem, we obtain

$$\begin{aligned}\big|\langle w_\varepsilon, y_{\varepsilon,1}\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}\big| &\overset{\text{by (5.35)}}{\le} C\|y_{\varepsilon,1}\|_{L^2(\Gamma_\varepsilon)}\big(\mathcal{H}^{N-1}(\Gamma_\varepsilon)\big)^{1/2}\\ &\overset{\text{by (2.8)}}{\le} C_1\,\|y_{\varepsilon,1}\|_{H_0^1(\Omega_\varepsilon;\partial\Omega)},\end{aligned} \tag{5.43}$$

$$\begin{aligned}\big|\langle \frac{\partial h}{\partial\nu_{A+C}}, y_{\varepsilon,2}\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}\big| &\le C\|y_{\varepsilon,2}\|_{L^2(\Gamma_\varepsilon)}\big(\mathcal{H}^{N-1}(\Gamma_\varepsilon)\big)^{1/2}\\ &\overset{\text{by (2.8)}}{\le} C_1\|y_{\varepsilon,2}\|_{H_0^1(\Omega_\varepsilon;\partial\Omega)}.\end{aligned} \tag{5.44}$$

As a result, we arrive at the a priori estimates

$$\Big(\int_\Omega \big\|\nabla y_{\varepsilon,1}\big\|^2_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx\Big)^{1/2} \le \alpha^{-1}\big(\|f_\varepsilon\chi_{\Omega_\varepsilon}\|_{H^{-1}(\Omega)} + M\big), \tag{5.45}$$

$$\Big(\int_\Omega \big\|\nabla y_{\varepsilon,2}\big\|^2_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx\Big)^{1/2}\le M\alpha^{-1}. \tag{5.46}$$

Hence, the sequences $\{y_{\varepsilon,1}\in H_0^1(\Omega_\varepsilon;\partial\Omega)\}_{\varepsilon>0}$ and $\{y_{\varepsilon,2}\in H_0^1(\Omega_\varepsilon;\partial\Omega)\}_{\varepsilon>0}$ are weakly compact with respect to the weak convergence in variable spaces [12], i.e., we may assume that there exists a couple of functions $\widehat{y}_1$ and $\widehat{y}_2$ in $H_0^1(\Omega)$ such that

$$\lim_{\varepsilon\to 0}\int_\Omega \big(\nabla\varphi,\nabla y_{\varepsilon,i}\big)_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx=\int_\Omega \big(\nabla\varphi,\nabla\widehat{y}_i\big)_{\mathbb{R}^N}\,dx, \tag{5.47}$$

for all $\varphi\in C_0^\infty(\Omega)$ and $i=1,2$.

Now we can pass to the limit in integral identities (5.38)–(5.39) as $\varepsilon\to 0$. Using (5.35), (5.47), (5.42), L^∞-property of $A\in\mathfrak{A}_{\rm ad}$, and the fact that $\chi_{\Omega_\varepsilon}f_\varepsilon\to f$ strongly in $H^{-1}(\Omega)$, we finally obtain

$$\int_\Omega \big(\nabla\varphi,A\nabla\widehat{y}_1+C\nabla\widehat{y}_1\big)_{\mathbb{R}^N}\,dx=\langle f,\varphi\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \tag{5.48}$$

$$\int_\Omega \big(\nabla\varphi,A\nabla\widehat{y}_2+C\nabla\widehat{y}_2\big)_{\mathbb{R}^N}\,dx=0 \tag{5.49}$$

for every $\varphi\in C_0^\infty(\Omega)$. Hence, $\widehat{y}_1$ and $\widehat{y}_2$ are weak solutions to the boundary-value problem (3.1)–(3.2) and (5.33), respectively. Hence, $\widehat{y}_2\in L(C)$ and $\widehat{y}_1\in D(C)$ by Proposition 3.3. As a result, we arrive at the conclusion: the pair $(A,\widehat{y}_1+h)$ belongs to the set Ξ, for every $h\in L(C)$. Since by the initial assumptions $(A,y)\in\Xi$, it follows that having set in (5.34)

$$h=y-\widehat{y}_1, \tag{5.50}$$

we obtain

$$h\in L(C)\quad\text{and } y_\varepsilon=y_{\varepsilon,1}+y_{\varepsilon,2}\rightharpoonup y\quad\text{in } H_0^1(\Omega_\varepsilon;\partial\Omega)\text{ as }\varepsilon\to 0. \tag{5.51}$$

Therefore, in view of (5.51), (5.42), and (5.35), we see that $(A_\varepsilon,v_\varepsilon,y_\varepsilon)\xrightarrow{w}(A,y)$ in the sense of Definition 5.6. Thus, the properties (2.13) hold.

It is worth to notice that in the Case 1, we can give the same conclusion, because we originally have $h\equiv 0$. Hence, the solutions to boundary-value problems (5.48)–(5.49) are unique and, therefore, we can claim that $y=\widehat{y}_1$, $\widehat{y}_2=0$, and $h=0$.

It remains to prove the inequality (2.14). To do so, it is sufficient to show that

$$\begin{aligned} I(A,y)&:=\|y-y_d\|^2_{H^1(\Omega)}=\lim_{\varepsilon\to 0}I_\varepsilon(A_\varepsilon,v_\varepsilon,y_\varepsilon)\\ &=\lim_{\varepsilon\to 0}\Big[\|y_\varepsilon-y_d\|^2_{H^1(\Omega_\varepsilon)}+\frac{1}{\varepsilon^\sigma}\|v_\varepsilon\|^2_{H^{-1/2}(\Gamma_\varepsilon)}\Big],\end{aligned} \tag{5.52}$$

where the sequence $\{(A_\varepsilon,v_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon\}_{\varepsilon>0}$ is defined by (5.34) and (5.50).

In view of this, we use the following relations

$$\begin{gathered} \|v_\varepsilon\|^2_{H^{-1/2}(\Gamma_\varepsilon)}\le 2\|w_\varepsilon\|^2_{H^{-1/2}(\Gamma_\varepsilon)}+2\|\frac{\partial h}{\partial\nu_{A+C}}\|^2_{H^{-1/2}(\Gamma_\varepsilon)}<+\infty,\\ \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^\sigma}\|w_\varepsilon\|^2_{H^{-1/2}(\Gamma_\varepsilon)}\overset{\text{by (5.35)}}{\le} M\lim_{\varepsilon\to 0}\frac{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}{\varepsilon^\sigma}=0,\\ \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^\sigma}\|\frac{\partial h}{\partial\nu_{A+C}}\|^2_{H^{-1/2}(\Gamma_\varepsilon)}\overset{\text{by (5.42)}}{\le} M\lim_{\varepsilon\to 0}\frac{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}{\varepsilon^\sigma}=0,\\ \lim_{\varepsilon\to 0}\|y_\varepsilon-y_d\|^2_{L^2(\Omega_\varepsilon)}\overset{\text{by (5.22) and (5.51)}}{=}\|y-y_d\|^2_{L^2(\Omega)}.\end{gathered} \tag{5.53}$$

To prove the convergence

$$\lim_{\varepsilon\to 0}\|\nabla y_\varepsilon-\nabla y_d\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^N)}=\|\nabla y-\nabla y_d\|^2_{L^2(\Omega;\mathbb{R}^N)}, \tag{5.54}$$

we will show, in fact, that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(\nabla y_\varepsilon, A\nabla y_\varepsilon)_{\mathbb{R}^N}\,dx=\int_\Omega(\nabla y, A\nabla y)_{\mathbb{R}^N}\,dx. \tag{5.55}$$

In view of the coerciveness property of matrix $A\in\mathbb{A}_{\rm ad}$, it is equivalent to the norm convergence $\|\nabla y_\varepsilon\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^N)}\to\|\nabla y\|^2_{L^2(\Omega;\mathbb{R}^N)}$ that implies relation (5.54) due to the weak convergence (5.51).

To prove (5.55), we apply the energy equality

$$\int_\Omega(\nabla y, A\nabla y)_{\mathbb{R}^N}\,dx=-[y,y]_C+\langle f,y\rangle_{H^{-1}(\Omega);H^1_0(\Omega)}, \tag{5.56}$$

which comes from the condition $(A,y)\in\Xi$. It is easy to see that the integral identity for the weak solutions y_ε to boundary-value problems (5.3) can be represented in the so-called extended form

$$\begin{aligned}
&\int_\Omega\big(\nabla\varphi, A\nabla y_\varepsilon+C\nabla y_\varepsilon\big)_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx\\
&=\int_\Omega f_\varepsilon\chi_{\Omega_\varepsilon}\varphi\,dx+\langle w_\varepsilon,\varphi\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}+\langle\frac{\partial h}{\partial\nu_{A+C}},\varphi\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}\\
&-\int_\Omega\big(\nabla\psi, A\nabla h^*\big)_{\mathbb{R}^N}\,dx-[h^*,\psi]_C,\quad\forall\varphi,\psi\in C_0^\infty(\Omega),
\end{aligned}\tag{5.57}$$

where h^* is an arbitrary element of $L(C)$. Indeed, because of the equality

$$\int_\Omega\big(\nabla\psi, A\nabla h^*\big)_{\mathbb{R}^N}\,dx+[h^*,\psi]_C\overset{\text{by (5.32)}}{=}0,\quad\forall\psi\in C_0^\infty(\Omega),$$

we have an equivalent identity to the classical definition of weak solutions of (5.3).

From (5.42), (5.51), and Sobolev Trace Theorem, it follows that the numerical sequences

$$\{\langle w_\varepsilon,y_\varepsilon\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}\}_{\varepsilon>0},\quad\{\langle\frac{\partial h}{\partial\nu_{A+C}},y_\varepsilon\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}\}_{\varepsilon>0}$$

are bounded. Therefore, we can assume, passing to a subsequence if necessary, that there exists a value $\xi_1\in\mathbb{R}$ such that

$$\langle w_\varepsilon,y_\varepsilon\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}+\langle\frac{\partial h}{\partial\nu_{A+C}},y_\varepsilon\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}\to\xi_1\quad\text{as }\varepsilon\to 0. \tag{5.58}$$

Since $y_\varepsilon\rightharpoonup y$ weakly in $H^1_0(\Omega_\varepsilon;\partial\Omega)$ and $y\in D(C)$, it follows that there exists a sequence of smooth functions $\{\psi_\varepsilon\in C_0^\infty(\Omega)\}_{\varepsilon>0}$ such that $\psi_\varepsilon\to y$ strongly in $H^1_0(\Omega)$. Therefore, following the extension rule (2.2), we have

$$\lim_{\varepsilon\to 0}\int_\Omega\big(\nabla\psi_\varepsilon, A\nabla h^*\big)_{\mathbb{R}^N}\,dx=\int_\Omega\big(\nabla y, A\nabla h^*\big)_{\mathbb{R}^N}\,dx, \tag{5.59}$$

$$\lim_{\varepsilon\to 0}[h^*,\psi_\varepsilon]_C=[h^*,y]_C. \tag{5.60}$$

Since the matrix $C\in L^2\big(\Omega;\mathbb{S}^N_{\rm skew}\big)$ is of the $\mathfrak{F}$-type, we can assume that the element $h^*\in L(A)$ is such that $[h^*,y]_C+\int_\Omega\big(\nabla y, A\nabla h^*\big)_{\mathbb{R}^N}\,dx\neq 0$. Now we specify the choice of element $h^*\in L(C)$ as follows

$$\widehat{h}^* = \frac{\xi_1+[y,y]_C}{\xi_2+\xi_3}\,h^*, \quad \text{where } \xi_3 := \int_\Omega \big(\nabla y, A\nabla h^*\big)_{\mathbb{R}^N}\,dx,\ \xi_2 := [h^*,y]_C,$$

or, in other words, we aim to ensure the condition $\xi_1-\xi_2-\xi_3+[y,y]_C=0$. As a result, we have: $\widehat{h}^*$ is an element of $L(C)$ such that

$$\lim_{\varepsilon\to 0}\int_\Omega \big(\nabla\psi_\varepsilon, \nabla\widehat{h}^*\big)_{\mathbb{R}^N}\,dx = \xi_2\frac{\xi_1+[y,y]_C}{\xi_2+\xi_3}, \quad \lim_{\varepsilon\to 0}[\widehat{h}^*,\psi_\varepsilon]_C = \xi_3\frac{\xi_1+[y,y]_C}{\xi_2+\xi_3}. \tag{5.61}$$

Putting $\varphi=y_\varepsilon$ and $h^*=\widehat{h}^*$ in (5.57) and using the fact that

$$\int_\Omega \big(\nabla y_\varepsilon, C\nabla y_\varepsilon\big)_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx = 0,$$

we arrive at the following energy equality for the boundary-value problem (5.3),

$$\begin{aligned}
&\int_\Omega \big(\nabla y_\varepsilon, A\nabla y_\varepsilon\big)_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx\\
&= \int_\Omega f_\varepsilon\chi_{\Omega_\varepsilon}y_\varepsilon\,dx + \langle w_\varepsilon, y_\varepsilon\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)} + \langle \frac{\partial h}{\partial\nu_{A+C}}, y_\varepsilon\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}\\
&\quad - \int_\Omega \big(\nabla\psi_\varepsilon, A\nabla\widehat{h}^*\big)_{\mathbb{R}^N}\,dx - [\widehat{h}^*,\psi_\varepsilon]_C.
\end{aligned} \tag{5.62}$$

As a result, taking into account the properties (5.22), (5.51), (5.61), we can pass to the limit as $\varepsilon\to 0$ in (5.62). This yields

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega \big(\nabla y_\varepsilon, A\nabla y_\varepsilon\big)_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx\\
&= \lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon\chi_{\Omega_\varepsilon}y_\varepsilon\,dx + \lim_{\varepsilon\to 0}\langle w_\varepsilon, y_\varepsilon\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)}\\
&\quad + \lim_{\varepsilon\to 0}\langle \frac{\partial h}{\partial\nu_{A+C}}, y_\varepsilon\rangle_{H^{-1/2}(\Gamma_\varepsilon);H^{1/2}(\Gamma_\varepsilon)} - \lim_{\varepsilon\to 0}\int_\Omega \big(\nabla\psi_\varepsilon, \nabla\widehat{h}^*\big)_{\mathbb{R}^N}\,dx - \lim_{\varepsilon\to 0}[\widehat{h}^*,\psi_\varepsilon]_C\\
&\overset{\text{by (5.61)}}{=} \langle f, y\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} - [y,y]_C \overset{\text{by (5.56)}}{=} \int_\Omega (\nabla y, A\nabla y)_{\mathbb{R}^N}\,dx.
\end{aligned} \tag{5.63}$$

Hence, turning back to (5.52), we see that this relation is a direct consequence of (5.53) and (5.63). Thus, the sequence $\{(A_\varepsilon,v_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon\}_{\varepsilon>0}$, which is defined by (5.34) and (5.50), is Γ-realizing. The property (ddd) is established.

Step 3. We prove the property (dd) of Definition 2.4. Let $\{(A_k,v_k,y_k)\}_{k\in\mathbb{N}}$ be a sequence such that $(A_k,v_k,y_k)\in\Xi_{\varepsilon_k}$ for some $\varepsilon_k\to 0$ as $k\to\infty$,

$$A_k \overset{*}{\rightharpoonup} A \text{ in } L^\infty(\Omega;\mathbb{S}^N_{\rm sym}), \quad y_k\rightharpoonup y \text{ in variable space } H_0^1(\Omega_{\varepsilon_k};\partial\Omega), \tag{5.64}$$

and the sequence of fictitious controls $\{v_k\in H^{-1/2}(\Gamma_{\varepsilon_k})\}_{k\in\mathbb{N}}$ satisfies inequality (5.27). In view of Definition 5.6 it means that $(A_k,v_k,y_k)\overset{w}{\to}(A,y)$ as $k\to\infty$.

Our aim is to show that

$$(A,y)\in\Xi \quad\text{and}\quad I(A,y)\le\liminf_{k\to\infty} I_{\varepsilon_k}(A_k,v_k,y_k). \tag{5.65}$$

Following the arguments of the proof of Theorem 4.1, it is easy to show that the limit matrix A is an admissible control to OCP (3.3)–(3.4).

Since the integral identity

$$\begin{aligned}&\int_\Omega \big(\nabla\varphi, A_k\nabla y_k + C\nabla y_k\big)_{\mathbb{R}^N}\chi_{\Omega_{\varepsilon_k}}\,dx\\ &= \int_\Omega f_{\varepsilon_k}\chi_{\Omega_{\varepsilon_k}}\varphi\,dx + \langle v_k,\varphi\rangle_{H^{-1/2}(\Gamma_{\varepsilon_k});H^{1/2}(\Gamma_{\varepsilon_k})},\quad \forall\varphi\in C_0^\infty(\Omega)\end{aligned} \tag{5.66}$$

holds for every $k\in\mathbb{N}$, we can pass to the limit in (5.66) as $k\to\infty$ using Definition 5.6 and estimate

$$\big|\langle v_k,\varphi\rangle_{H^{-1/2}(\Gamma_{\varepsilon_k});H^{1/2}(\Gamma_{\varepsilon_k})}\big| \le C(\Omega)\,\|\varphi\|_{H_0^1(\Omega)}\,\big(\mathcal{H}^{N-1}(\Gamma_{\varepsilon_k})\big)^{1/2},\quad \forall\varphi\in C_0^\infty(\Omega)$$

coming from inequality (5.27). Then proceeding as on the Step 2, it can easily be shown that the limit pair (A,y) is admissible to OCP (3.3)–(3.4). Hence, the condition $(5.65)_1$ is valid.

As for the inequality $(5.65)_2$, it immediately follows from the following reasoning. Since $\chi_{\Omega_{\varepsilon_k}}\nabla y_k \rightharpoonup \chi_\Omega\nabla y$ in $L^2(\Omega;\mathbb{R}^N)$, it remains to apply the lower semicontinuity of H^1-norm with respect to the weak convergence and take into account the estimate

$$\frac{1}{(\varepsilon_k)^\sigma}\|v_k\|^2_{H^{-1/2}(\Gamma_{\varepsilon_k})} \le C\frac{\mathcal{H}^{N-1}(\Gamma_{\varepsilon_k})}{(\varepsilon_k)^\sigma}\to 0\quad\text{as } k\to\infty. \tag{5.67}$$

The proof is complete. □

We are now in a position to show that Hypothesis (H2) is valid.

Theorem 5.8. *Assume that the matrix $C\in L^2(\Omega;\mathbb{S}^N_{\rm skew})$ is of the $\mathfrak{F}$-type and the parameter σ in* (5.1) *satisfies condition* (5.28). *Then for each $f\in H^{-1}(\Omega)$ and $y_d\in H^1(\Omega)$ the set of admissible solutions to OCP* (3.3)–(3.5) *is sequentially τ-closed, i.e., Hypothesis (H2) is satisfied.*

Proof. Let $\{(A_k,y_k)\}_{k\in\mathbb{N}}\subset\Xi\subset L^\infty(\Omega;\mathbb{S}^N_{\rm sym})\times H_0^1(\Omega)$ be an arbitrary τ-convergent sequence of admissible pairs, and let $(\widehat{A},\widehat{y})\in L^\infty(\Omega;\mathbb{S}^N_{\rm sym})\times H_0^1(\Omega)$ be its τ-limit, i.e.

$$A_k\overset{*}{\rightharpoonup}\widehat{A}\quad\text{in } L^\infty(\Omega;\mathbb{S}^N_{\rm sym}), y_k\rightharpoonup\widehat{y}\quad\text{in } H_0^1(\Omega).$$

Then $\widehat{A}\in\mathfrak{A}_{\rm ad}$ by Proposition 3.1. The main point is to show that the limit pair $(\widehat{A},\widehat{y})$ is admissible to the problem (3.3)–(3.5).

Indeed, as follows from Theorem 5.7 (see also property (ddd) of Definition 2.4), for each $k\in\mathbb{N}$, there exists a Γ-realizing sequence $\{(A_{k,m},v_{k,m},y_{k,m})\}_{m\in\mathbb{N}}$ such that $(A_{k,m},v_{k,m},y_{k,m})\in\Xi_{\varepsilon_m}$ with $\varepsilon_m=1/m$ for all $m\in\mathbb{N}$ and $(A_{k,m},v_{k,m},y_{k,m})\overset{w}{\to}(A_k,y_k)$ as $m\to\infty$, i.e.

$$\begin{gathered}A_{k,m}\overset{*}{\rightharpoonup}A_k\quad\text{in } L^\infty(\Omega;\mathbb{S}^N_{\rm sym}),\\ \chi_{\Omega_{1/m}}y_{k,m}\rightharpoonup y_k\quad\text{in } H_0^1(\Omega),\\ \sup_{m\in\mathbb{N}}\frac{1}{\mathcal{H}^{N-1}(\Gamma_{1/m})}\|v_{k,m}\|^2_{H^{-1/2}(\Gamma_{1/m})}<+\infty.\end{gathered}$$

Since $\chi_{\Omega_{1/m}}\to\chi_\Omega$ strongly in $L^2(\Omega)$, by Cantor's diagonal arguments, we conclude that $(A_{k,k},v_{k,k},y_{k,k})\overset{w}{\to}(\widehat{A},\widehat{y})$ as $k\to\infty$. Hence, Theorem 5.7 (see also property (dd) of Definition 2.4) implies $(\widehat{A},\widehat{y})\in\Xi$. The proof is complete. □

Taking into account this result, we can now specify the statement of Theorem 3.7 as follows.

Theorem 5.9. *If the matrix $C \in L^2(\Omega; \mathbb{S}^N_{\mathrm{skew}})$ is of the $\mathfrak{F}$-type and the parameter σ in (5.1) satisfies condition (5.28), then, for each $f \in H^{-1}(\Omega)$ and $y_d \in H^1(\Omega)$, the optimal control problem (3.3)–(3.5) has a nonempty set of solutions.*

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