Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 17, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF MULTIPLE SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^N$

HONGHUI YIN, ZUODONG YANG

Abstract. In this article, we establish the multiplicity of positive weak solution for the quasilinear elliptic equation

$$\begin{gathered}
-\Delta_p u+\lambda|u|^{p-2}u=f(x)|u|^{s-2}u+h(x)|u|^{r-2}u \quad x\in\mathbb{R}^N,\\
u>0 \quad x\in\mathbb{R}^N,\\
u\in W^{1,p}(\mathbb{R}^N)
\end{gathered}$$

We show how the shape of the graph of f affects the number of positive solutions. Our results extend the corresponding results in [21].

1. Introduction

In this article we consider the existence of solutions for the nonlinear quasilinear problem

$$\begin{gathered}
-\Delta_p u+\lambda|u|^{p-2}u=f(x)|u|^{s-2}u+h(x)|u|^{r-2}u \quad x\in\mathbb{R}^N,\\
u>0 \quad x\in\mathbb{R}^N\\
u\in W^{1,p}(\mathbb{R}^N)
\end{gathered}\tag{1.1}$$

where $1\le r<p<s<p^*$, $p<N$, $p^*=\frac{pN}{N-p}$ denotes the critical Sobolev exponent, $\lambda>0$ is a parameter, $h\in L^{\frac{p}{p-r}}(\mathbb{R}^N)\backslash\{0\}$ is nonnegative. For the function f, we assume the following conditions:

- (C1) $f\in C(\mathbb{R}^N)$ and is nonnegative in $\mathbb{R}^N$;
- (C2) $f^\infty=\lim_{|x|\to\infty}f(x)>0$;
- (C3) There exist some points $x^1,x^2,\dots,x^k$ in $\mathbb{R}^N$ such that $f(x^i)$ are some strict maxima and satisfy
$$f^\infty<f(x^i)=f_{\max}\equiv\max\{f(x)|x\in\mathbb{R}^N\}$$
for $i=1,2,\dots,k$.

Associated with (1.1), we consider the energy functional

$$I_\lambda(u)=\frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p+\lambda|u|^pdx-\frac{1}{s}\int_{\mathbb{R}^N}f(x)|u|^sdx-\frac{1}{r}\int_{\mathbb{R}^N}h(x)|u|^rdx.$$

2000 *Mathematics Subject Classification.* 35J62, 35J50 35J92.
Key words and phrases. Nehari manifold; quasilinear; positive solution; (PS)-sequence.

Submitted August 4, 2013. Published January 10, 2014.

It is well known that the functional $I_\lambda \in C^1(W^{1,p}(\mathbb{R}^N), R)$, and that the solutions of (1.1) are the critical points of the energy functional I_λ.

When $p = 2$ and $h(x) \equiv 0$, Equation (1.1) becomes

$$\begin{gathered} -\Delta u + \lambda u = f(x)|u|^{s-2}u \quad x \in \mathbb{R}^N, \\ u > 0 \quad x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{gathered} \tag{1.2}$$

It is known that the existence of positive solutions of (1.2) is affected by the shape of the graph of $f(x)$. This has been the focus of a great deal of research by several authors [3, 4, 8, 18]. Specially, if f is a positive constant, then (1.2) has a unique positive solution [15] Adachi and Tanaka [1] showed that there exist at least four positive solutions of the equation

$$\begin{gathered} -\Delta u + \lambda u = f(x)|u|^{s-2}u + h(x) \quad x \in \mathbb{R}^N, \\ u > 0 \quad x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{gathered} \tag{1.3}$$

under the assumptions $0 < f(x) \le f^\infty = \lim_{|x|\to\infty} f(x)$, $h \in H^{-1}(R^N)\backslash\{0\}$ is nonnegative and $\|h\|_{H^{-1}}$ is sufficiently small. Several authors have studied a generalized version of (1.3),

$$\begin{gathered} -\Delta u + \lambda u = f(x,u) + h(x) \quad x \in \mathbb{R}^N, \\ u > 0 \quad x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{gathered} \tag{1.4}$$

where $f(x,u)$ and $h(x)$ satisfy some suitable conditions. They showed the existence of at least two positive solutions when $\|h\|_{H^{-1}}$ is sufficiently small, see [2, 9, 14].

Wu [21] considered the problem (1.1) with $p = 2$, under some suitable assumptions on $f(x), h(x)$. The author obtained the existence of multiple positive solution by variational methods. Several publications [5, 6, 10, 22] show results about the quasilinear elliptic equation

$$\begin{gathered} -\Delta_p u + \lambda|u|^{p-2}u = f(x,u) \quad x \in \Omega, \\ u \in W_0^{1,p}(\Omega),\ u \neq 0 \end{gathered} \tag{1.5}$$

where $1 < p < N$, $N \ge 3$, Ω is an unbounded domain in $\mathbb{R}^N$. Because of the unboundedness of the domain, the Sobolev compact embedding does not hold. There are many methods to overcome the difficulty. In [22], the authors used the concentration-compactness principle posed by Lions and the mountain pass lemma to solve problem (1.5). In [5, 6], the authors studied the problem in symmetric Sobolev spaces which posses Sobolev compact embedding.

Especially, when $\lambda = 1$, $f(x,u) = q(x)u^\alpha$ and Ω is replaced by $\mathbb{R}^N$, using a min-max procedure formulated by Bahri and Li [4], Citti and Uguzzoni [10] obtained the existence of a solution $u \in W^{1,p}(\mathbb{R}^N) \cap C^{1+\beta}_{\rm loc}(\mathbb{R}^N)$ of (1.5) when $p \in (1,2)$, and $\beta \in (0,1)$ is constant. In [19], the authors studied the problem

$$\begin{gathered} -\Delta_p u + \lambda a(x)u^{p-1} = f(x)u^{p^*-1} + g(x)u^q \quad x \in \mathbb{R}^N, \\ u \in D_0^{1,p}(\mathbb{R}^N) \cap C^{1+\beta}_{loc}(\mathbb{R}^N), \quad \lim_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{1.6}$$

which is a general case of (1.1). The authors proved that there exists a positive solution of (1.6) for all λ in some interval $[0, \lambda_0)$.

In this article, we consider show the existence of multiple positive solutions of (1.1). Our arguments are based on a combination of the concentration-compactness principle of Lions [16], and Ekeland's variational principle [13]. Our main result is the following theorem.

Theorem 1.1. *Assume* (C1)–(C3) *hold, and* $h \in L^{\frac{p}{p-r}}(\mathbb{R}^N)\backslash\{0\}$ *is nonnegative. Then there exists* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$*, Equation* (1.1) *has at least* $k+1$ *positive solutions.*

The rest of this article is organized as follows. In Section 2, we give some preliminaries and some properties of Nehri manifold. In Section 3, we prove the main result, Theorem 1.1.

2. Preliminaries

Throughout the paper, C, c will denote various positive constants, their values may vary from place to anther. By the change of variables $\eta = \lambda^{-1/p}$, $v(x) = \eta^{p/(s-p)}u(\eta x)$, Equation (1.1) can be transformed into

$$\begin{gathered} -\Delta_p v + |v|^{p-2}v = f_\eta |v|^{s-2}v + \eta^{\frac{p(s-r)}{s-p}} h_\eta |v|^{r-2}v \quad x \in \mathbb{R}^N, \\ v > 0 \quad x \in \mathbb{R}^N, \\ v \in W^{1,p}(\mathbb{R}^N) \end{gathered} \tag{2.1}$$

where $f_\eta = f(\eta x), h_\eta = h(\eta x)$.

For $u \in W^{1,p}(\mathbb{R}^N)$, $c \in R$, $a \in C(\mathbb{R}^N)$ nonnegative and bounded, and $b \in L^{\frac{p}{p-r}}(\mathbb{R}^N)$ non-negative, we define

$$\begin{gathered} I_{a,b}(u) = \frac{1}{p}\|u\|^p - \frac{1}{s}\int_{\mathbb{R}^N} a|u|^s dx - \eta^{\frac{p(s-r)}{s-p}}\frac{1}{r}\int_{\mathbb{R}^N} b|u|^r dx; \\ M_{a,b}(c) = \{u \in W^{1,p}(\mathbb{R}^N)\backslash\{0\} | \langle I'_{a,b}(u), u\rangle = c\}; \\ \alpha_{a,b}(c) = \inf\{I_{a,b}(u) | u \in M_{a,b}(c)\}, \end{gathered}$$

where $\|u\| = (\int_\Omega |\nabla u|^p + |u|^p dx)^{1/p}$ is a standard norm in $W^{1,p}(\mathbb{R}^N)$ and $I'_{a,b}$ denote the Fréchet derivative of $I_{a,b}$. We shall write $M_{a,b}(0), \alpha_{a,b}(0)$ as $M_{a,b}, \alpha_{a,b}$ respectively. Then, we have the following results.

Lemma 2.1. *Suppose* a *is a continuous bounded and nonnegative function on* $\mathbb{R}^N$*, then* $\alpha_{a,0}(c) = \frac{c}{p}$ *for* $c > 0$ *and*

$$\alpha_{a,0} \le \alpha_{a,0}(c) + \alpha_{a,0}(-c) - \frac{s-p}{sp}|c| \quad \textit{for all } c \in \mathbb{R}.$$

Proof. The case $p = 2$ was proved by Cao-Noussair [8, Lemma 2.2]. By a modification of the method given in [8], we obtain our result. For the readers convenience, we give a sketch here. For any $c > 0$, let $u \in M_{a,0}(c)$. Then

$$\|u\|^p = \int_{\mathbb{R}^N} a|u|^s dx + c \ge c.$$

Thus

$$I_{a,0}(u) = \frac{1}{p}\|u\|^p - \frac{1}{s}\int_{\mathbf{R^N}} a|u|^s dx = (\frac{1}{p} - \frac{1}{s})\|u\|^p + \frac{c}{s} \ge \frac{c}{p}.$$

To show that the equality holds, choose $v \in W^{1,p}(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} |\nabla v|^p dx = c$, for any $\sigma > 0$, define

$$u_\sigma(x) = \sigma^{\frac{N-p}{p}} v(\sigma x), \quad w_\sigma(x) = (1+\theta)u_\sigma$$

where $\theta > 0$ being selected so that $w_\sigma \in M_{a,0}(c)$. It is easy to see that

$$\int_{\mathbb{R}^N} |\nabla u_\sigma|^p dx = c,$$
$$\int_{\mathbb{R}^N} |u_\sigma|^q dx = \sigma^{\frac{(N-p)q}{p} - N} \int_{\mathbb{R}^N} |v|^q dx \to 0 \quad \text{as } \sigma \to \infty$$

for $q < p^*$. Obviously, such a $\theta = \theta(\sigma)$ exists when σ large enough and $\theta \to 0$ as $\sigma \to +\infty$. Therefore,

$$I_{a,0}(w_\sigma) = \frac{1}{p}\|w_\sigma\|^p - \frac{1}{s}\int_{\mathbb{R}^N} a|w_\sigma|^s dx \to \frac{c}{p} \quad \text{as } \sigma \to +\infty.$$

Hence

$$\alpha_{a,0}(c) = \frac{c}{p}.$$

To complete the proof of Lemma 2.1, let $c > 0$ and $u \in M_{a,0}(-c)$. Then

$$\|u\|^p = \int_{\mathbb{R}^N} a|u|^s dx - c < \int_{\mathbb{R}^N} a|u|^s dx.$$

It is easy to see that there exist unique $t \in (0,1)$ such that $v = tu \in M_{a,0}$. Then we have

$$\begin{aligned}
I_{a,0}(v) &= (\frac{1}{p} - \frac{1}{s})\|v\|^p \\
&= (\frac{1}{p} - \frac{1}{s})t^p\|u\|^p \\
&< (\frac{1}{p} - \frac{1}{s})\|u\|^p + \frac{c}{s} - \frac{c}{s} \\
&= I_{a,0}(u) + \frac{c}{p} + (\frac{1}{s} - \frac{1}{p})c \\
&\le I_{a,0}(u) + \alpha_{a,0}(c) - \frac{s-p}{sp}c.
\end{aligned}$$

The required inequality then follows by taking the infimum over $M_{a,0}(-c)$. $\square$

Define

$$\psi(u) = \langle I'_{f_\eta,h_\eta}(u), u\rangle = \|u\|^p - \int_{\mathbb{R}^N} f_\eta|u|^s dx - \eta^{\frac{p(s-r)}{s-p}} \int_{\mathbb{R}^N} h_\eta|u|^r dx.$$

Then for $u \in M_{f_\eta,h_\eta}$, we have

$$\begin{aligned}
\langle\psi'(u), u\rangle &= p\|u\|^p - s\int_{\mathbb{R}^N} f_\eta|u|^s dx - r\eta^{\frac{p(s-r)}{s-p}} \int_{\mathbb{R}^N} h_\eta|u|^r dx \\
&= (p-r)\|u\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta|u|^s dx.
\end{aligned}$$

Using the same methods as [20], we split M_{f_η,h_η} into three parts:

$$M^+_{f_\eta,h_\eta} = \{u \in M_{f_\eta,h_\eta} | (p-r)\|u\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta|u|^s dx > 0\};$$

$$M^0_{f_\eta,h_\eta} = \{u \in M_{f_\eta,h_\eta} | (p-r)\|u\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta |u|^s dx = 0\};$$
$$M^-_{f_\eta,h_\eta} = \{u \in M_{f_\eta,h_\eta} | (p-r)\|u\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta |u|^s dx < 0\}.$$

Then we have the following result.

Lemma 2.2. *There exists $\eta_1 > 0$ such that for all $\eta \in (0, \eta_1)$, we have $M^0_{f_\eta,h_\eta} = \emptyset$.*

Proof. Assume the contrary, that is $M^0_{f_\eta,h_\eta} \neq \emptyset$ for all $\eta > 0$. Then for $u \in M^0_{f_\eta,h_\eta}$, we have

$$\|u\|^p = \frac{s-r}{p-r}\int_{\mathbb{R}^N} f_\eta |u|^s dx \tag{2.2}$$
$$\eta^{\frac{p(s-r)}{s-p}} \int_{\mathbb{R}^N} h_\eta |u|^r dx = \|u\|^p - \int_{\mathbb{R}^N} f_\eta |u|^s dx = \frac{s-p}{p-r}\int_{\mathbb{R}^N} f_\eta |u|^s dx. \tag{2.3}$$

Moreover,

$$\begin{aligned}
\frac{s-p}{s-r}\|u\|^p &= \|u\|^p - \int_{\mathbb{R}^N} f_\eta |u|^s dx \le \eta^{\frac{p(s-r)}{s-p}} \|h_\eta\|_{L^{\frac{p}{p-r}}} \|u\|^r \\
&= \eta^\beta \|h\|_{L^{\frac{p}{p-r}}} \|u\|^r,
\end{aligned}$$

where $\beta = \frac{p(s-r)}{s-p} - \frac{p-r}{p}N$. Also we have

$$\|u\| \le [\frac{s-r}{s-p}\eta^\beta \|h\|_{L^{\frac{p}{p-r}}}]^{\frac{1}{p-r}}. \tag{2.4}$$

Let $K : M_{f_\eta,h_\eta} \to R$ be given by

$$K(u) = c(s,r)(\frac{\|u\|^{p\frac{s-1}{p-1}}}{\int_{\mathbb{R}^N} f_\eta |u|^s dx})^{\frac{p-1}{s-p}} - \eta^{\frac{p(s-r)}{s-p}} \int_{\mathbb{R}^N} h_\eta |u|^r dx,$$

where $c(s,r) = (\frac{s-r}{p-r})^{\frac{1-s}{s-p}}\frac{s-p}{p-r}$. Then $K(u) = 0$ for all $\eta > 0$ and $u \in M^0_{f_\eta,h_\eta}$. From (2.2) and (2.3), it follows that for $u \in M^0_{f_\eta,h_\eta}$, and

$$K(u) = c(s,r)[\frac{(\frac{s-r}{p-r}\int_{\mathbb{R}^N} f_\eta |u|^s dx)^{\frac{s-1}{p-1}}}{\int_{\mathbb{R}^N} f_\eta |u|^s dx}]^{\frac{p-1}{s-p}} - \frac{s-p}{p-r}\int_{\mathbb{R}^N} f_\eta |u|^s dx = 0. \tag{2.5}$$

However, by (2.4), the Hölder and Sobolev inequalities and

$$(\frac{\|u\|^s}{\int_{\mathbb{R}^N} f_{\max} |u|^s dx})^{\frac{p-1}{s-p}} > (\frac{S^s}{f_{\max}})^{\frac{p-1}{s-p}} \quad \text{for all } u \in M_{f_\eta,h_\eta},$$

where $S = \inf_{u \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|}{\|u\|_{L^s}}$ is the best Sobolev constant. Also we have

$$\begin{aligned}
K(u) &\ge c(s,r)(\frac{\|u\|^{p\frac{s-1}{p-1}}}{\int_{\mathbb{R}^N} f_\eta |u|^s dx})^{\frac{p-1}{s-p}} - \eta^\beta \|h\|_{L^{\frac{p}{p-r}}} \|u\|^r \\
&\ge \|u\|^r [c(s,r)(\frac{S^s}{f_{\max}})^{\frac{p-1}{s-p}} \|u\|^{1-r} - \eta^\beta \|h\|_{L^{\frac{p}{p-r}}}] \\
&\ge \|u\|^r [c(s,r)(\frac{S^s}{f_{\max}})^{\frac{p-1}{s-p}} (\eta^\beta \frac{s-r}{s-p}\|h\|_{L^{\frac{p}{p-r}}})^{\frac{1-r}{p-r}} - \eta^\beta \|h\|_{L^{\frac{p}{p-r}}}]
\end{aligned}$$

for all $u \in M^0_{f_\eta,h_\eta}$, where $\beta = \frac{p(s-r)}{s-p} - \frac{p-r}{p}N > 0$ (see Lemma 3.11). Since $\frac{1-r}{p-r} \le 0$, there exists $\eta_1 > 0$ such that for each $\eta \in (0,\eta_1)$ and $u \in M^0_{f_\eta,h_\eta}$, we have $K(u) > 0$, this contradicts to (2.5). We can conclude that $M^0_{f_\eta,h_\eta} = \emptyset$ for all $\eta \in (0,\eta_1)$. $\square$

By Lemma 2.2 for $\eta \in (0,\eta_1)$ we write $M_{f_\eta,h_\eta} = M^+_{f_\eta,h_\eta} \cup M^-_{f_\eta,h_\eta}$ and define

$$\alpha^+_{f_\eta,h_\eta} = \inf_{u\in M^+_{f_\eta,h_\eta}} I_{f_\eta,h_\eta}, \quad \alpha^-_{f_\eta,h_\eta} = \inf_{u\in M^-_{f_\eta,h_\eta}} I_{f_\eta,h_\eta}.$$

The following Lemma shows that the minimizers on M_{f_η,h_η} are "usually" critical points for I_{f_η,h_η}.

Lemma 2.3. *For $\eta \in (0,\eta_1)$, if u_0 is a local minimizer for I_{f_η,h_η} on M_{f_η,h_η}, then $I'_{f_\eta,h_\eta}(u_0) = 0$ in $W^{-1}(\mathbb{R}^N)$, where $W^{-1}(\mathbb{R}^N)$ is the dual space of $W^{1,p}(\mathbb{R}^N)$.*

Proof. If u_0 is a local minimizer for I_{f_η,h_η} on M_{f_η,h_η}, then u_0 is a solution of the optimization problem

$$\text{minimize } I_{f_\eta,h_\eta}(u) \text{ subject to } \psi(u) = 0.$$

Hence, by the theory of Lagrange multipliers, there exists $\theta \in \mathbb{R}$ such that

$$I'_{f_\eta,h_\eta}(u_0) = \theta\psi'(u_0) \quad \text{in } W^{-1}(\mathbb{R}^N).$$

This implies

$$\langle I'_{f_\eta,h_\eta}(u_0), u_0\rangle = \theta\langle\psi'(u_0), u_0\rangle.$$

Since $u_0 \in M_{f_\eta,h_\eta}$ and by Lemma 2.2, $M^0_{f_\eta,h_\eta} = \emptyset$ when $\eta \in (0,\eta_1)$, we have

$$\langle I'_{f_\eta,h_\eta}(u_0), u_0\rangle = 0 \text{ and } \langle\psi'(u_0), u_0\rangle \ne 0.$$

So we obtain $\theta = 0$. This completes the proof. $\square$

For each $u \in W^{1,p}(\mathbb{R}^N)\backslash\{0\}$, we define

$$t_{\max} = \Big(\frac{p-r}{s-r}\frac{\|u\|^p}{\int_{\mathbb{R}^N} f_\eta|u|^s dx}\Big)^{\frac{1}{s-p}} > 0.$$

Then we have the following Lemma.

Lemma 2.4. *There exists $\eta_2 > 0$ such that for each $u \in W^{1,p}(\mathbb{R}^N)\backslash\{0\}$ and $\eta \in (0,\eta_2)$, we have*

(i) there is a unique $t^- = t^-(u) > t_{\max} > 0$ such that $t^-u \in M^-_{f_\eta,h_\eta}$ and $I_{f_\eta,h_\eta}(t^-u) = \max_{t\ge t_{\max}} I_{f_\eta,h_\eta}(tu)$;

(ii) if $\int_{\mathbb{R}^N} h_\eta|u|^r dx > 0$, then there is a unique $0 < t^+ = t^+(u) < t_{\max} < t^- = t^-(u)$ such that $t^+u \in M^+_{f_\eta,h_\eta}$, $t^-u \in M^-_{f_\eta,h_\eta}$ and

$$I_{f_\eta,h_\eta}(t^+u) = \min_{t^-\ge t\ge 0} I_{f_\eta,h_\eta}(tu), \quad I_{f_\eta,h_\eta}(t^-u) = \max_{t\ge t_{\max}} I_{f_\eta,h_\eta}(tu).$$

Proof. (i) Since $h(x)$ is nonnegative, then $\int_{\mathbb{R}^N} h_\eta|u|^r dx \ge 0$. Let

$$m(t) = t^{p-r}\|u\|^p - t^{s-r}\int_{\mathbb{R}^N} f_\eta|u|^s dx,$$

clearly, $m(t)$ is increasing in $(0, t_{\max})$ and is decreasing in $(t_{\max}, +\infty)$, also, we have

$$m(0) = 0, \lim_{t\to+\infty} m(t) = -\infty,$$

i.e. $m(t)$ is concave and achieve its maximum at $t_{\max}$. Moreover,

$$\begin{aligned}
m(t_{\max}) &= (\frac{p-r}{s-r}\frac{\|u\|^p}{\int_{\mathbb{R}^N} f_\eta |u|^s dx})^{\frac{p-r}{s-p}}\|u\|^p - (\frac{p-r}{s-r}\frac{\|u\|^p}{\int_{\mathbb{R}^N} f_\eta |u|^s dx})^{\frac{s-r}{s-p}}\int_{\mathbb{R}^N} f_\eta |u|^s dx \\
&= \frac{\|u\|^{p\frac{s-r}{s-p}}}{(\int_{\mathbb{R}^N} f_\eta |u|^s dx)^{\frac{p-r}{s-p}}}[(\frac{p-r}{s-r})^{\frac{p-r}{s-p}} - (\frac{p-r}{s-r})^{\frac{s-r}{s-p}}] \\
&= \frac{s-p}{s-r}(\frac{p-r}{s-r})^{\frac{p-r}{s-p}}(\frac{\|u\|^s}{\int_{\mathbb{R}^N} f_\eta |u|^s dx})^{\frac{p-r}{s-p}}\|u\|^r \\
&\geq \frac{s-p}{s-r}(\frac{p-r}{s-r})^{\frac{p-r}{s-p}}(\frac{S^s}{f_{\max}})^{\frac{p-r}{s-p}}\|u\|^r = C\|u\|^r.
\end{aligned}$$

Since $C > 0$,

$$0 \leq \eta^{\frac{p(s-r)}{s-p}}\int_{\mathbb{R}^N} h_\eta |u|^r dx \leq \eta^\beta \|h\|_{L^{\frac{p}{p-r}}}\|u\|^r$$

and $\beta > 0$, there exists $\eta_2 > 0$, such that for any $\eta \in (0, \eta_2)$, we have

$$m(t_{\max}) > \eta^{\frac{p(s-r)}{s-p}}\int_{R^N} h_\eta |u|^r dx.$$

Case (a): $\int_{R^N} h_\eta |u|^r dx = 0$. Then there is unique $t^- > t_{\max}$ such that $m(t^-) = 0$ and $m'(t^-) < 0$. Now

$$\langle \psi'(t^-u), t^-u\rangle = (p-r)\|t^-u\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta |t^-u|^s dx = (t^-)^{r+1}m'(t^-) < 0$$

and

$$\begin{aligned}
\langle I'_{f_\eta,h_\eta}(t^-u), t^-u\rangle &= \|t^-u\|^p - \int_{\mathbb{R}^N} f_\eta |t^-u|^s dx - \eta^{\frac{p(s-r)}{s-p}}\int_{\mathbb{R}^N} h_\eta |t^-u|^r dx \\
&= (t^-)^r[(t^-)^{p-r}\|u\|^p - (t^-)^{s-r}\int_{\mathbb{R}^N} f_\eta |u|^s dx] \\
&= (t^-)^r m(t^-) = 0.
\end{aligned}$$

Thus, $t^-u \in M^-_{f_\eta,h_\eta}$. Moreover, we have

$$\frac{d}{dt}I_{f_\eta,h_\eta}(tu) = 0, \quad \frac{d^2}{dt^2}I_{f_\eta,h_\eta}(tu) < 0, \quad \text{for } t = t^-.$$

Then we have $I_{f_\eta,h_\eta}(t^-u) = \max_{t \geq t_{\max}} I_{f_\eta,h_\eta}(tu)$.

Case (b): $\int_{\mathbb{R}^N} h_\eta |u|^r dx > 0$. There are unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$ such that

$$m(t^+) = \eta^{\frac{p(s-r)}{s-p}}\int_{\mathbb{R}^N} h_\eta |u|^r dx = m(t^-)$$

and $m'(t^+) > 0 > m'(t^-)$. Similar to the argument in Case a, we have $t^\pm u \in M^\pm_{f_\eta,h_\eta}$, and $I_{f_\eta,h_\eta}(t^-u) \geq I_{f_\eta,h_\eta}(tu) \geq I_{f_\eta,h_\eta}(t^+u)$ for each $t \in [t^+, t^-]$, and $I_{f_\eta,h_\eta}(tu) \geq I_{f_\eta,h_\eta}(t^+u)$ for each $t \in [0, t^+]$.

(ii) By case (b) it follows part (i) $\square$

To establish the existence of a local minimum for I_{f_η,h_η} on M_{f_η,h_η}, we need the following results.

Lemma 2.5. *(i) For each $u \in M^+_{f_\eta,h_\eta}$, we have $\int_{\mathbb{R}^N} h_\eta |u|^r dx > 0$ and $I_{f_\eta,h_\eta}(u) < 0$. In particular $\alpha_{f_\eta,h_\eta} \le \alpha^+_{f_\eta,h_\eta} < 0$.*

(ii) I_{f_η,h_η} is coercive and bounded below on M_{f_η,h_η} for all $\eta \in (0, (\frac{s-p}{s-r})^{\frac{1}{\beta}})$. Moreover, $\alpha_{f_\eta,h_\eta} \to 0$ as $\eta \to 0$.

Proof. (i) For each $u \in M^+_{f_\eta,h_\eta}$, we have

$$(p-r)\|u\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta |u|^s dx > 0,$$
$$\|u\|^p = \int_{\mathbb{R}^N} f_\eta |u|^s dx + \eta^{\frac{p(s-r)}{s-p}} \int_{\mathbb{R}^N} h_\eta |u|^r dx.$$

By (C1), we have

$$\eta^{\frac{p(s-r)}{s-p}} \int_{\mathbb{R}^N} h_\eta |u|^r dx = \|u\|^p - \int_{\mathbb{R}^N} f_\eta |u|^s dx > \frac{s-p}{p-r}\int_{\mathbb{R}^N} f_\eta |u|^s dx \ge 0$$

and

$$\begin{aligned}
I_{f_\eta,h_\eta}(u) &= (\frac{1}{p} - \frac{1}{s})\int_{\mathbb{R}^N} f_\eta |u|^s dx + (\frac{1}{p} - \frac{1}{r})\eta^{\frac{p(s-r)}{s-p}} \int_{\mathbb{R}^N} h_\eta |u|^r dx \\
&< (\frac{1}{p} - \frac{1}{s})\int_{\mathbb{R}^N} f_\eta |u|^s dx + (\frac{1}{p} - \frac{1}{r})\frac{s-p}{p-r}\int_{\mathbb{R}^N} f_\eta |u|^s dx \\
&= (s-p)(\frac{1}{ps} - \frac{1}{pr})\int_{\mathbb{R}^N} f_\eta |u|^s dx \le 0
\end{aligned}$$

(ii) For each $u \in M_{f_\eta,h_\eta}$, we have $\|u\|^p = \int_{\mathbb{R}^N} f_\eta |u|^s dx + \eta^{\frac{p(s-r)}{s-p}} \int_{\mathbb{R}^N} h_\eta |u|^r dx$. Then by the Hölder and Young inequalities,

$$\begin{aligned}
I_{f_\eta,h_\eta}(u) &\ge \frac{s-p}{ps}\|u\|^p - \frac{s-r}{rs}\eta^\beta \|h\|_{L^{\frac{p}{p-r}}}\|u\|^r \\
&\ge (\frac{s-p}{ps} - \frac{s-r}{ps}\eta^\beta)\|u\|^p - \eta^\beta \frac{(p-r)(s-r)}{prs}\|h\|^{\frac{p}{p-r}}_{L^{\frac{p}{p-r}}}.
\end{aligned}$$

Thus, I_{f_η,h_η} is coercive and bounded below on M_{f_η,h_η} for all $\eta \in (0, (\frac{s-p}{s-r})^{\frac{1}{\beta}})$ and $\alpha_{f_\eta,h_\eta} \to 0$ as $\eta \to 0$, where $\beta = \frac{p(s-r)}{s-p} - \frac{p-r}{p}N > 0$ as above. □

3. Proofs of main results

Now, we use the graph of the coefficient f to find some Palais-Smale sequences which are used to prove Theorem 1.1. For $a > 0$, let $C_a(x^i)$ denote the hypercube $\Pi_{j=1}^N (x^i_j - a, x^i_j + a)$ centered at $x^i = (x^i_1, x^i_2, \dots, x^i_N)$ for $i = 1, 2, \dots, k$. Let $\overline{C_a(x^i)}$ and $\partial C_a(x^i)$ denote the closure and the boundary of $C_a(x^i)$ respectively. By the conditions (C1) and (C3), we can choose numbers $K, l > 0$ such that $C_l(x^i)$ are disjoint, $f(x) < f(x^i)$ for $x \in \partial C_l(x^i)$ for all $i = 1, 2, \dots, k$ and $\cup_{i=1}^k C_l(x^i) \subset \Pi_{i=1}^N (-K, K)$.

Define $\phi_\eta \in C(R, R)$, $g_\eta \in (W^{1,p}(\mathbb{R}^N), \mathbb{R}^N)$ by

$$\phi_\eta(t) = \begin{cases} \frac{2K}{\eta} & t > \frac{2K}{\eta}, \\ t & -\frac{2K}{\eta} \le t \le \frac{2K}{\eta}, \\ -\frac{2K}{\eta} & t < -\frac{2K}{\eta}. \end{cases}$$

$$g_\eta^j(u) = \frac{\int_{\mathbb{R}^N} \phi_\eta(x_j)|u|^s dx}{\int_{\mathbb{R}^N} |u|^s dx} \quad \text{for } j = 1, 2, \dots, N$$
$$g_\eta(u) = (g_\eta^1(u), g_\eta^2(u), \dots, g_\eta^N(u)) \in \mathbb{R}^N.$$

Let $C^i_{l/\eta} \equiv C_{l/\eta}(x^i/\eta)$,

$$N_\eta^i = \{u \in M^-_{f_\eta,h_\eta} "u \geq 0 \text{ and } g_\eta(u) \in C^i_{l/\eta}\},$$
$$\partial N_\eta^i = \{u \in M^-_{f_\eta,h_\eta} : u \geq 0 \text{ and } g_\eta(u) \in \partial C^i_{l/\eta}\}$$

for $i = 1, 2, \dots, k$. It is easy to verify that N_η^i and ∂N_η^i are nonempty sets for all $i = 1, 2, \dots, k$. Consider the minimization problems in N_η^i and ∂N_η^i for I_{f_η,h_η},

$$\gamma_\eta^i = \inf_{u \in N_\eta^i} I_{f_\eta,h_\eta}(u), \quad \overline{\gamma}_\eta^i = \inf_{u \in \partial N_\eta^i} I_{f_\eta,h_\eta}(u).$$

Using the results in [19], we can assume w be a unique positive radial solution of

$$-\Delta_p u + |u|^{p-2}u = f_{\max}|u|^{s-2}u \quad x \in \mathbb{R}^N,$$
$$u > 0 \ \ x \in \mathbb{R}^N,$$
$$u \in W^{1,p}(\mathbb{R}^N)$$

and that $I_{f_{\max},0}(w) = \alpha_{f_{\max},0}$. By (C3) and the routine computations, we have

$$\alpha_{f_{\max},0} < \alpha_{f^\infty,0}.$$

For small $\eta > 0$ satisfying $2\sqrt{\eta} < 1$, we define a function $\psi_\eta \in C^1(\mathbb{R}^N, [0,1])$ such that

$$\psi_\eta(x) = \begin{cases} 1 & |x| < \frac{1}{2\sqrt{\eta}} - 1, \\ 0 & |x| > \frac{1}{2\sqrt{\eta}}, \end{cases}$$

and $|\nabla\psi_\eta| \leq 2$ in $\mathbb{R}^N$. Let $x^\eta = \frac{1}{2\sqrt{\eta}}(1, 1, \dots, 1) \in \mathbb{R}^N$ and

$$w_\eta(x) = t_\eta^- w(x - \frac{x^i}{\eta} + x^\eta)\psi_\eta(x - \frac{x^i}{\eta} + x^\eta),$$

where $t_\eta^- > 0$ are selected such that $w_\eta \in M^-_{f_\eta,h_\eta}$. Then we have the following results.

Lemma 3.1. *As $\eta \to 0$, we have*

(i) $\eta^{\frac{p(s-r)}{s-p}} \int_{\mathbb{R}^N} h_\eta w^r(x - \frac{x^i}{\eta} + x^\eta)\psi_\eta^r(x - \frac{x^i}{\eta} + x^\eta)dx \to 0$;
(ii) $t_\eta^- \to 1$.

Proof. (i) Since $\beta = \frac{p(s-r)}{s-p} - \frac{p-r}{p}N > 0$ and $h_\eta(x) \geq 0$, we have

$$\begin{aligned} 0 \leq\ & eta^{\frac{p(s-r)}{s-p}} \int_{\mathbb{R}^N} h_\eta w^r(x - \frac{x^i}{\eta} + x^\eta)\psi_\eta^r(x - \frac{x^i}{\eta} + x^\eta)dx \\ &\leq \eta^\beta \|h\|_{L^{\frac{p}{p-r}}} \|w(x - \frac{x^i}{\eta} + x^\eta)\psi_\eta(x - \frac{x^i}{\eta} + x^\eta)\|^r \end{aligned}$$

and

$$\|w(x - \frac{x^i}{\eta} + x^\eta)\psi_\eta(x - \frac{x^i}{\eta} + x^\eta)\|^p \to \frac{sp}{s-p}\alpha_{f_{\max},0}.$$

Thus (i) holds.

(ii) Since $w_\eta \in M^-_{f_\eta,h_\eta}$, we have

$$\begin{aligned}
&(t^-_\eta)^p[\int_{\mathbb{R}^N} |\nabla(w(x-\frac{x^i}{\eta}+x^\eta)\psi_\eta(x-\frac{x^i}{\eta}+x^\eta))|^p\\
&+(w(x-\frac{x^i}{\eta}+x^\eta)\psi_\eta(x-\frac{x^i}{\eta}+x^\eta))^p]\\
&=(t^-_\eta)^s\int_{\mathbb{R}^N} f_\eta w^s(x-\frac{x^i}{\eta}+x^\eta)\psi^s_\eta(x-\frac{x^i}{\eta}+x^\eta)dx\\
&\quad+\eta^{\frac{p(s-r)}{s-p}}(t^-_\eta)^r\int_{\mathbb{R}^N} h_\eta w^r(x-\frac{x^i}{\eta}+x^\eta)\psi^r_\eta(x-\frac{x^i}{\eta}+x^\eta)dx.
\end{aligned}$$

When $\eta\to 0$, from part (i) it follows that

$$\begin{aligned}
(t^-_\eta)^p(\|w\|^p+o(\eta)) &= (t^-_\eta)^p\|w(x-\frac{x^i}{\eta}+x^\eta)\psi_\eta(x-\frac{x^i}{\eta}+x^\eta)\|^p+o(\eta)\\
&=(t^-_\eta)^s\int_{\mathbb{R}^N} f_\eta w^s(x-\frac{x^i}{\eta}+x^\eta)\psi^s_\eta(x-\frac{x^i}{\eta}+x^\eta)dx+o(\eta)\\
&=(t^-_\eta)^s\int_{\mathbb{R}^N} f(\eta x+x^i-\eta x^\eta)w^s dx+o(\eta).
\end{aligned}$$

Moreover, $\eta x^\eta\to 0$ as $\eta\to 0$, and from $\|w\|^p=\int_{\mathbb{R}^N} f_{\max} w^s dx$, we have

$$\begin{aligned}
t^-_\eta > t_{\max} &= (\frac{p-r}{s-r}\frac{\|w(x-\frac{x^i}{\eta}+x^\eta)\psi_\eta(x-\frac{x^i}{\eta}+x^\eta)\|^p}{\int_{\mathbb{R}^N} f_\eta|w(x-\frac{x^i}{\eta}+x^\eta)\psi_\eta(x-\frac{x^i}{\eta}+x^\eta)|^s dx})^{\frac{1}{s-p}}\\
&\to(\frac{p-r}{s-r})^{\frac{1}{s-p}}>0.
\end{aligned}$$

Thus, $t^-_\eta\to 1$ as $\eta\to 0$ and (ii) holds. □

Let $\eta_*=\min\{\eta_1,\eta_2,(\frac{s-p}{s-r})^{\frac{1}{\beta}}\}$, then we have the following result.

Lemma 3.2. *For each $\varepsilon>0$, there exists $\eta_\varepsilon\in(0,\eta_*]$ such that*

$$\alpha^-_{f_\eta,h_\eta}\le\gamma^i_\eta<\min\{\alpha_{f_{\max},0}+\varepsilon,\alpha_{f_\eta,h_\eta}+\alpha_{f^\infty,0}\},\quad i=1,2,\dots,k,\ \eta\in(0,\eta_\varepsilon).$$

Proof. For $i=1,2,\dots,k$, obviously we have $\alpha^-_{f_\eta,h_\eta}\le\gamma^i_\eta$.

Now we show the second inequality hold. First, we prove that $g_\eta(w_\eta)\in C^i_{l/\eta}$. For $j=1,2,\dots,N$, since

$$g^j_\eta(w_\eta)=\frac{\int_{\mathbb{R}^N}\phi_\eta(x_j)w^s(x-\frac{x^i}{\eta}+x^\eta)\psi^s_\eta(x-\frac{x^i}{\eta}+x^\eta)dx}{\int_{\mathbb{R}^N} w^s(x-\frac{x^i}{\eta}+x^\eta)\psi^s_\eta(x-\frac{x^i}{\eta}+x^\eta)dx}$$

and

$$\psi_\eta(x-\frac{x^i}{\eta}+x^\eta)=0\quad\text{if } |x_j-\frac{x^i_j}{\eta}|>\frac{1}{\sqrt{\eta}}.$$

By the definition of ψ_η, we have

$$g^j_\eta(w_\eta)=\frac{\int_{C^i_{l/\eta}}\phi_\eta(x_j)w^s(x-\frac{x^i}{\eta}+x^\eta)\psi^s_\eta(x-\frac{x^i}{\eta}+x^\eta)dx}{\int_{C^i_{l/\eta}} w^s(x-\frac{x^i}{\eta}+x^\eta)\psi^s_\eta(x-\frac{x^i}{\eta}+x^\eta)dx}$$

provided $\frac{1}{\sqrt{\eta}} < \frac{l}{\eta}$. From the definition of ϕ_η and g_η we conclude that $g_\eta(w_\eta) \in C^i_{l/\eta}$. Thus, $w_\eta \in N^i_\eta$. Moreover, by Lemma 3.1, we obtain

$$\begin{aligned} I_{f_\eta,h_\eta}(w_\eta) &= \frac{(t^-_\eta)^p}{p}\Big[\int_{\mathbb{R}^N} |\nabla(w(x-\frac{x^i}{\eta}+x^\eta)\psi_\eta(x-\frac{x^i}{\eta}+x^\eta))|^p dx \\ &\quad + \int_{\mathbb{R}^N} |w(x-\frac{x^i}{\eta}+x^\eta)\psi_\eta(x-\frac{x^i}{\eta}+x^\eta)|^p dx\Big] \\ &\quad - \frac{(t^-_\eta)^s}{s}\int_{\mathbb{R}^N} f_\eta w^s(x-\frac{x^i}{\eta}+x^\eta)\psi^s_\eta(x-\frac{x^i}{\eta}+x^\eta)dx \\ &\quad - \eta^{\frac{p(s-r)}{s-p}}\frac{(t^-_\eta)^r}{r}\int_{\mathbb{R}^N} h_\eta w^r(x-\frac{x^i}{\eta}+x^\eta)\psi^r_\eta(x-\frac{x^i}{\eta}+x^\eta)dx \\ &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla w|^p + |w|^p dx - \frac{1}{s}\int_{\mathbb{R}^N} f(\eta x + x^i - \eta x^\eta) w^s dx + o(\eta). \end{aligned}$$

Since $\eta x^\eta \to 0$ as $\eta \to 0$ and from the above, we have

$$I_{f_\eta,h_\eta}(w_\eta) = I_{f_{\max},0}(w) + o(\eta) = \alpha_{f_{\max},0} + o(\eta).$$

Therefore, for any $\varepsilon > 0$ there exists $\eta_3 > 0$ such that

$$\gamma^i_\eta < \alpha_{f_{\max},0} + \varepsilon,\ i = 1,2,\dots,k,\ \eta \in (0,\eta_3).$$

Moreover, $\alpha_{f_{\max},0} < \alpha_{f^\infty,0}$ and $\alpha_{f_\eta,h_\eta} \to 0$ as $\eta \to 0$, then there exists $\eta_4 > 0$ such that

$$\gamma^i_\eta < \alpha_{f_\eta,h_\eta} + \alpha_{f^\infty,0}, \quad i = 1,2,\dots,k,\ \eta \in (0,\eta_4).$$

We take $\eta_\varepsilon = \min\{\eta_3,\eta_4\}$, this implies

$$\gamma^i_\eta < \min\{\alpha_{f_{\max},0} + \varepsilon, \alpha_{f_\eta,h_\eta} + \alpha_{f^\infty,0}\},$$

for $i = 1,2,\dots,k$ and $\eta \in (0,\eta_\varepsilon)$. This completes the proof. $\square$

Since $W^{1,p}(\mathbb{R}^N)$ is not a Hilbert space in general, even if the (PS) sequence $\{u_n\}$ of $I_\lambda(u)$ is bounded, hence there exists $u \in W^{1,p}(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\mathbb{R}^N),$$

we can not ensure

$$|\nabla u_{n_k}|^{p-2}\nabla u_{n_k} \rightharpoonup |\nabla u|^{p-2}\nabla u \ \text{ in } \ L^{\frac{p}{p-1}}(\mathbb{R}^N)$$

for some subsequence $\{u_{n_k}\}$ of $\{u_n\}$, so we can not use Brezis-Lieb lemma [20] directly. We use the following results.

Lemma 3.3. *If $\{u_n\} \subset W^{1,p}(\mathbb{R}^N)$ is a $(PS)_c$ sequence of I_{f_η,h_η}, then there exists a subsequence $\{u_k\}$ such that $u_k \rightharpoonup u_0$ in $W^{1,p}(\mathbb{R}^N)$ for some $u_0 \in W^{1,p}(\mathbb{R}^N)$, and $I'(u_0) = 0$, $\nabla u_k \to \nabla u_0$ a.e. in $\mathbb{R}^N$.*

The proof of the above lemma was given in [12, Lemma 2.1], also in [17]. We omit it here.

Lemma 3.4. *There are positive numbers δ and $\eta_\delta \in (0,\eta_*]$ such that for $i = 1,2,\dots,k$, we have*

$$\widetilde{\gamma^i_\eta} > \alpha_{f_{\max},0} + \delta \quad \textit{for all } \eta \in (0,\eta_\delta).$$

Proof. Fix $i \in \{1, 2, \dots, k\}$. Suppose the contrary that there exists a sequence $\{\eta_n\}$ with $\eta_n \to 0$ as $n \to \infty$ such that $\widetilde{\gamma^i_{\eta_n}} \to c \le \alpha_{f_{\max},0}$. Consequently, there exists a sequence $\{u_n\} \subset \partial N^i_{\eta_n}$ such that $g_{\eta_n}(u_n) \in \partial C^i_{\frac{l}{\eta_n}}$ and

$$\langle I'_{f_{\eta_n},h_{\eta_n}}(u_n), u_n\rangle = 0,$$
$$I_{f_{\eta_n},h_{\eta_n}}(u_n) \to c \le \alpha_{f_{\max},0}.$$

By Lemma 2.5, $\{u_n\}$ is uniformly bounded in $W^{1,p}(\mathbb{R}^N)$. For $u_n \in M^-_{f_{\eta_n},h_{\eta_n}}$, we deduce from the Sobolev imbedding theorem that there exists a constant $\nu > 0$ such that $\|u_n\| > \nu$ for all n. Applying the concentration-compactness principle of Lions [16] to $|u_n|^s$, there are positive constants R, μ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B^N(y_n,R)} |u_n|^s dx \ge \mu \quad \text{for all } n,$$

where $B^N(y_n, R) = \{x \in \mathbb{R}^N | |x - y_n| < R\}$. Let $\widetilde{u}_n = u_n(x + y_n)$, and define

$$\widetilde{f}_{\eta_n}(x) = f(\eta_n x + \eta_n y_n),\ \widetilde{h}_{\eta_n}(x) = h(\eta_n x + \eta_n y_n).$$

Then we have

$$\begin{gathered}\langle I'_{\widetilde{f}_{\eta_n},\widetilde{h}_{\eta_n}}(\widetilde{u}_n), \widetilde{u}_n\rangle = 0,\\ I_{\widetilde{f}_{\eta_n},\widetilde{h}_{\eta_n}}(\widetilde{u}_n) \to c.\end{gathered} \tag{3.1}$$

By Lemma 3.3, Sobolev imbedding theorem and Riesz's theorem, there is a $u_0 \in W^{1,p}(\mathbb{R}^N)$ and a subsequence of $\{\widetilde{u}_n\}$, still denoted by $\{\widetilde{u}_n\}$ such that

$$\widetilde{u}_n \rightharpoonup u_0 \quad \text{in } W^{1,p}(\mathbb{R}^N),$$
$$\widetilde{u}_n \to u_0 \quad \text{a.e. in } \mathbb{R}^N,$$
$$\int_{B^N(0,R)} |\widetilde{u}_n|^s dx \to \int_{B^N(0,R)} |u_0|^s dx \ge \mu,$$

and

$$\nabla\widetilde{u}_n \to \nabla u_0 \quad \text{a.e. in } \mathbb{R}^N,$$
$$|\nabla\widetilde{u}_n|^{p-2}\nabla\widetilde{u}_n \rightharpoonup |\nabla u_0|^{p-2}\nabla u_0 \quad \text{in } L^{\frac{p}{p-1}}(\mathbb{R}^N).$$

Set $w_n = \widetilde{u}_n - u_0$. By the Brezis-Lieb lemma [20], we have

$$\int_{\mathbb{R}^N} \widetilde{f}_{\eta_n}|\widetilde{u}_n|^s dx = \int_{\mathbb{R}^N} \widetilde{f}_{\eta_n}|u_0|^s dx + \int_{\mathbb{R}^N} \widetilde{f}_{\eta_n}|w_n|^s dx + o(1). \tag{3.2}$$

Since $\{u_n\}$ is uniformly bounded and $\widetilde{u}_n \rightharpoonup u_0$, we obtain

$$\eta_n^{\frac{p(s-r)}{s-p}} \int_{\mathbb{R}^N} h_{\eta_n}|u_n|^r dx = \eta_n^{\frac{p(s-r)}{s-p}} \int_{\mathbb{R}^N} \widetilde{h}_{\eta_n}|\widetilde{u}_n|^r dx \to 0 \quad \text{as } n \to \infty \tag{3.3}$$

and

$$\|\widetilde{u}_n\|^p = \|u_0\|^p + \|w_n\|^p + o(1). \tag{3.4}$$

Combining (3.1)-(3.4), we have

$$\|w_n\|^p - \int_{\mathbb{R}^N} \widetilde{f}_{\eta_n}|w_n|^s dx = -(\|u_0\|^p - \int_{\mathbb{R}^N} \widetilde{f}_{\eta_n}|u_0|^s dx) + o(1). \tag{3.5}$$

We distinguish the two cases: (A) $\|w_n\| \to 0$ and (B) $\|w_n\| \to c > 0$.
Case (A): From condition (C3) we can choose a positive constant δ such that

$$f(x) < f_{\max} \text{ for } x \in \overline{C}^i_{l+\delta} \backslash C^i_{l-\delta}.$$

We complete the proof by establishing the contradiction

$$\lim_{n\to\infty} I_{f_{\eta_n},h_{\eta_n}}(u_n) > \alpha_{f_{\max},0}.$$

Consider the sequence $\{\eta_n y_n\}$. By passing to a subsequence if necessary, we may assume that one of the following cases occur:

(A1) $\{\eta_n y_n\} \subset \overline{C}^i_{l+\delta} \backslash C^i_{l-\delta}$,
(A2) $\{\eta_n y_n\} \subset \overline{C}^i_{l-\delta}$,
(A3) $\{\eta_n y_n\} \subset \mathbb{R}^N \backslash C^i_{l+\delta}$ and $\{\eta_n y_n\}$ is bounded;
(A4) $\{\eta_n y_n\}$ is unbounded.

Let $\epsilon > 0$ and $R_\epsilon > 0$ be such that

$$\frac{\int_{|x|\geq R_\epsilon} |\widetilde{u}_n|^s dx}{\int_{\mathbb{R}^N} |\widetilde{u}_n|^s dx} \leq \epsilon. \tag{3.6}$$

In case (A1), we assume $\eta_n y_n \to \widetilde{y} \in \overline{C}^i_{l+\delta} \backslash C^i_{l-\delta}$ and $f(\widetilde{y}) < f_{\max}$. Consequently by (3.3) and (3.4), we have

$$\begin{aligned}
\lim_{n\to\infty} I_{f_{\eta_n},h_{\eta_n}}(u_n) &= \lim_{n\to\infty} [\frac{1}{p}\|\widetilde{u}_n\|^p - \frac{1}{s}\int_{\mathbb{R}^N} \widetilde{f}_{\eta_n}(x)|\widetilde{u}_n|^s dx - \eta_n^{\frac{p(s-r)}{s-p}} \int_{\mathbb{R}^N} \widetilde{h}_{\eta_n}|\widetilde{u}_n|^r dx] \\
&= \frac{1}{p}\|u_0\|^p - \frac{1}{s}\int_{\mathbb{R}^N} f(\widetilde{y})|u_0|^s dx \\
&\geq \alpha_{f(\widetilde{y}),0} > \alpha_{f_{\max},0},
\end{aligned}$$

we also have

$$\|u_0\|^p - \int_{\mathbb{R}^N} f(\widetilde{y})|u_0|^s dx = 0,$$

which is a contradiction.

In case (A2),

$$\begin{aligned}
g^j_{\eta_n}(u_n) &= \frac{\int_{\mathbb{R}^N} \phi_{\eta_n}(x_j + (y_n)_j)|\widetilde{u}_n|^s dx}{\int_{\mathbb{R}^N} |\widetilde{u}_n|^s dx} \\
&= \frac{\int_{|x|\leq R_\epsilon} \phi_{\eta_n}(x_j + (y_n)_j)|\widetilde{u}_n|^s dx + \int_{|x|\geq R_\epsilon} \phi_{\eta_n}(x_j + (y_n)_j)|\widetilde{u}_n|^s dx}{\int_{\mathbb{R}^N} |\widetilde{u}_n|^s dx}.
\end{aligned}$$

In the region $|x| \leq R_\epsilon$, when n is sufficiently large, we have

$$x_j + (y_n)_j \in (\frac{x^i_j - (l-\delta)}{\eta_n} - R_\epsilon, \frac{x^i_j + (l-\delta)}{\eta_n} + R_\epsilon) \subset (-\frac{2K}{\eta_n}, \frac{2K}{\eta_n}).$$

It follows from (3.6) and the definition of ϕ_{η_n} that

$$\begin{aligned}
g^j_{\eta_n}(u_n) &> (\frac{x^i_j - (l-\delta)}{\eta_n} - R_\epsilon)(1-\epsilon) - \frac{2K}{\eta_n}\epsilon, \\
g^j_{\eta_n}(u_n) &< (\frac{x^i_j + (l-\delta)}{\eta_n} + R_\epsilon)(1-\epsilon) + \frac{2K}{\eta_n}\epsilon.
\end{aligned}$$

It is clear from the above inequalities that we can choose $\epsilon > 0$, $\delta > \epsilon$ sufficiently small such that

$$g^j_{\eta_n}(u_n) \in (\frac{x^i_j - l}{\eta_n}, \frac{x^i_j + l}{\eta_n})$$

for n large enough, which contradicts $g_{\eta_n}(u_n) \in \partial C^i_{l/\eta_n}$.

In case (A3), we assume that $\eta_n y_n \to \widetilde{y} \notin C^i_{l+\delta}$ as $n \to \infty$, then for some $j \in \{1, 2, \dots, N\}$, we have $\widetilde{y}_j \geq x^i_j + (l+\delta)$ or $\widetilde{y}_j \leq x^i_j - (l+\delta)$.

First, we assume $\widetilde{y}_j \geq x^i_j + (l+\delta)$ occurs, then $(y_n)_j > \frac{x^i_j + (l+\frac{\delta}{2})}{\eta_n}$ for all n. When $|x_j| \leq R_\epsilon$, we have

$$x_j + (y_n)_j > \frac{x^i_j + (l+\frac{\delta}{2})}{\eta_n} - R_\epsilon$$

and

$$g^j_{\eta_n}(u_n) > (\frac{x^i_j + (l+\frac{\delta}{2})}{\eta_n} - R_\epsilon)(1-\epsilon) - \frac{2K}{\eta_n}\epsilon > \frac{x^i_j + l}{\eta_n},$$

for sufficiently small $\epsilon > 0$, $\delta > \epsilon$ and n large enough. This contradicts to $g_{\eta_n}(u_n) \in \partial C^i_{\frac{l}{\eta_n}}$. When $\widetilde{y}_j \leq x^i_j - (l+\delta)$, the argument is similar.

In case (A4), we assume $\eta_n y_n \to \infty$ as $n \to \infty$, using a similar argument to case (A1) and condition (C3), we can also obtain a contradiction.

Case (B): Set

$$\|u_0\|^p - \int_{\mathbb{R}^N} \widetilde{f}_{\eta_n} |u_0|^s dx = A + o(1),$$

then by (3.5),

$$\|w_n\|^p - \int_{\mathbb{R}^N} \widetilde{f}_{\eta_n} |w_n|^s dx = -A + o(1).$$

Without loss of generality, we may assume that $A > 0$($A < 0$ can be considered similarly). We can choose a sequence $\{t_n\}$ with $t_n \to 1$ as $n \to \infty$ such that $v_n = t_n w_n$ satisfies

$$\|v_n\|^p - \int_{\mathbb{R}^N} \widetilde{f}_{\eta_n} |v_n|^s dx = -A.$$

Since $u_0 \in M_{\widetilde{f}_{\eta_n},0}(A + o(1))$, by (3.2)-(3.4) and Lemma 2.1 we have

$$\begin{aligned}
I_{f_{\eta_n}, h_{\eta_n}}(u_n) &= \frac{1}{p}\|u_0\|^p - \frac{1}{s}\int_{\mathbb{R}^N} \widetilde{f}_{\eta_n}(x)|u_0|^s dx \\
&\quad + \frac{1}{p}\|w_n\|^p - \frac{1}{s}\int_{\mathbb{R}^N} \widetilde{f}_{\eta_n}(x)|w_n|^s dx + o(1) \\
&\geq \frac{A + o(1)}{p} + \frac{1}{p}\|v_n\|^p - \frac{1}{s}\int_{\mathbb{R}^N} \widetilde{f}_{\eta_n}(x)|v_n|^s dx + o(1) \\
&\geq \alpha_{\widetilde{f}_{\eta_n},0}(A) + \alpha_{\widetilde{f}_{\eta_n},0}(-A) + o(1) \\
&> \alpha_{\widetilde{f}_{\eta_n},0} + \frac{s-p}{2sp}A + o(1) \\
&\geq \alpha_{f_{\max},0} + \frac{s-p}{2sp}A + o(1),
\end{aligned}$$

which is a contradiction. If $A = 0$, we can find two sequences $\{t_n\}$ and $\{s_n\}$ with $t_n \to 1$, $s_n \to 1$ as $n \to \infty$ such that $\overline{w}_n = t_n w_n$, $\overline{v}_n = s_n u_0$ satisfy

$$\|\overline{w}_n\|^p - \int_{\mathbb{R}^N} \widetilde{f}_{\eta_n} |\overline{w}_n|^s dx = 0,$$
$$\|\overline{v}_n\|^p - \int_{\mathbb{R}^N} \widetilde{f}_{\eta_n} |\overline{v}_n|^s dx = 0.$$

Thus

$$\lim_{n\to\infty} I_{f_{\eta_n}, h_{\eta_n}}(u_n)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Big[\frac{1}{p}\|\overline{w}_n\|^p - \frac{1}{s}\int_{\mathbb{R}^N} \widetilde{f}_{\eta_n}|\overline{w}_n|^s dx + \frac{1}{p}\|\overline{v}_n\|^p - \frac{1}{s}\int_{\mathbb{R}^N} \widetilde{f}_{\eta_n}|\overline{v}_n|^s dx\Big] \\
&> \alpha_{f_{\max},0},
\end{aligned}
$$

which is a contradiction. This completes the proof. □

From now on, taking $\delta > 0$ as in Lemma 3.4, and fixing $\varepsilon > 0$ such that $\varepsilon \le \delta$, consider η_ε as in Lemma 3.2, η_δ as in Lemma 3.4, and denote $\eta_0 = \min\{\eta_\varepsilon, \eta_\delta\}$.

Lemma 3.5. *If $\eta \in (0, \eta_0)$, then for each $u \in M_{f_\eta,h_\eta}$, there exist $\varepsilon_u > 0$ and a differentiable function $\xi_u : B(0,\varepsilon_u) \subset W^{1,p}(\mathbb{R}^N) \to R^+$ such that $\xi_u(0) = 1$, $\xi_u(v)(u-v) \in M_{f_\eta,h_\eta}$, and*

$$
\begin{aligned}
\langle \xi_u'(0), v\rangle = \Big[&p\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv dx \\
&- s\int_{\mathbb{R}^N} f_\eta|u|^{s-2}uv dx - \eta^{\frac{p(s-r)}{s-p}} r\int_{\mathbb{R}^N} h_\eta|u|^{r-2}uv dx\Big] \\
&\div \Big[(p-r)\|u\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta|u|^s dx\Big]
\end{aligned}
$$

for all $v \in W^{1,p}(\mathbb{R}^N)$.

Proof. For $u \in M_{f_\eta,h_\eta}$, define a function $F : R \times W^{1,p}(\mathbb{R}^N) \to R$ by

$$
\begin{aligned}
F_u(\xi_u, w) &= \langle I'_{f_\eta,h_\eta}(\xi_u(u-w)), \xi_u(u-w)\rangle \\
&= \xi_u^p \int_{\mathbb{R}^N} |\nabla(u-w)|^p + |u-w|^p dx - \xi_u^s\int_{\mathbb{R}^N} f_\eta|u-w|^s dx \\
&\quad - \eta^{\frac{p(s-r)}{s-p}}\xi_u^r\int_{\mathbb{R}^N} h_\eta|u-w|^r dx.
\end{aligned}
$$

Then $F_u(1,0) = \langle I'_{f_\eta,h_\eta}(u), u\rangle = 0$ and

$$
\begin{aligned}
\frac{d}{d\xi_u}F_u(1,0) &= p\|u\|^p - s\int_{\mathbb{R}^N} f_\eta|u|^s dx - \eta^{\frac{p(s-r)}{s-p}} r\int_{\mathbb{R}^N} h_\eta|u|^r dx \\
&= (p-r)\|u\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta|u|^s dx \neq 0.
\end{aligned}
$$

According to the implicit function theorem, there exist $\varepsilon_u > 0$ and a differentiable function $\xi_u : B(0,\varepsilon_u) \subset W^{1,p}(\mathbb{R}^N) \to R^+$ such that $\xi_u(0) = 1$, and

$$
\begin{aligned}
\langle \xi_u'(0), v\rangle = \Big[&p\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv dx \\
&- s\int_{\mathbb{R}^N} f_\eta|u|^{s-2}uv dx - \eta^{\frac{p(s-r)}{s-p}} r\int_{\mathbb{R}^N} h_\eta|u|^{r-2}uv dx\Big] \\
&\div \Big[(p-r)\|u\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta|u|^s dx\Big],
\end{aligned}
$$

and $F_u(\xi_u(v), v) = 0$ for all $v \in B(0,\varepsilon_u)$, which is equivalent to

$$
\langle I'_{f_\eta,h_\eta}(\xi_u(v)(u-w)), \xi_u(v)(u-w)\rangle = 0, \quad \forall v \in B(0,\varepsilon_u).
$$

That is, $\xi_u(v)(u-v) \in M_{f_\eta,h_\eta}$. □

Lemma 3.6. *If $\eta \in (0,\eta_0)$, then for each $u \in N_\eta^i$, there exist $\varepsilon_u > 0$ and a differentiable function $\xi_u^- : B(0,\varepsilon_u) \subset W^{1,p}(\mathbb{R}^N) \to R^+$ such that $\xi_u^-(0) = 1$, $\xi_u^-(v)(u-v) \in N_\eta^i$ for all $v \in B(0,\varepsilon_u)$, and*

$$\begin{aligned}\langle (\xi_u^-)'(0), v\rangle = \Big[p\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uvdx\\ - s\int_{\mathbb{R}^N} f_\eta |u|^{s-2}uvdx - \eta^{\frac{p(s-r)}{s-p}} r\int_{\mathbb{R}^N} h_\eta |u|^{r-2}uvdx\Big]\\ \div \Big[(p-r)\|u\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta |u|^s dx\Big]\end{aligned}$$

for all $v \in W^{1,p}(R^N)$.

Proof. Similar to the argument in Lemma 3.5, there exist $\varepsilon_u > 0$ and a differentiable function $\xi_u^- : B(0,\varepsilon_u) \subset W^{1,p}(\mathbb{R}^N) \to R^+$ such that $\xi_u^-(0) = 1$, $\xi_u^-(v)(u-v) \in M_{f_\eta,h_\eta}$ for all $v \in B(0,\varepsilon_u)$, Since

$$(p-r)\|u\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta |u|^s dx < 0,$$

thus, if ε_u small enough, by the continuity of the functions ξ_u^- and g_η, we have

$$\begin{aligned}&\langle \psi'(\xi_u^-(v)(u-v))), \xi_u^-(v)(u-v)\rangle\\ &= (p-r)\|\xi_u^-(v)(u-v)\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta |\xi_u^-(v)(u-v)|^s dx < 0.\end{aligned}$$

and $g_\eta(\xi_u^-(v)(u-v)) \in C_{l/\eta}^i$. ☐

Proposition 3.7. *(i) If $\eta \in (0,\eta_0)$, then there exists a $(PS)_{\alpha_{f_\eta,h_\eta}}$ sequence $\{u_n\} \subset M_{f_\eta,h_\eta}$ in $W^{1,p}(\mathbb{R}^N)$ for I_{f_η,h_η}.*

(ii) If $\eta \in (0,\eta_0)$, then there exists a $(PS)_{\gamma_\eta^i}$ sequence $\{u_n\} \subset N_\eta^i$ in $W^{1,p}(\mathbb{R}^N)$ for I_{f_η,h_η}, $i = 1,2,\dots,k$.

Proof. Since the proof of (i) is similar to that of (ii), but simpler, we only prove (ii) here. We denote by $\overline{N_\eta^i}$ the closure of N_η^i, then we note that

$$\overline{N_\eta^i} = N_\eta^i \cup \partial N_\eta^i, \quad \text{for each } i = 1,2,\dots,k.$$

From Lemma 3.2 and Lemma 3.4, we obtain

$$\gamma_\eta^i < \min\{\alpha_{f_\eta,h_\eta} + \alpha_{f^\infty,0}, \widetilde{\gamma_\eta^i}\}, \quad i = 1,2,\dots,k,\ \eta \in (0,\eta_0). \tag{3.7}$$

Hence

$$\gamma_\eta^i = \inf\{I_{f_\eta,h_\eta}(u) : u \in \overline{N_\eta^i}\} \quad \text{for } i = 1,2,\dots,k.$$

Fix some $i \in \{1,2,\dots,k\}$. Applying the Ekeland variational principle [17] there exists a minimizing sequence $\{u_n\} \subset \overline{N_\eta^i}$ such that

$$I_{f_\eta,h_\eta}(u_n) < \gamma_\eta^i + \frac{1}{n}, \tag{3.8}$$

$$I_{f_\eta,h_\eta}(u_n) < I_{f_\eta,h_\eta}(w) + \frac{1}{n}\|w-u_n\| \quad \text{for all } w \in \overline{N_\eta^i}. \tag{3.9}$$

From (3.7) we may assume that $u_n \in N_\eta^i$ for n sufficiently large. Applying Lemma 3.6 with $u = u_n$ we obtain the functional $\xi_{u_n}^- : B(0,\varepsilon_{u_n}) \to R$ for some $\varepsilon_{u_n} > 0$ such that $\xi_{u_n}^-(w)(u_n - w) \in N_\eta^i$. Choose $0 < \rho < \varepsilon_{u_n}$ and $u \in W^{1,p}(\mathbb{R}^N)$ with

$u \not\equiv 0$. Set $w_\rho = \frac{\rho u}{\|u\|}$ and $z_\rho^n = \xi_{u_n}^-(w_\rho)(u_n - w_\rho)$. Since $z_\rho^n \in N_\eta^i$, we deduce from (3.9) that

$$I_{f_\eta,h_\eta}(z_\rho^n) - I_{f_\eta,h_\eta}(u_n) \geq -\frac{1}{n}\|z_\rho^n - u_n\|.$$

By the mean value theorem, we have

$$\langle I'_{f_\eta,h_\eta}(u_n), z_\rho^n - u_n\rangle + o(\|z_\rho^n - u_n\|) \geq -\frac{1}{n}\|z_\rho^n - u_n\|.$$

Thus,

$$\langle I'_{f_\eta,h_\eta}(u_n), -w_\rho\rangle + (\xi_{u_n}^-(w_\rho) - 1)\langle I'_{f_\eta,h_\eta}(u_n), u_n - w_\rho\rangle \geq -\frac{1}{n}\|z_\rho^n - u_n\| + o(\|z_\rho^n - u_n\|). \tag{3.10}$$

Since $\xi_{u_n}^-(w_\rho)(u_n - w_\rho) \in N_\eta^i$ and consequently from (3.10) we obtain

$$\begin{aligned}
&- \rho\langle I'_{f_\eta,h_\eta}(u_n), \frac{u}{\|u\|}\rangle + (\xi_{u_n}^-(w_\rho) - 1)\langle I'_{f_\eta,h_\eta}(u_n) - I'_{f_\eta,h_\eta}(z_\rho^n), u_n - w_\rho\rangle \\
&\geq -\frac{1}{n}\|z_\rho^n - u_n\| + o(\|z_\rho^n - u_n\|).
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle I'_{f_\eta,h_\eta}(u_n), \frac{u}{\|u\|}\rangle &\leq \frac{(\xi_{u_n}^-(w_\rho) - 1)}{\rho}\langle I'_{f_\eta,h_\eta}(u_n) - I'_{f_\eta,h_\eta}(z_\rho^n), u_n - w_\rho\rangle \\
&\quad + \frac{\|z_\rho^n - u_n\|}{n\rho} + \frac{o(\|z_\rho^n - u_n\|)}{\rho}.
\end{aligned} \tag{3.11}$$

Since

$$\|z_\rho^n - u_n\| \leq \rho|\xi_{u_n}^-(w_\rho)| + |\xi_{u_n}^-(w_\rho) - 1|\|u_n\|$$

and

$$\lim_{\rho\to 0}\frac{|\xi_{u_n}^-(w_\rho) - 1|}{\rho} \leq \|(\xi_{u_n}^-)'(0)\|,$$

if we let $\rho \to 0$ in (3.11) for a fixed n, and by Lemma 2.5 (ii) we can find a constant $C > 0$, independent of ρ, such that

$$\langle I'_{f_\eta,h_\eta}(u_n), \frac{u}{\|u\|}\rangle \leq \frac{C}{n}(1 + \|(\xi_{u_n}^-)'(0)\|).$$

We are done once we show that $\|(\xi_{u_n}^-)'(0)\|$ is uniformly bounded in n. By Lemma 2.5 (ii), Lemma 3.6 and the Hölder inequality, we have

$$\langle(\xi_{u_n}^-)'(0), v\rangle \leq \frac{b\|v\|}{|(p-r)\|u_n\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta|u_n|^s dx|} \quad \text{for some } b > 0.$$

We only need to show that

$$|(p-r)\|u_n\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta|u_n|^s dx| > C$$

for some $C > 0$ and n large. We argue by way of contradiction. Assume that there exists a subsequence $\{u_n\}$ satisfy

$$(p-r)\|u_n\|^p - (s-r)\int_{\mathbb{R}^N} f_\eta|u_n|^s dx = o(1). \tag{3.12}$$

By the fact that $u_n \in M_{f_\eta,h_\eta}^-(u_n)$ and (3.12), we obtain that

$$\int_{\mathbb{R}^N} f_\eta|u_n|^s dx > 0.$$

So we have
$$\|u_n\| \le [\frac{s-r}{s-p}\eta^\beta \|h\|_{L^{\frac{p}{p-r}}}]^{\frac{1}{p-r}} + o(1) \tag{3.13}$$
$$\|u_n\| > (\frac{p-r}{s-r}\frac{S^s}{f_{\max}})^{\frac{1}{s-p}} + o(1). \tag{3.14}$$
Then
$$\begin{aligned} K(u_n) &= c(s,r)\Big[\frac{(\frac{s-r}{p-r}\int_{\mathbb{R}^N} f_\eta |u_n|^s dx + o(1))^{\frac{s-1}{p-1}}}{\int_{\mathbb{R}^N} f_\eta |u_n|^s dx}\Big]^{\frac{p-1}{s-p}} - \frac{s-p}{p-r}\int_{\mathbb{R}^N} f_\eta |u_n|^s dx \\ &= o(1). \end{aligned} \tag{3.15}$$
However, by (3.13)-(3.14), the Hölder and Sobolev inequalities, combining with $\beta > 0$ and $\eta \in (0, \eta_0)$, we have
$$\begin{aligned} K(u_n) &\ge c(s,r)(\frac{\|u_n\|^{p\frac{s-1}{p-1}}}{\int_{\mathbb{R}^N} f_\eta |u_n|^s dx})^{\frac{p-1}{s-p}} - \eta^\beta \|h\|_{L^{\frac{p}{p-r}}} \|u_n\|^r + o(1) \\ &\ge \|u_n\|^r [c(s,r)(\frac{S^s}{f_{\max}})^{\frac{p-1}{s-p}} \|u_n\|^{1-r} - \eta^\beta \|h\|_{L^{\frac{p}{p-r}}}] + o(1) \\ &\ge \|u_n\|^r [c(s,r)(\frac{S^s}{f_{\max}})^{\frac{p-1}{s-p}} (\eta^\beta \frac{s-r}{s-p}\|h\|_{L^{\frac{p}{p-r}}})^{\frac{1-r}{p-r}} - \eta^\beta \|h\|_{L^{\frac{p}{p-r}}}] + o(1) \\ &\ge d \end{aligned}$$
for some $d > 0$ and n large enough. This is a contradiction to (3.15). So we have
$$I_{f_\eta, h_\eta}(u_n) = \gamma_\eta^i + o(1)$$
and $I'_{f_\eta, h_\eta}(u_n) = 0$ in $W^{-1}(\mathbb{R}^N)$. Thus we complete the proof of (ii). □

Theorem 3.8. *For each $\eta \in (0, \eta_0)$, Equation (2.1) has a positive solution $u_\eta \in M^+_{f_\eta, h_\eta}$ such that $I_{f_\eta, h_\eta}(u_\eta) = \alpha_{f_\eta, h_\eta} = \alpha^+_{f_\eta, h_\eta}$.*

Proof. By Proposition 3.7 (i), there exists a $(PS)_{\alpha_{f_\eta, h_\eta}}$ sequence $\{u_n\} \subset M_{f_\eta, h_\eta}$, by Lemma 2.5 (ii) and Lemma 3.3, there exist a subsequence $\{u_n\}$ and u_η in $W^{1,p}(\mathbb{R}^N)$ such that
$$\begin{gathered} u_n \rightharpoonup u_\eta \quad \text{weakly in } W^{1,p}(\mathbb{R}^N), \\ u_n \to u_\eta \quad \text{a.e. in } \mathbb{R}^N, \\ u_n \to u_\eta \quad \text{in } L^q(\mathbb{R}^N) \text{ for} 1 \le q \le p^*, \\ \nabla u_n \to \nabla u_\eta \quad \text{a.e. in } \mathbb{R}^N, \\ |\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u_\eta|^{p-2}\nabla u_\eta \quad \text{in } L^{\frac{p}{p-1}}(\mathbb{R}^N), \end{gathered}$$
It is easy to see that u_η is a solution of (2.1)

Moreover, by the Egorov theorem and the Hölder inequality and condition $h \in L^{\frac{p}{p-r}}(\mathbb{R}^N)$, we obtain
$$\int_{\mathbb{R}^N} h_\eta |u_n|^r dx \to \int_{\mathbb{R}^N} h_\eta |u_\eta|^r dx.$$
We claim that $\int_{\mathbb{R}^N} h_\eta |u_\eta|^r dx \ne 0$. If not,
$$\|u_n\|^p = \int_{\mathbb{R}^N} f_\eta |u_n|^s dx + o(1),$$

and

$$
\begin{aligned}
&(\frac{1}{p}-\frac{1}{s})\int_{\mathbb{R}^N} f_\eta|u_n|^s dx\\
&=\frac{1}{p}\|u_n\|^p-\frac{1}{s}\int_{\mathbb{R}^N} f_\eta|u_n|^s dx-\eta^{\frac{p(s-r)}{s-p}}\frac{1}{r}\int_{\mathbb{R}^N} h_\eta|u_n|^r dx+o(1)\\
&=\alpha_{f_\eta,h_\eta}+o(1),
\end{aligned}
$$

this contradicts $\alpha_{f_\eta,h_\eta}<0$. Thus, u_η is a nontrivial solution of (2.1). Now we show that $u_n\to u_\eta$ strongly in $W^{1,p}(\mathbb{R}^N)$. If not, $\|u_\eta\|<\liminf_{n\to\infty}\|u_n\|$, so we have

$$
\begin{aligned}
\alpha_{f_\eta,h_\eta}&\le I_{f_\eta,h_\eta}(u_\eta)=(\frac{1}{p}-\frac{1}{s})\|u_\eta\|^p-(\frac{1}{r}-\frac{1}{s})\eta^{\frac{p(s-r)}{s-p}}\int_{\mathbb{R}^N} h_\eta|u_\eta|^r dx\\
&<\lim_{n\to\infty} I_{f_\eta,h_\eta}(u_n)=\alpha_{f_\eta,h_\eta},
\end{aligned}
$$

this is a contradiction. Thus $I_{f_\eta,h_\eta}(u_\eta)=\alpha_{f_\eta,h_\eta}$. At last, we show $u_\eta\in M^+_{f_\eta,h_\eta}$. If not, by Lemma 2.2, we know that $u_\eta\in M^-_{f_\eta,h_\eta}$, by Lemma 2.4, there exist unique t_0^+ and t_0^- such that $t_0^+u_\eta\in M^+_{f_\eta,h_\eta}$ and $t_0^-u_\eta\in M^-_{f_\eta,h_\eta}$, and $t_0^+<t_0^-=1$. Since

$$
\frac{d}{dt}I_{f_\eta,h_\eta}(t_0^+u_\eta)=0,\quad \frac{d^2}{dt^2}I_{f_\eta,h_\eta}(t_0^+u_\eta)>0,
$$

there exists $\widetilde{t}\in(t_0^+,t_0^-]$ such that $I_{f_\eta,h_\eta}(t_0^+u_\eta)<I_{f_\eta,h_\eta}(\widetilde{t}u_\eta)$. By Lemma 2.4,

$$
I_{f_\eta,h_\eta}(t_0^+u_\eta)<I_{f_\eta,h_\eta}(\widetilde{t}u_\eta)\le I_{f_\eta,h_\eta}(t_0^-u_\eta)=I_{f_\eta,h_\eta}(u_\eta),
$$

which is a contradiction. Thus, $I_{f_\eta,h_\eta}(u_\eta)=\alpha_{f_\eta,h_\eta}=\alpha^+_{f_\eta,h_\eta}$. Since $I_{f_\eta,h_\eta}(u_\eta)=I_{f_\eta,h_\eta}(|u_\eta|)$ and $|u_\eta|\in M^+_{f_\eta,h_\eta}$, by Lemma 2.3 and the maximum principle, we may assume that u_η is a positive solution of (2.1). □

Proposition 3.9. *Assume that $\{u_n\}\subset M^-_{f_\eta,h_\eta}$ is a $(PS)_c$ sequence, where $c<\alpha_{f_\eta,h_\eta}+\alpha_{f^\infty,0}$. Then there exists a subsequence, still denoted by $\{u_n\}$, and u_0 in $W^{1,p}(\mathbb{R}^N)$ such that $u_n\to u_0$ strongly in $W^{1,p}(\mathbb{R}^N)$ and $I_{f_\eta,h_\eta}(u_0)=c$.*

Proof. By Lemma 2.5 (ii), there exists a subsequence $\{u_n\}$ and u_0 in $W^{1,p}(\mathbb{R}^N)$ such that

$$
u_n\rightharpoonup u_0\quad\text{weakly in } W^{1,p}(\mathbb{R}^N).
$$

First, we claim that $u_0\equiv 0$ is impossible. If not, by $h\in L^{\frac{p}{p-r}}(\mathbb{R}^N)$, the Egorov theorem and the Hölder inequality, we have

$$
\|u_n\|^p=o(1). \tag{3.16}
$$

Moreover, $\{u_n\}\subset M^-_{f_\eta,h_\eta}$, we deduce from the Sobolev imbedding theorem that

$$
\|u_n\|>C\quad\text{for some } C>0,\ n=1,2,\dots.
$$

which contradicts to (3.16). Thus, by Lemma 3.3, u_0 is a nontrivial solution of (2.1) and $I_{f_\eta,h_\eta}(u_0)\ge\alpha_{f_\eta,h_\eta}$. We write $u_n=u_0+v_n$ with $v_n\rightharpoonup 0$ weakly in $W^{1,p}(\mathbb{R}^N)$. By the Brezis-Lieb lemma [16], we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} f_\eta|u_n|^p dx&=\int_{\mathbb{R}^N} f_\eta|u_0|^p dx+\int_{\mathbb{R}^N} f_\eta|v_n|^p dx+o(1)\\
&=\int_{\mathbb{R}^N} f_\eta|u_0|^p dx+\int_{\mathbb{R}^N} f^\infty|v_n|^p dx+o(1).
\end{aligned}
$$

Since $\{u_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$, we have $\{v_n\}$ is also a bounded sequence in $W^{1,p}(\mathbb{R}^N)$. Moreover, by $h \in L^{\frac{p}{p-r}}(\mathbb{R}^N)$, the Egorov theorem and the Hölder inequality, we have

$$\int_{\mathbb{R}^N} h_\eta |v_n|^r dx = \int_{\mathbb{R}^N} h_\eta |u_n|^r dx - \int_{\mathbb{R}^N} h_\eta |u_0|^r dx + o(1) = o(1).$$

Hence, for n large enough, we can conclude that

$$\begin{aligned}
\alpha_{f_\eta,h_\eta} + \alpha_{f^\infty,0} &> I_{f_\eta,h_\eta}(u_0 + v_n) \\
&\geq I_{f_\eta,h_\eta}(u_0) + \frac{1}{p}\|v_n\|^p - \frac{1}{s}\int_{\mathbb{R}^N} f^\infty |v_n|^s dx + o(1) \\
&\geq \alpha_{f_\eta,h_\eta} + \frac{1}{p}\|v_n\|^p - \frac{1}{s}\int_{\mathbb{R}^N} f^\infty |v_n|^s dx + o(1),
\end{aligned}$$

we obtain

$$\frac{1}{p}\|v_n\|^p - \frac{1}{s}\int_{\mathbb{R}^N} f^\infty |v_n|^s dx < \alpha_{f^\infty,0} + o(1). \tag{3.17}$$

Also from $I'_{f_\eta,h_\eta}(u_n) = o(1)$ in $W^{-1}(\mathbb{R}^N)$, $\{u_n\}$ is uniformly bounded and u_0 is a solution of (2.1), we obtain

$$\langle I'_{f_\eta,h_\eta}(u_n), u_n\rangle = \|v_n\|^p - \int_{\mathbb{R}^N} f^\infty |v_n|^s dx + o(1) = o(1). \tag{3.18}$$

We claim that (3.17) and (3.18) can be hold simultaneously only if $\{v_n\}$ admits a subsequence which converges strongly to zero. If not, then $\|v_n\|$ is bounded away from zero; that is,

$$\|v_n\| \geq C \quad \text{for some } C > 0.$$

From (3.18), it follows that

$$\int_{\mathbb{R}^N} f^\infty |v_n|^s dx \geq \frac{sp}{s-p}\alpha_{f^\infty,0} + o(1).$$

By (3.17) and (3.18), for n large enough

$$\begin{aligned}
\alpha_{f^\infty,0} &\leq (\frac{1}{p} - \frac{1}{s})\int_{\mathbb{R}^N} f^\infty |v_n|^s dx + o(1) \\
&= \frac{1}{p}\|v_n\|^p - \frac{1}{s}\int_{\mathbb{R}^N} f^\infty |v_n|^s dx + o(1) < \alpha_{f^\infty,0},
\end{aligned}$$

which is a contradiction. Therefore, $u_n \to u_0$ strongly in $W^{1,p}(R^N)$ and $I_{f_\eta,h_\eta}(u_0) = c$. □

Proof of Theorem 1.1. By Lemma 3.2, Proposition 3.7 and Proposition 3.9, for each $\eta \in (0, \eta_0)$ and $i \in \{1, 2, \dots, k\}$, there exist a sequence $\{u_n^i\} \subset N_\eta^i$ and $u_0^i \in W^{1,p}(\mathbb{R}^N)\backslash\{0\}$ such that

$$\begin{gathered}
I_{f_\eta,h_\eta}(u_n^i) = \gamma_\eta^i + o(1), \\
I'_{f_\eta,h_\eta}(u_n^i) = o(1)
\end{gathered}$$

and $u_n^i \to u_0^i$ strongly in $W^{1,p}(\mathbb{R}^N)$. Obviously, the function u_0^i is a solution of the equation (2.1) and $I_{f_\eta,h_\eta}(u_0^i) = \gamma_\eta^i$. Similar to the argument in Theorem 3.8, we have u_0^i is positive. Since $g_\eta^i(u_0^i) \in \overline{C_{l/\eta}(x^i)}$, $u_\eta \in M^+_{f_\eta,h_\eta}$ and $u_0^i \in M^-_{f_\eta,h_\eta}$, where u_η is a positive solution of Eq.(2.1) as in Theorem 3.8. This implies u_η, u_0^i and u_0^j are different for $i \neq j$.

Letting $\lambda_0 = \eta_0^{-p}$, $U_\lambda(x) = \lambda^{\frac{1}{s-p}} u_\eta(\lambda^{1/p}x)$ and $U_i(x) = \lambda^{\frac{1}{s-p}} u_0^i(\lambda^{1/p}x)$. We obtain U_λ and U_i are positive solutions of the (1.1) with $i = 1, 2, \dots, k$. This completes the proof. □

Remark 3.10. It is easy to see from the proof of Theorem 1.1 that the solutions U_λ, $U_i (i = 1, 2, \dots, k)$ satisfy

(1) $\|U_\lambda\|_{L^\infty(\mathbb{R}^N)}$, $\|U_i\|_{L^\infty(\mathbb{R}^N)} \to \infty$ as $\lambda \to \infty$;
(2) $\|U_\lambda\|_{L^p(\mathbb{R}^N)}$, $\|U_i\|_{L^p(\mathbb{R}^N)} \to \infty$ as $\lambda \to \infty$ if $p < s < \frac{p^2}{N} + p$;
(3) $\|U_\lambda\|_{L^p(\mathbb{R}^N)}$, $\|U_i\|_{L^p(\mathbb{R}^N)} \to 0$ as $\lambda \to \infty$ if $\frac{p^2}{N} + p < s < p^*$.

Lemma 3.11. *When $1 \le r < p < s < p^*$ and $N \ge 1$, we have* $\frac{p(s-r)}{s-p} - \frac{(p-r)N}{p} > 0$.

Proof. When $N \le p$ and $1 \le r < p < s < p^*$, obviously, we have

$$\frac{p(s-r)}{s-p} - \frac{(p-r)N}{p} > 0.$$

We consider only the case $N > p$. Set

$$L(s) = p^2(s-r) - (p-r)N(s-p), \quad s \in (p, p^*).$$

Then it is easy to see that

$$L(s) \ge \min\{L(p), L(p^*)\} = \min\{p^2(p-r), \frac{p^3 r}{N-p}\} > 0.$$

This completes the proof. □

Acknowledgements. The authors wish to thank the anonymous reviewers and the editor for their helpful comments. We would like to thank Professot D. M. Cao and T. F. Wu for their help and advice for completing this article. This research was supported by the National Natural Science Foundation of China (No. 11171092); the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (No. 08KJB110005); the Natural Science Foundation of Jiangsu Education Office (No. 12KJB110002)

References

[1] S. Adachi, K. Tanaka; *Multiple positive solutions for nonhomogeneous equations*, Nonlinear Analysis 47 (2001), 3787-3793.

[2] S. Adachi, K. Tanaka; *Four positive solutions for the semilinear elliptic equation $-\Delta u + u = a(x)u^p + f(x)$ in $\mathbb{R}^N$*, Calc. Var. 11 (2000), 63-95.

[3] A. Bahri, P. L. Lions; *On the existence of positive solutions of semilinear elliptic equations in unbounded domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (3) (1997), 365-443.

[4] A.Bahri, Y. Y. Li; *On the min-max procedure for the existence of a position solution for certain scalar field equations in $\mathbb{R}^N$*, Rev. Mat. Iberoamericana 6 (1990), 1-15.

[5] H. Berestycki, P. L. Lions; *Nonlinear scalar field equations-I:existence of a ground state, Archive for Rational Mechanics and Analysis* 82 (4) (1983), 313-345.

[6] H. Berestycki, P. L. Lions; *Nonlinear scalar field equations-II:existence of infinitely many solutions*, Archive for Rational Mechanics and Analysis 82 (4) (1983), 347-375.

[7] H. Brezis, E. Lieb; *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. 88(1983), 486-490.

[8] D. Cao, E. S. Noussair; *Multiplicity of positive and nodal solutions for nonlinear elliptic problems in $\mathbb{R}^N$*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 13 (5) (1996), 567-588.

[9] D. M. Cao, H. S. Zhou; *Multiple positive solutions of nonhomogeneous semilinear elliptic equations in $\mathbb{R}^N$*, Proc. Roy. Soc. Edinburgh Sect. A 126 (1996), 443-463.

[10] G. Citti, F. Uguzzoni; *Positive solutions of $-\Delta_p u + u^{p-1} = q(x)u^\alpha$ in $\mathbb{R}^N$*, Nonlinear Differential Equations and Applications 9 (1) (2002), 1-14.

[11] G. Citti; *A uniqueness theorem for radial ground states of the equation* $\Delta_p u + f(u) = 0$, Boll. Un. Mat. Ital. (7) 7-B (1993), 283-310.
[12] D. M. Cao, G. B. Li, H. S. Zhou; *The existence of two solutions to quasilinear elliptic equations on* $\mathbb{R}^N$, Chinese J. Contemp. Math. 17 (3) (1996), 277-285.
[13] I. Ekeland; *On the variational principle*, J. Math. Anal. Appl. 17 (1974), 324-353.
[14] L. Jeanjean; *Two positive solutions for a class of nonhomogeneous elliptic equations*, Differential and Integral Equations 10 (1997), 609-624.
[15] M. K. Kwong; *Uniqueness of positive solution of* $\Delta u - u + u^p = 0$ *in* $\mathbb{R}^N$, Arch. Ra. Math. Anal. 105 (1989), 243-266.
[16] P. L. Lions; *The concentration-compactness principle in the calculus of variations. The locally compact case. I, II*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), 109-145, 223-283.
[17] G. B. Li; *The existence of a weak solution of quasilinear elliptic equation with critical Sobolev exponent on unbounded domains*, Acta. Math. Scientia 14 (1) (1994), 64-74.
[18] Y. Li; *Remarks on a semilinear elliptic equation on* $\mathbb{R}^N$, J. Differential Equations 74 (1988), 34-49.
[19] C. A. Swanson, L. S. Yu; *Critical p-Laplacian Problems in* $\mathbb{R}^N$, Annali di Matematica pura ed applicata (IV), vol. CLXIX (1995), 233-250.
[20] G.T arantello; *On nonhomogenous elliptic equations involving the critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire 9 (3) (1992), 281-304.
[21] T. F. Wu; *Multiplicity of positive solutions for semilinear elliptic equations in* $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh, Sect. A 138 (2008), 647-670.
[22] J. Yang, X. Zhu; *On the existence of nontrivial solution of a quasilinear elliptic boundary value problem for unbounded domains*, Acta Mathematica Scientia 7 (3) (1987), 341-359.

Honghui Yin
Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Jiangsu Nanjing 210023, China.
School of Mathematical Sciences, Huaiyin Normal University, Jiangsu Huaian 223001, China
E-mail address: yinhh771109@163.com

Zuodong Yang
Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Jiangsu Nanjing 210023, China.
School of Teacher Education, Nanjing Normal University, Jiangsu Nanjing 210097, China
E-mail address: zdyang_jin@263.net