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INVERSE PROBLEM OF DETERMINING THE COEFFICIENTS IN A DEGENERATE PARABOLIC EQUATION

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Abstract. We consider the inverse problem of identifying the time-dependent coefficients in a degenerate parabolic equation. The conditions of existence and uniqueness of the classical solution to this problem are established. We investigate the case of weak power degeneration.

1. Introduction

Inverse problems of determining simultaneously several coefficients in parabolic equations without degeneration are studied in many articles. These unknown parameters can depend on spatial variables [1]-[3] or on time variables [10]-[11].

The inverse problems for the degenerate parabolic equation are rarely investigated. Sufficient conditions for the existence and uniqueness of classical solutions to inverse problems of identification for the time-dependent leading coefficient in a degenerate parabolic equation in a domain with known boundary are established in [13, 12], and for a free boundary domain in [4, 9]. Both cases of weak and strong power degeneration are investigated. The conditions of determination for the time-dependent lower coefficients in the parabolic equations with weak power degeneration in a fixed boundary domain are found in [5] and in a free boundary domain in [6].

In this article we consider an inverse problem of identifying simultaneously the two unknown time-dependent parameters in a one-dimensional degenerate parabolic equation. It is known that the leading coefficient of this equation is the product of the power function which caused degeneration and an unknown function of time. Our aim is to establish the conditions of existence and uniqueness of the classical solution to this problem in the case of weak degeneration.

2. Statement of the problem

In a domain $Q_T = \{(x,t) : 0 < x < h, 0 < t < T\}$ we consider an inverse problem of determining the time dependent coefficients $a = a(t)$ and $b = b(t)$ in the one-dimensional degenerate parabolic equation

$$u_t = a(t)t^{\beta}u_{xx} + b(t)u_x + c(x,t)u + f(x,t) \tag{2.1}$$

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with initial condition

$$u(x,0)=\varphi(x), \quad x\in[0,h], \tag{2.2}$$

boundary conditions

$$u(0,t)=\mu_1(t), \quad u(h,t)=\mu_2(t), \quad t\in[0,T] \tag{2.3}$$

and over-determination conditions

$$a(t)t^{\beta}u_x(0,t)=\mu_3(t), \quad t\in[0,T], \tag{2.4}$$

$$\int_0^h u(x,t)dx=\mu_4(t), \quad t\in[0,T], \tag{2.5}$$

where β is a known number.

Definition 2.1. A triplet of functions $(a,b,u)\in(C[0,T])^2\times C^{2,1}(Q_T)\cap C^{1,0}(\overline{Q}_T)$, with $a(t)>0$, $t\in[0,T]$ is called a solution to the problem (2.1)-(2.5) if it verifies the equation (2.1) and conditions (2.2)-(2.5).

We will investigate the case of weak power degeneration, when $0<\beta<1$.

3. Existence of a solution

We use the following assumptions:

- (A1) $\varphi\in C^2[0,h]$, $\mu_3(t)=\mu_{3,0}(t)t^{\beta}$, $\mu_{3,0}\in C[0,T]$, $\mu_i\in C^1[0,T]$, $i=1,2,4$, $c,f\in C(\overline{Q}_T)$ and satisfy the Hölder condition with respect to x uniformly to t;
- (A2) $\varphi'(x)>0$, $x\in[0,h]$, $\mu_{3,0}(t)>0$, $\mu_2(t)-\mu_1(t)\neq 0$, $t\in[0,T]$;
- (A3) $\varphi(0)=\mu_1(0)$, $\varphi(h)=\mu_2(0)$, $\int_0^h\varphi(x)dx=\mu_4(0)$.

Theorem 3.1. *Under assumptions* (A1)–(A3), *problem* (2.1)–(2.5) *has a solution* (a,b,u) *for* $x\in[0,h]$ *and* $t\in[0,T_0]$, *where the number* T_0, $0<T_0\le T$, *is defined by the data.*

Proof. First of all we reduce the problem (2.1)–(2.5) to the equivalent system of equations. Suppose temporary that $a=a(t)$ and $b=b(t)$ are the known functions. Making the substitution

$$u(x,t)=\widetilde{u}(x,t)+\varphi(x)+\mu_1(t)-\mu_1(0)+\frac{x}{h}\Big(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)\Big) \tag{3.1}$$

we reduce the direct problem (2.1)–(2.3) to the problem with respect to function $\widetilde{u}=\widetilde{u}(x,t)$ with homogeneous initial and boundary conditions:

$$\begin{aligned}\widetilde{u}_t&=a(t)t^{\beta}\widetilde{u}_{xx}+b(t)\widetilde{u}_x+c(x,t)\widetilde{u}+f(x,t)-\mu_1'(t)-\frac{x}{h}(\mu_2'(t)-\mu_1'(t))\\&\quad+a(t)t^{\beta}\varphi''(x)+b(t)\Big(\varphi'(x)+\frac{1}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0))\Big)\\&\quad+c(x,t)\Big(\varphi(x)+\mu_1(t)-\mu_1(0)+\frac{x}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0))\Big),\\&\quad(x,t)\in Q_T,\end{aligned} \tag{3.2}$$

$$\widetilde{u}(x,0)=0, \quad x\in[0,h], \tag{3.3}$$

$$\widetilde{u}(0,t)=\widetilde{u}(h,t)=0, \quad t\in[0,T]. \tag{3.4}$$

For arbitrary functions $b = b(t)$ and $a = a(t) > 0$ continuous on $[0,T]$ problem (3.2)–(3.4) is equivalent to the equation

$$\begin{aligned}\widetilde{u}(x,t) = \int_0^t \int_0^h G_1(x,t,\xi,\tau)\Big(& b(\tau)\widetilde{u}_\xi + c(\xi,\tau)\widetilde{u} + f(\xi,\tau) - \mu_1'(\tau) \\ & - \frac{\xi}{h}(\mu_2'(\tau) - \mu_1'(\tau)) + a(\tau)\tau^\beta\varphi''(\xi) + b(\tau)\Big(\varphi'(\xi) + \frac{1}{h}(\mu_2(\tau) \\ & - \mu_1(\tau) - \mu_2(0) + \mu_1(0))\Big) + c(\xi,\tau)\Big(\varphi(\xi) + \mu_1(\tau) - \mu_1(0) \\ & + \frac{\xi}{h}(\mu_2(\tau) - \mu_1(\tau) - \mu_2(0) + \mu_1(0))\Big)d\xi\,d\tau,\end{aligned} \tag{3.5}$$

where $G_1 = G_1(x,t,\xi,\tau)$ is a Green function of the first value-boundary problem for the heat equation

$$\widetilde{u}_t = a(t)t^\beta\widetilde{u}_{xx}. \tag{3.6}$$

It is known [7, p. 13], that the Green functions $G_k = G_k(x,t,\xi,\tau)$, $k = 1,2$ of the first ($k = 1$) or the second ($k = 2$) value-boundary problem for (3.6) have the form

$$\begin{aligned}G_k(x,t,\xi,\tau) = \frac{1}{2\sqrt{\pi(\theta(t)-\theta(\tau))}} \sum_{n=-\infty}^{+\infty} \Big(& \exp\Big(-\frac{(x-\xi+2nh)^2}{4(\theta(t)-\theta(\tau))}\Big) \\ & + (-1)^k \exp\Big(-\frac{(x+\xi+2nh)^2}{4(\theta(t)-\theta(\tau))}\Big)\Big), \quad k = 1,2,\end{aligned} \tag{3.7}$$

where $\theta(t) = \int_0^t a(\tau)\tau^\beta d\tau$.

Put $v(x,t) \equiv u_x(x,t)$. Using (3.1), (3.5), we reduce the direct problem (2.1)-(2.3) to the system of integral equations

$$\begin{aligned}u(x,t) = {} & \varphi(x) + \mu_1(t) - \mu_1(0) + \frac{x}{h}\Big(\mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0)\Big) \\ & + \int_0^t \int_0^h G_1(x,t,\xi,\tau)\Big(b(\tau)v(\xi,\tau) + c(\xi,\tau)u(\xi,\tau) + f(\xi,\tau) \\ & - \mu_1'(\tau) - \frac{\xi}{h}(\mu_2'(\tau) - \mu_1'(\tau)) + a(\tau)\tau^\beta\varphi''(\xi)\Big)d\xi\,d\tau, \quad (x,t) \in \overline{Q}_T,\end{aligned} \tag{3.8}$$

$$\begin{aligned}v(x,t) = {} & \varphi'(x) + \frac{1}{h}\Big(\mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0)\Big) \\ & + \int_0^t \int_0^h G_{1x}(x,t,\xi,\tau)\Big(b(\tau)v(\xi,\tau) + c(\xi,\tau)u(\xi,\tau) + f(\xi,\tau) \\ & - \mu_1'(\tau) - \frac{\xi}{h}(\mu_2'(\tau) - \mu_1'(\tau)) + a(\tau)\tau^\beta\varphi''(\xi)\Big)d\xi\,d\tau, \quad (x,t) \in \overline{Q}_T.\end{aligned} \tag{3.9}$$

Note that we differentiate (3.8) with respect to x in order to find $v = v(x,t)$. Let us study the behavior of the integrals on the right-hand sides of the formulas (3.8), (3.9). Using (3.7), it is easy to verify that

$$\int_0^h G_1(x,t,\xi,\tau)d\xi \le 1, \quad \int_0^h |G_{1x}(x,t,\xi,\tau)|d\xi \le \frac{C_1}{\sqrt{\theta(t)-\theta(\tau)}}. \tag{3.10}$$

Then from (3.8) and (3.9), we deduce that

$$I_1 \equiv \Big|\int_0^t \int_0^1 G_1(x,t,\xi,\tau)\Big(b(\tau)v(\xi,\tau) + c(\xi,\tau)u(\xi,\tau) + f(\xi,\tau) - \mu_1'(\tau)$$

$$
\begin{aligned}
&\quad - \frac{\xi}{h}(\mu_2'(\tau)-\mu_1'(\tau)) + a(\tau)\tau^\beta\varphi''(\xi)\Big)d\xi\,d\tau\Big| \\
&\le C_2 t, \\
I_2 \equiv &\Big|\int_0^t\int_0^1 G_{1x}(x,t,\xi,\tau)\Big(b(\tau)v(\xi,\tau)+c(\xi,\tau)u(\xi,\tau)+f(\xi,\tau)-\mu_1'(\tau) \\
&\quad - \frac{\xi}{h}(\mu_2'(\tau)-\mu_1'(\tau)) + a(\tau)\tau^\beta\varphi''(\xi)\Big)d\xi\,d\tau\Big| \\
&\le C_3\int_0^t \frac{d\tau}{\sqrt{\theta(t)-\theta(\tau)}} \le C_4\int_0^t\frac{d\tau}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}} \\
&\le C_5 t^{\frac{1-\beta}{2}}\int_0^1\frac{dz}{\sqrt{1-z^{\beta+1}}} \le C_6 t^{\frac{1-\beta}{2}}.
\end{aligned}
$$

Taking into account that $0<\beta<1$, we conclude that the integrals on the right-hand sides of the formulas (3.8), (3.9) tend to zero as t tends to zero.

Under the conditions of the Theorem 3.1 we can represent equations (2.4), (2.5) in the form

$$a(t) = \frac{\mu_3(t)}{t^\beta v(0,t)}, \quad t\in[0,T], \tag{3.11}$$

$$
\begin{aligned}
b(t) = \frac{1}{\mu_2(t)-\mu_1(t)}&\Big(\mu_4'(t)+\mu_3(t)-a(t)t^\beta v(h,t) \\
&-\int_0^h (c(x,t)u(x,t)+f(x,t))dx\Big), \quad t\in[0,T].
\end{aligned}\tag{3.12}
$$

To this end, it suffices to differentiate (2.5) with respect to t.

Consequently, problem (2.1)–(2.5) is reduced to the system of equations (3.8), (3.9), (3.11), (3.12) with respect to the unknowns $u=u(x,t)$, $v=v(x,t)$, $a=a(t)$, $b=b(t)$. It follows from the way of derivation of (3.8), (3.9), (3.11), (3.12) that if (a,b,u) is a solution to the problem (2.1)-(2.5) then (u,v,a,b) is a continuous solution to the system of equations (3.8), (3.9), (3.11), (3.12). On the other hand, if $(u,v,a,b)\in(C(\overline{Q}_T))^2\times(C[0,T])^2$, $a(t)>0$, $t\in[0,T]$ is a solution to the system (3.8), (3.9), (3.11), (3.12), than (a,b,u) is a solution to the inverse problem (2.1)-(2.5) in the sense of the above definition. Indeed, using the uniqueness properties of the solutions to the system of Volterra integral equations of the second kind it is easy to see that $v(x,t)\equiv u_x(x,t)$. So it follows from (3.8) that function $u=u(x,t)$ is a solution to the equation

$$
\begin{aligned}
u(x,t) = \varphi(x)+\mu_1(t)-\mu_1(0)&+\frac{x}{h}\Big(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)\Big) \\
&+\int_0^t\int_0^h G_1(x,t,\xi,\tau)\Big(b(\tau)u_\xi(\xi,\tau)+c(\xi,\tau)u(\xi,\tau)+f(\xi,\tau) \\
&-\mu_1'(\tau)-\frac{\xi}{h}(\mu_2'(\tau)-\mu_1'(\tau))+a(\tau)\tau^\beta\varphi''(\xi)\Big)d\xi\,d\tau.
\end{aligned}
$$

This means that $u\in C^{2,1}(Q_T)\cap C^{1,0}(\overline{Q}_T)$ is a solution to (2.1)–(2.3). Then we can rewrite (3.12) in the form

$$a(t)t^\beta(u_x(h,t)-u_x(0,t))+b(t)(\mu_2(t)-\mu_1(t))+\int_0^h(c(x,t)u(x,t)+f(x,t))dx=\mu_4'(t)$$

or , using (2.1), (2.3), in the form

$$\int_0^h u_t(x,t)dx = \mu_4'(t).$$

We integrate this equality with respect to time variable from 0 to t. Taking into account compatibility conditions, we obtain (2.5). The validity of (2.4) follows from (3.11).

Now we study the system of equations (3.8), (3.9), (3.11), (3.12). We will apply the Schauder fixed point theorem for a compact operator to this system. For this aim we first establish a priori estimates for the solutions to (3.8), (3.9), (3.11), (3.12).

We estimate the functions $u = u(x,t)$, $v = v(x,t)$ taking into account (3.8), (3.9). We conclude from the assumption of Theorem 3.1 that only the first terms on the right-hand side of (3.8), (3.9) respectively are non equiv to zero. The sum of the rest terms tend to zero when t tends to zero. So, there exists the number $t_1, 0 < t_1 \le T$, such that

$$\begin{aligned} &\Big|\mu_1(t) - \mu_1(0) + \frac{x}{h}\Big(\mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0)\Big) \\ &+ \int_0^t \int_0^h G_1(x,t,\xi,\tau)\Big(b(\tau)v(\xi,\tau) + c(\xi,\tau)u(\xi,\tau) + f(\xi,\tau) \\ &- \mu_1'(\tau) - \frac{\xi}{h}(\mu_2'(\tau) - \mu_1'(\tau)) + a(\tau)\tau^\beta\varphi''(\xi)\Big)d\xi\, d\tau\Big| \\ &\le \frac{M_0}{2}, \quad (x,t) \in [0,h]\times[0,t_1], \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\frac{\varphi'(x)}{2} + \frac{\mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0)}{h} \\ &+ \int_0^t \int_0^h G_{1x}(x,t,\xi,\tau)\Big(b(\tau)v(\xi,\tau) + c(\xi,\tau)u(\xi,\tau) + f(\xi,\tau) - \mu_1'(\tau) \\ &- \frac{\xi}{h}(\mu_2'(\tau) - \mu_1'(\tau)) + a(\tau)\tau^\beta\varphi''(\xi)\Big)d\xi\, d\tau \ge 0, \quad (x,t) \in [0,h]\times[0,t_1], \end{aligned} \tag{3.14}$$

where $M_0 \equiv \max_{x\in[0,h]} |\varphi(x)| > 0$. Then from (3.8), we obtain

$$|u(x,t)| \le \frac{3M_0}{2} \equiv M_1, \quad (x,t) \in [0,h]\times[0,t_1]. \tag{3.15}$$

In addition, from (3.9) we conclude that

$$v(x,t) \ge \frac{\varphi'(x)}{2} \ge \frac{\min_{x\in[0,h]}\varphi'(x)}{2} \equiv M_2 > 0, \quad (x,t) \in [0,h]\times[0,t_1] \tag{3.16}$$

and from (3.11),

$$a(t) \le \frac{\max_{[0,T]}\mu_{3,0}(t)}{M_2} \equiv A_1, \quad t \in [0,t_1]. \tag{3.17}$$

Denote $V(t) = \max_{x\in[0,h]} v(x,t)$. Taking into account (3.15), from (3.12) we obtain

$$|b(t)| \le C_7 + C_8 a(t) t^\beta V(t), \quad t \in [0,t_1]. \tag{3.18}$$

Taking into account (3.15), (3.18), from (3.9) we obtain the following inequality for $V = V(t)$,

$$\begin{aligned} V(t) \le C_9 + C_{10}\int_0^t \frac{1}{\sqrt{\theta(t)-\theta(\tau)}}d\tau + C_{11}\int_0^t \frac{V(\tau)}{\sqrt{\theta(t)-\theta(\tau)}}d\tau \\ + C_{12}\int_0^t \frac{a(\tau)\tau^\beta V^2(\tau)}{\sqrt{\theta(t)-\theta(\tau)}}d\tau, \quad t\in[0,t_1]. \end{aligned} \tag{3.19}$$

On the other hand, from (3.11) we find that

$$a(t) \ge \frac{\mu_3(t)}{t^\beta V(t)}, \quad t\in[0,T], \tag{3.20}$$

or

$$\frac{1}{a(t)} \le \frac{t^\beta V(t)}{\mu_3(t)}.$$

Using this inequality, from (3.19) we obtain

$$V_1(t) \le C_{13} + C_{14}\int_0^t \frac{a(\tau)\tau^\beta V_1^2(\tau)}{\mu_3(\tau)\sqrt{\theta(t)-\theta(\tau)}}d\tau, \quad t\in[0,t_1], \tag{3.21}$$

where $V_1(t) = V(t) + \frac{1}{2}$.

Applying the Cauchy and Cauchy-Buniakowski inequalities to the squared inequality (3.21), we conclude

$$V_1^2(t) \le 2C_{13}^2 + 2C_{14}^2\int_0^t \frac{V_1^4(\tau)d\tau}{\sqrt{\theta(t)-\theta(\tau)}}\int_0^t \frac{a^2(\tau)\tau^{2\beta}d\tau}{\mu_3^2(\tau)\sqrt{\theta(t)-\theta(\tau)}}. \tag{3.22}$$

Let us consider the integral

$$J_1 = \int_0^t \frac{a^2(\tau)\tau^{2\beta}d\tau}{\mu_3^2(\tau)\sqrt{\theta(t)-\theta(\tau)}}.$$

Using (3.17) and the definition of the function $\theta = \theta(t)$, we find that

$$\theta(t) \le \frac{A_1}{(1+\beta)}t^{1+\beta}.$$

Then

$$t \ge (\theta(t))^{\frac{1}{1+\beta}}\Big(\frac{(1+\beta)}{A_1}\Big)^{\frac{1}{1+\beta}}.$$

Taking into account (3.17) and the conditions of Theorem 3.1, we obtain

$$J_1 \le C_{15}\int_0^t \frac{a(\tau)\tau^\beta d\tau}{\tau^\beta\sqrt{\theta(t)-\theta(\tau)}} \le C_{16}\int_0^t \frac{a(\tau)\tau^\beta d\tau}{\theta(\tau)^{\frac{\beta}{1+\beta}}\sqrt{\theta(t)-\theta(\tau)}}.$$

Making the substitution $z = \theta(\tau)/\theta(t)$, we obtain

$$J_1 \le C_{17}(\theta(t))^{\frac{1-\beta}{2(1+\beta)}}\int_0^1 \frac{dz}{z^{\frac{\beta}{1+\beta}}\sqrt{1-z}} \le C_{18}. \tag{3.23}$$

Using (3.23) in (3.22), we derive the inequality

$$V_1^2(t) \le C_{19} + C_{20}\int_0^t \frac{V_1^4(\tau)d\tau}{\sqrt{\theta(t)-\theta(\tau)}}.$$

We replace t by σ , then multiply it by $\frac{a(\sigma)\sigma^\beta}{\mu_3(\sigma)\sqrt{\theta(t)-\theta(\sigma)}}$, and then integrate with respect to σ from 0 to t. Taking into account (3.23) and

$$\int_\tau^t \frac{a(\sigma)\sigma^\beta d\sigma}{\sqrt{(\theta(t)-\theta(\sigma))(\theta(\sigma)-\theta(\tau))}} = \pi,$$

we deduce that

$$\int_0^t \frac{a(\sigma)\sigma^\beta V_1^2(\sigma)}{\mu_3(\sigma)\sqrt{\theta(t)-\theta(\sigma)}} d\sigma \le C_{21} + C_{22}\int_0^t \frac{V_1^4(\tau)}{\tau^\beta} d\tau.$$

Using this inequality in (3.21), we derive the inequality

$$V_1(t) \le C_{23} + C_{24}\int_0^t \frac{V_1^4(\tau)}{\tau^\beta} d\tau.$$

Denote $H(t) = C_{23} + C_{24}\int_0^t \frac{V_1^4(\tau)}{\tau^\beta} d\tau$. Differentiating, we find

$$H'(t) = C_{24}\frac{V_1^4(t)}{t^\beta} \le C_{24}\frac{H^4(t)}{t^\beta}.$$

We integrate to obtain

$$H(t) \le \frac{C_{23}\sqrt[3]{(1-\beta)}}{\sqrt[3]{1-\beta-3C_{23}^3C_{24}t^{1-\beta}}} \le M_3, \quad t\in[0,t_2],$$

where the number t_2, $0<t_2<T$ satisfies

$$1-\beta-3C_{23}^3C_{24}t_2^{1-\beta} > 0. \tag{3.24}$$

This implies

$$v(x,t) \le M_3, \quad (x,t)\in[0,h]\times[0,t_2], \tag{3.25}$$

where the constant M_3 depends on the data.

Using (3.25), from (3.18), (3.20) we obtain the estimates

$$a(t) \ge A_2 > 0, \quad t\in[0,t_2], \tag{3.26}$$

$$|b(t)| \le M_4, \quad t\in[0,t_2], \tag{3.27}$$

where A_2 and M_4 are the known constants. Thus, an a priori estimates of the solutions to system (3.8), (3.9), (3.11), (3.12) are established.

Put $T_0 = \min\{t_1,t_2\}$. Denote $\omega = (u,v,a,b)$, $\mathcal{N} = \{(u,v,a,b) \in (C(\overline{Q}_{T_0}))^2 \times (C[0,T_0])^2 : |u(x,t)| \le M_1, M_2 \le v(x,t) \le M_3, A_2 \le a(t) \le A_1, |b(t)| \le M_4\}$. We rewrite system (3.8), (3.9), (3.11), (3.12) as the operator equation

$$\omega = P\omega,$$

where the operator P is defined by the right-hand sides of these equations. The obtained estimates (3.15), (3.16), (3.17), (3.25), (3.26), (3.27) of the functions (u,v,a,b) guarantee that P maps $\mathcal{N}$ into $\mathcal{N}$. The compactness of operator P can be established as in non degenerate case [7, p. 20]. Now the Schauder fixed-point theorem yields the existence of the continuous solution to system (3.8), (3.9), (3.11), (3.12) on $[0,h]\times[0,T_0]$. This means that exists the solution (a,b,u) to the inverse problem (2.1)-(2.5). This completes the proof of Theorem 3.1. $\square$

4. Uniqueness of a solution

Theorem 4.1. *Assume that the following conditions hold:*

(B1) $c, f \in C^{1,0}([0,h] \times [0,T])$, $\varphi \in C^3[0,h]$;

(B2) $\mu_{3,0}(t) \neq 0$, $\mu_2(t) - \mu_1(t) \neq 0$, $t \in [0,T]$

Then the solution to (2.1)-(2.5) *is unique.*

Proof. Suppose that (2.1)-(2.5) has two solutions (a_i, b_i, u_i), $i = 1, 2$. Denote $a(t) = a_1(t) - a_2(t)$, $b(t) = b_1(t) - b_2(t)$, $u(x,t) = u_1(x,t) - u_2(x,t)$. From (2.1)-(2.5), we obtain

$$u_t = a_1(t)t^\beta u_{xx} + b_1(t)u_x + c(x,t)u + a(t)t^\beta u_{2xx} + b(t)u_{2x}, \quad (x,t) \in Q_T, \tag{4.1}$$

$$u(x,0) = 0, \quad x \in [0,h], \tag{4.2}$$

$$u(0,t) = u(h,t) = 0, \quad t \in [0,T], \tag{4.3}$$

$$a_1(t)t^\beta u_x(0,t) + a(t)t^\beta u_{2x}(0,t) = 0, \quad t \in [0,T], \tag{4.4}$$

$$\int_0^h u(x,t)dx = 0, \quad t \in [0,T]. \tag{4.5}$$

With the aid of the Green function $G^*(x,t,\xi,\tau)$ of the first boundary-value problem for the equation

$$u_t = a_1(t)t^\beta u_{xx} + b_1(t)u_x + c(x,t)u$$

we represent the solution to (4.1)-(4.3) in the form

$$u(x,t) = \int_0^t \int_0^h G^*(x,t,\xi,\tau)(a(\tau)\tau^\beta u_{2\xi\xi}(\xi,\tau) + b(\tau)u_{2\xi}(\xi,\tau))d\xi\, d\tau, \tag{4.6}$$

$(x,t) \in \overline{Q}_T$. Differentiating (4.6) with respect to x, we find

$$u_x(x,t) = \int_0^t \int_0^h G_x^*(x,t,\xi,\tau)(a(\tau)\tau^\beta u_{2\xi\xi}(\xi,\tau) + b(\tau)u_{2\xi}(\xi,\tau))d\xi\, d\tau, \tag{4.7}$$

$(x,t) \in \overline{Q}_T$.

From (4.4), we deduce

$$a(t) = -\frac{a_1(t)}{u_{2x}(0,t)}u_x(0,t), \quad t \in [0,T]. \tag{4.8}$$

Note that the condition $u_{2x}(0,t) \neq 0$, $t \in [0,T]$ is satisfied because of the assumption $\mu_{3,0}(t) \neq 0$, $t \in [0,T]$ in Theorem 4.1.

Differentiating (4.5) with respect to t and using (4.1)-(4.4), we obtain

$$b(t) = -\frac{1}{\mu_2(t) - \mu_1(t)}\Big(a_1(t)t^\beta u_x(h,t) + a(t)t^\beta u_{2x}(h,t) + \int_0^h c(x,t)u(x,t)dx\Big), \tag{4.9}$$

for $t \in [0,T]$. Using (4.6), (4.7), we reduce (4.8), (4.9) to the system of homogeneous integral Volterra equations of the second kind

$$a(t) = \int_0^t (K_{11}(t,\tau)a(\tau) + K_{12}(t,\tau)b(\tau))d\tau, \tag{4.10}$$

$$b(t) = \int_0^t (K_{21}(t,\tau)a(\tau) + K_{22}(t,\tau)b(\tau))d\tau, \tag{4.11}$$

where

$$K_{11}(t,\tau) = -\frac{a_1(t)}{u_{2x}(0,t)} \int_0^h G^*_x(0,t,\xi,\tau)\tau^\beta u_{2\xi\xi}(\xi,\tau)d\xi,$$
$$K_{12}(t,\tau) = -\frac{a_1(t)}{u_{2x}(0,t)} \int_0^h G^*_x(0,t,\xi,\tau)u_{2\xi}(\xi,\tau)d\xi,$$
$$\begin{aligned} K_{21}(t,\tau) = -\frac{1}{\mu_2(t)-\mu_2(\tau)}\Big(& a_1(t)t^\beta \int_0^h G^*_x(h,t,\xi,\tau)\tau^\beta u_{2\xi\xi}(\xi,\tau)d\xi \\ & + \int_0^h\int_0^h G^*(x,t,\xi,\tau)c(x,t)\tau^\beta u_{2\xi\xi}(\xi,\tau)d\xi dx \\ & - \frac{t^\beta a_1(t)u_{2x}(h,t)}{u_{2x}(0,t)} \int_0^h G^*_x(0,t,\xi,\tau)\tau^\beta u_{2\xi\xi}(\xi,\tau)d\xi\Big), \end{aligned}$$
$$\begin{aligned} K_{22}(t,\tau) = -\frac{1}{\mu_2(t)-\mu_2(\tau)}\Big(& a_1(t)t^\beta \int_0^h G^*_x(h,t,\xi,\tau)u_{2\xi}(\xi,\tau)d\xi \\ & + \int_0^h\int_0^h G^*(x,t,\xi,\tau)c(x,t)u_{2\xi}(\xi,\tau)d\xi dx \\ & - \frac{t^\beta a_1(t)u_{2x}(h,t)}{u_{2x}(0,t)} \int_0^h G^*_x(0,t,\xi,\tau)u_{2\xi}(\xi,\tau)d\xi\Big). \end{aligned}$$

To show that the kernels K_{11}, K_{21} have an integrable singularity, we estimate the function $u_{2xx}(x,t)$. To this end, we consider the corresponding problem (2.1)-(2.3). For the function $u_2(x,t), u_{2x}(x,t)$ the equalities analogous to (3.8), (3.9) respectively take place. We denote by $G_1^{(2)} = G_1^{(2)}(x,t,\xi,\tau)$ the Green function of the first value-boundary problem for the equation

$$u_t = a_2(t)t^\beta u_{xx}.$$

This function defines by the formula analogous to (3.7) with $\theta^{(2)}(t) = \int_0^t a_2(\tau)\tau^\beta d\tau$. Differentiating the corresponding formula (3.8) twice with respect to x and using the relationship $G^{(2)}_{1xx}(x,t,\xi,\tau) = G^{(2)}_{1\xi\xi}(x,t,\xi,\tau)$ we arrive to the equation

$$\begin{aligned} u_{2xx}(x,t) = \varphi''(x) + \int_0^t & G^{(2)}_{1\xi}(x,t,h,\tau)\Big(b(\tau)u_{2\xi}(h,\tau) + c(h,\tau)u_2(h,\tau) \\ & + f(h,\tau) - \mu_2'(\tau) + a_2(\tau)\tau^\beta\varphi''(h)\Big)d\tau \\ & - \int_0^t G^{(2)}_{1\xi}(x,t,0,\tau)\Big(b(\tau)u_{2\xi}(0,\tau) + c(0,\tau)u_2(0,\tau) \\ & + f(0,\tau) - \mu_1'(\tau) + a_2(\tau)\tau^\beta\varphi''(0)\Big)d\tau \\ & - \int_0^t\int_0^h G^{(2)}_{1\xi}(x,t,\xi,\tau)\Big(c(\xi,\tau)u_{2\xi}(\xi,\tau) + c_\xi(\xi,\tau)u_2(\xi,\tau) \\ & + f_\xi(\xi,\tau) - \frac{1}{h}(\mu_2'(\tau)-\mu_1'(\tau)) + a_2(\tau)\tau^\beta\varphi'''(\xi)\Big)d\xi\, d\tau \\ & - \int_0^t\int_0^h G^{(2)}_{1\xi}(x,t,\xi,\tau)b(\tau)u_{2\xi\xi}(\xi,\tau)d\xi\, d\tau, \quad (x,t)\in\overline{Q}_T. \end{aligned} \tag{4.12}$$

Taking into account (3.7), we have

$$\begin{aligned}J &\equiv \int_0^t |G_{1\xi}^{(2)}(x,t,0,\tau)|d\tau \\ &= \frac{1}{2\sqrt{\pi}}\int_0^t \frac{1}{(\theta^{(2)}(t)-\theta^{(2)}(\tau))^{3/2}}\sum_{n=-\infty}^{+\infty}|x+2nh|\exp\Big(-\frac{(x+2nh)^2}{4(\theta^{(2)}(t)-\theta^{(2)}(\tau))}\Big)d\tau.\end{aligned}$$

Using the definition of $\theta^{(2)}=\theta^{(2)}(t)$ and substituting $z=1-\frac{\tau}{t}$, we obtain

$$J \le C_{25}t^{-\frac{3\beta+1}{2}}\int_0^1 z^{-\frac{3}{2}}\sum_{n=-\infty}^{+\infty}|x+2nh|\exp\Big(-\frac{C_{26}(x+2nh)^2}{t^{1+\beta}z}\Big)dz.$$

Let us change the variable $\sigma=\sqrt{\frac{C_{26}}{t^{1+\beta}z}}(x+2nh)$. We get as a result

$$J \le \frac{C_{27}}{t^\beta}\int_{-\infty}^{+\infty} e^{-\sigma^2}d\sigma \le \frac{C_{28}}{t^\beta}.$$

We evaluate the rest integrals which the formula (4.12) contains by a similar way. So from (4.12) we deduce

$$|u_{2xx}(x,t)| \le \frac{C_{29}}{t^\beta}. \tag{4.13}$$

This implies that the kernels of system (4.10), (4.11) have the integrable singularities and respectively this system have only trivial solution

$$a(t)\equiv 0, \quad b(t)\equiv 0, \quad t\in[0,T].$$

Using this fact in the problem (4.1)-(4.3) we obtain that

$$u(x,t)\equiv 0,\ (x,t)\in \overline{Q}_T.$$

The proof is complete. □

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