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EXISTENCE OF SOLUTIONS FOR THREE-POINT BVPS ARISING IN BRIDGE DESIGN

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Abstract. This article deals with a class of three-point nonlinear boundary-value problems (BVPs) with Neumann type boundary conditions which arises in bridge design. The source term (nonlinear term) depends on the derivative of the solution, it is also Lipschitz continuous. We use monotone iterative technique in the presence of upper and lower solutions for both well order and reverse order case. Under some sufficient conditions we prove existence results. We also construct two examples to validate our results. These result can be used to generate a user friendly package in Mathematica or MATLAB so that solutions of nonlinear boundary-value problems can be computed.

1. Introduction

In the past few years, there has been much attention focused on multipoint BVPs for nonlinear ordinary differential equations, see [1, 4, 6, 9, 16, 17]. Multipoint BVPs have lots of applications in modern science and engineering.

It is observed [7] that a linear model is insufficient to explain the large oscillatory behavior in suspension bridges also suspension bridges have other nonlinear behaviors such as traveling waves. If the roadbed of a suspension bridge is treated as a one-dimensional vibrating beam the following equation is derived (see [7, Section 3])

$$u_{tt} + EIu_{xxxx} + \delta u_t = -ku^+ + W(x) + \varepsilon f(x,t), \tag{1.1}$$

$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0. \tag{1.2}$$

Thus the suspension bridge is seen as a beam of length L, with hinged ends, whose downward deflection is measured by $u(x,t)$, with a small viscous damping term, subject to three separate forces; the stays, holding it up as nonlinear springs with spring constant k, the weight per unit length of the bridge $W(x)$ pushing it down, and the external forcing term $\varepsilon f(x,t)$. The loading $W(x)$ would usually be constant.

If W is replaced by the term $W(x) = W_0 \sin(x/L)$, an error of magnitude around 10% is introduced in the loading and little less in the steady-state deflection. Second, if the forcing term is given by $f(x,t) = f(t)\sin(x/L)$ and general solutions of (1.1)-(1.2), is of the form $u(x,t) = y(t)\sin(x/L)$. These no-nodal solutions were the

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most commonly observed type for low velocities on the Tacoma Narrows Bridge. When this $u(x,t)$ is substituted into (1.1), this results into the differential equation

$$-y''(t) = f(t, y, y') \tag{1.3}$$

were $f(t,y,y') = \delta y' + EI(\pi/L)^4 y + ky^+ - W_0 - \varepsilon f(t)$, where y^+ denotes y if y is non-negative, and zero if y is negative.

Large size bridges are sometimes contrived with multi-point supports, which gives rise to multi-point boundary condition.

In this paper we focus on monotone iterative technique related to upper and lower solutions. Lot of aspects of this technique has been explored for two point BVPs. When upper β_0 and lower α_0 solution are well ordered; i.e., $\beta_0 \geq \alpha_0$ lots of results are available (see [10]-[11] and the references there in). When the upper and lower solutions are in reverse order, that is $\beta_0 \leq \alpha_0$ some results are available (see [3, 8, 14, 15] and the references there in).

As far as three-point BVPs are concerned, different techniques are used to prove existence results. The case when source function is independent from first derivative results are available. But there are very few results when the first order derivative is involved explicitly in the nonlinear term f.

Guo et al [5] consider the three point BVPs

$$y''(t) + f(t, y, y') = 0, \quad 0 < t < 1, \tag{1.4}$$

$$y(0) = 0, \quad y(1) = \delta y(\eta) \tag{1.5}$$

where f is a nonnegative continuous function, $\delta > 0$, $\eta \in (0,1)$ and $\alpha\eta < 1$. They used a new fixed point theorem in a cone. Bao et al [2] proved some existence results for three-point BVPs (1.4)-(1.5). They used fixed point index method under a non-well-ordered upper and lower solution condition.

Recently Singh et al [12, 13] used monotone iterative method with upper and lower solutions when they are well ordered as well as reverse ordered, and proved some existence results for a class of three point non-linear BVPs.

The aim of this article is to explore the monotone iterative method with upper and lower solutions (for both well order as well as reverse order case) for the Neumann type nonlinear three point BVPs given by

$$y''(t) + f(t, y, y') = 0, \quad 0 < t < 1, \tag{1.6}$$

$$y'(0) = 0, \quad y'(1) = \delta y'(\eta), \tag{1.7}$$

where $f \in C(I \times R, R)$, $I = [0,1]$, $0 < \eta < 1$, $0 < \delta < 1$.

This article is organized in six sections. In Section 2 we discuss the corresponding linear case of nonlinear three point BVPs and construct Green's function. In Sections 3 and 4 we discuss some other important Lemmas and maximum and anti-maximum principles. In Section 5 we derive sufficient conditions which guarantee the existence of solutions of nonsingular nonlinear three point BVPs for both case; i.e., when upper and lower solutions are well ordered and also when reverse ordered. Finally in Section 6 two examples are constructed to validate our results.

2. Linear Case and Green's Function

This section deals with linear three-point BVPs with Neumann type conditions. We construct Green's function for these problems to be investigated in later sections.

2.1. **Construction of the Green's function.** Consider the corresponding linear three-point BVPs as given by

$$Ly \equiv -y''(t) - \lambda y(t) = h(t), \quad 0 < t < 1, \tag{2.1}$$

$$y'(0) = 0, \quad y'(1) = \delta y'(\eta) + b, \tag{2.2}$$

where $h \in C(I)$ and b is any constant. Based on the sign of λ we can divide the construction of Green's function into two cases. In one case $\lambda > 0$ we get Green's function in terms of cos and sin. In the case when $\lambda < 0$ we get Green's function in terms of cosh and sinh.

Lemma 2.1. *Assume*

(H0) $\lambda \in (0, \pi^2/4)$, $\sin(\sqrt{\lambda}) - \delta \sin(\eta\sqrt{\lambda}) > 0$ *and* $\cos(\sqrt{\lambda}) - \delta\cos(\eta\sqrt{\lambda}) \geq 0$.

When $\lambda > 0$, *Green's function for the three-point BVPs*

$$y''(t) + \lambda y(t) = 0, \quad 0 < t < 1, \tag{2.3}$$

$$y'(0) = 0, \quad y'(1) = \delta y'(\eta), \tag{2.4}$$

is

$$G(t,s) = \begin{cases} \frac{\cos(\sqrt{\lambda}t)(\cos(\sqrt{\lambda}(s-1)) - \delta\cos(\sqrt{\lambda}(s-\eta)))}{\sqrt{\lambda}(\sin(\sqrt{\lambda}) - \delta\sin(\eta\sqrt{\lambda}))}, & 0 \leq t \leq s \leq \eta, \\ \frac{\cos(\sqrt{\lambda}s)(\cos(\sqrt{\lambda}(t-1)) - \delta\cos(\sqrt{\lambda}(t-\eta)))}{\sqrt{\lambda}(\sin(\sqrt{\lambda}) - \delta\sin(\eta\sqrt{\lambda}))}, & s \leq t,\ s \leq \eta, \\ \frac{\cos(\sqrt{\lambda}(s-1))\cos(\sqrt{\lambda}t)}{\sqrt{\lambda}(\sin(\sqrt{\lambda}) - \delta\sin(\eta\sqrt{\lambda}))}, & t \leq s,\ \eta \leq s, \\ \frac{\delta\sin(\eta\sqrt{\lambda})\sin(\sqrt{\lambda}(s-t)) + \cos(\sqrt{\lambda}s)\cos(\sqrt{\lambda}(1-t))}{\sqrt{\lambda}(\sin(\sqrt{\lambda}) - \delta\sin(\eta\sqrt{\lambda}))}, & \eta \leq s \leq t \leq 1. \end{cases}$$

It can be checked that (H0) can be satisfied for some sub interval of $\lambda \in (0, \pi^2/4)$.

For a proof of the above lemma, see [12, Lemma 2.1].

Lemma 2.2. *Let* $\lambda > 0$. *If* $y \in C^2(I)$ *is the solution of the three-point BVPs* (2.1) *and* (2.2), *then it can be expressed as*

$$y(t) = \frac{b\cos\sqrt{\lambda}t}{\sqrt{\lambda}(\delta\sin\sqrt{\lambda}\eta - \sin\sqrt{\lambda})} - \int_0^1 G(t,s)h(s)ds. \tag{2.5}$$

For a proof of the above lemma, see [12, Lemma 2.2].

Lemma 2.3. *Assume*

(H0′) $\lambda < 0$, $\sinh\sqrt{|\lambda|} - \delta\sinh\sqrt{|\lambda|}\eta\ > 0$ *and* $\delta\cosh\sqrt{|\lambda|}\eta - \cosh\sqrt{|\lambda|} \leq 0$.

Then for $\lambda < 0$, *the Green's function of the three-point BVPs*

$$y''(t) + \lambda y(t) = 0, \quad 0 < t < 1,$$

$$y'(0) = 0, \quad y'(1) = \delta y'(\eta)$$

is

$$G(t,s) = \begin{cases} \frac{\cosh(\sqrt{|\lambda|}t)(\delta\cosh(\sqrt{|\lambda|}(s-\eta)) - \cosh(\sqrt{|\lambda|}(s-1)))}{\sqrt{|\lambda|}(\sinh(\sqrt{|\lambda|}) - \delta\sinh(\eta\sqrt{|\lambda|}))}, & 0 \leq t \leq s \leq \eta, \\ \frac{\cosh(\sqrt{|\lambda|}s)(\delta\cosh(\sqrt{|\lambda|}(t-\eta)) - \cosh(\sqrt{|\lambda|}(t-1)))}{\sqrt{|\lambda|}(\sinh(\sqrt{|\lambda|}) - \delta\sinh(\eta\sqrt{|\lambda|}))}, & s \leq t,\ s \leq \eta, \\ -\frac{\cosh(\sqrt{|\lambda|}(s-1))\cosh(\sqrt{|\lambda|}t)}{\sqrt{|\lambda|}(\sinh(\sqrt{|\lambda|}) - \delta\sinh(\eta\sqrt{|\lambda|}))}, & t \leq s,\ \eta \leq s, \\ \frac{\delta\sinh(\eta\sqrt{|\lambda|})\sinh(\sqrt{|\lambda|}(s-t)) - \cosh(\sqrt{|\lambda|}s)\cosh(\sqrt{|\lambda|}(1-t))}{\sqrt{|\lambda|}(\sinh(\sqrt{|\lambda|}) - \delta\sinh(\eta\sqrt{|\lambda|}))}, & \eta \leq s \leq t \leq 1. \end{cases}$$

It can be easily checked that (H0′) can be satisfied for some values of $\lambda \in (-\infty, 0)$. For a proof of the above lemma, see [12, Lemma 2.3].

Lemma 2.4. *Let $\lambda < 0$. If $y \in C^2(I)$ is a solution of the three-point BVPs (2.1) and (2.2), then it is given by*

$$y(t) = \frac{b\cosh\sqrt{|\lambda|}t}{\sqrt{|\lambda|}(\sinh(\sqrt{|\lambda|}) - \delta\sinh(\eta\sqrt{|\lambda|}))} - \int_0^1 G(t,s)h(s)ds. \tag{2.6}$$

For a proof of the above lemma, see [12, Lemma 2.4].

3. Some inequalities

Lemma 3.1. *Let $\lambda \in (0, \pi^2/4)$ and $\lambda - M \geq 0$. Further if*

$$(\lambda - M)\cos\sqrt{\lambda} - N\sqrt{\lambda}\sin\sqrt{\lambda} \geq 0,$$

then for all $t \in [0,1]$

$$(\lambda - M)\cos\sqrt{\lambda}t - N\sqrt{\lambda}\sin\sqrt{\lambda}t \geq 0,$$

where $M, N \in \mathbb{R}^+$.

Proof. Using monotonicity of sin and cos, we derive that for all $t \in [0,1]$,

$$(\lambda - M)\cos\sqrt{\lambda}t - N\sqrt{\lambda}\sin\sqrt{\lambda}t \geq (\lambda - M)\cos\sqrt{\lambda} - N\sqrt{\lambda}\sin\sqrt{\lambda} \geq 0.$$

Which completes the proof. □

Lemma 3.2. *If $\lambda < 0$ is such that $M + \lambda \leq 0$, and*

$$\lambda \leq -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2 + 4M},$$

then for all $t \in [0,1]$,

$$(M + \lambda)\cosh\sqrt{|\lambda|}t + N\sqrt{|\lambda|}\sinh\sqrt{|\lambda|}t \leq 0,$$

where $M, N \in \mathbb{R}^+$.

Proof. As

$$(M + \lambda)\cosh\sqrt{|\lambda|}t + N\sqrt{|\lambda|}\sinh\sqrt{|\lambda|}t \leq [(M + \lambda) + N\sqrt{|\lambda|}]\cosh\sqrt{|\lambda|}t.$$

We will have $[(M + \lambda) + N\sqrt{|\lambda|}]\cosh\sqrt{|\lambda|}t \leq 0$ for all $t \in [0,1]$ if

$$[(M + \lambda) + N\sqrt{|\lambda|}] \leq 0.$$

The above inequality will be satisfied if

$$\lambda \leq -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2 + 4M}.$$

This completes the proof. □

Lemma 3.3. *Let* (H0) *be satisfied. Then*

(i) $G(t,s) \geq 0$,
(ii) $\frac{\partial G(t,s)}{\partial t} \leq 0$ *and*
(iii) $(\lambda - M)G(t,s) + N(\operatorname{sign} y')\frac{\partial G(t,s)}{\partial t} \geq 0$

for any $t, s \in [0,1]$ and $M, N \in \mathbb{R}^+$.

Proof. The conditions assumed in (H0) ensure that $G(t,s) \geq 0$. Since $G(t,s)$ is the solution of (2.3)–(2.4), we deduce that $\frac{\partial G(t,s)}{\partial t} \leq 0$. For part (iii) we prove that for all $t, s \in [0,1]$,

$$(\lambda - M)G + N\frac{\partial G(t,s)}{\partial t} \geq 0.$$

Which can easily be deduced by using Lemmas 2.1, 2.2 and 3.1. □

Lemma 3.4. *Assume* (H0′). *Then for any* $t, s \in [0,1]$, *we have*

(i) $G(t,s) \leq 0$,
(ii) $\frac{\partial G(t,s)}{\partial t} \leq 0$ *and*
(iii) $(M+\lambda)G(t,s) + N(\operatorname{sign} y')\frac{\partial G(t,s)}{\partial t} \geq 0$ *whenever we have* $M + \lambda - N\lambda \leq 0$

where $M, N \in \mathbb{R}^+$.

Proof. Parts (i) and (ii) follow the analysis of Lemma 3.3. For part (iii), it will be sufficient to prove that for all $t, s \in [0,1]$,

$$(M+\lambda)G(t,s) + N\frac{\partial G(t,s)}{\partial t} \geq 0.$$

Since $G(t,s)$ is the the Green function for (2.1)–(2.2), we have

$$\frac{\partial G(t,s)}{\partial t} \geq -\lambda G(t,s).$$

The above inequality along with condition $M + \lambda - N\lambda \leq 0$ gives

$$(M+\lambda)G(t,s) + N\frac{\partial G(t,s)}{\partial t} \geq (M+\lambda - N\lambda)G(t,s) \geq 0.$$

□

4. Anti-maximum and maximum principle

Proposition 4.1. *Let* $b \geq 0$, $h(t) \in C[0,1]$ *be such that* $h(t) \geq 0$, *and* (H0) *hold. Then the solution of* $Ly = h$ *and* (2.2) *is non-positive.*

Proposition 4.2. *Let* $b \geq 0$, $h(t) \in C[0,1]$ *be such that* $h(t) \geq 0$ *and* (H0′) *hold. Then the solution of* $Ly = h$ *and* (2.2) *is non-negative.*

5. Existence results for nonlinear three-point BVP

In this section, we prove two existence results for the nonlinear three-point Neumann type BVPs. On the basis of the order of upper and lower solutions, we divide this section into the following subsections.

5.1. **Reverse ordered case.**

Definition 5.1. The functions $\alpha, \beta \in C^2[0,1]$ are called lower and upper solutions for the class of three-point BVPs (1.6)-(1.7) if they satisfy the following inequalities:

$$\begin{gathered}
-\alpha''(t) \leq f(t, \alpha, \alpha'), \quad 0 < t < 1, \\
\alpha'(0) = 0, \quad \alpha'(1) \leq \delta\alpha'(\eta). \\
-\beta''(t) \geq f(t, \beta, \beta'), \quad 0 < t < 1, \\
\beta'(0) = 0, \quad \beta'(1) \geq \delta\beta'(\eta).
\end{gathered}$$

The sequences $(\alpha_n)_n$ and $(\beta_n)_n$ are defined by the following iterative schemes

$$\alpha_0 = \alpha, \quad -\alpha''_{n+1}(t) - \lambda\alpha_{n+1}(t) = f(t, \alpha_n, \alpha'_n) - \lambda\alpha_n, \tag{5.1}$$

$$\alpha'_{n+1}(0) = 0, \quad \alpha'_{n+1}(1) = \delta\alpha'_{n+1}(\eta), \tag{5.2}$$

$$\beta_0 = \beta, \quad -\beta''_{n+1}(t) - \lambda\beta_{n+1}(t) = f(t, \beta_n, \beta'_n) - \lambda\beta_n, \tag{5.3}$$

$$\beta'_{n+1}(0) = 0, \quad \beta'_{n+1}(1) = \delta\beta'_{n+1}(\eta), \tag{5.4}$$

where $\lambda \neq 0$.

Theorem 5.2. *Assume that* (H0)*and following hypothesis hold:*

- (H1) *there exist α and $\beta \in C^2[0,1]$ lower and upper solutions of* (1.6) *and* (1.7) *such that for all $t \in [0,1]$, $\alpha \geq \beta$;*
- (H2) *the function $f : D \to R$ is continuous on $D := \{(t,u,v) \in [0,1] \times \mathbb{R}^2 : \beta(t) \leq u \leq \alpha(t)\}$;*
- (H3) *there exist $M \geq 0$ such that for all $(t, u_1, v), (t, u_2, v) \in D$*

$$u_1 \leq u_2 \to f(t, u_2, v) - f(t, u_1, v) \leq M(u_2 - u_1);$$

- (H4) *there exist $N \geq 0$ such that for all $(t, u, v_1), (t, u, v_2) \in D$*

$$|f(t, u, v_2) - f(t, u, v_1)| \leq N|v_2 - v_1|;$$

- (H5) *for all $(t,u,v) \in D$, $|f(t,u,v)| \leq \varphi(|v|)$, such that $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and satisfies*

$$\max_{t\in[0,1]} \alpha - \min_{t\in[0,1]} \beta \leq \int_{l_0}^{\infty} \frac{s\ ds}{\varphi(s)}$$

where $l_0 = \sup_{[0,1]}[2|\alpha(t)|]$.

Further if $(\lambda - M)\cos\sqrt{\lambda} - N\sqrt{\lambda}\sin\sqrt{\lambda} \geq 0$ and for all $t \in [0,1]$

$$f(t, \beta(t), \beta'(t)) - f(t, \alpha(t), \alpha'(t)) - \lambda(\beta - \alpha) \geq 0$$

then the sequences (α_n) and (β_n) defined by (5.1)*,* (5.2) *and* (5.3)*,* (5.4) *converge monotonically in $C^1([0,1])$ to the solutions v and u of the nonlinear boundary-value problem* (1.6) *and* (1.7)*, such that for all $t \in [0,1]$*

$$\beta \leq u \leq v \leq \alpha.$$

The proof of the above theorem can be divided into several small results stated as follows.

Lemma 5.3. *If α_n is a lower solution of* (1.6) *and* (1.7)*, α_{n+1} is defined by* (5.1) *and* (5.2) *where $\lambda \in (0, \pi^2/4)$, then $\alpha_{n+1} \leq \alpha_n$.*

Proof. Since $y(t) = \alpha_{n+1} - \alpha_n$ satisfies $Ly \geq 0$, (2.2) with $b \geq 0$, the result can be concluded by Proposition 4.1. □

Proposition 5.4. *Let* (H0)–(H4) *hold and $(\lambda - M)\cos\sqrt{\lambda} - N\sqrt{\lambda}\sin\sqrt{\lambda} \geq 0$ then the function α_n defined by* (5.1) *and* (5.2) *are such that for all $n \in N$,*

- (i) *α_n is a lower solution of* (1.6)-(1.7)*; and*
- (ii) *$\alpha_{n+1} \leq \alpha_n$.*

Proof. We prove it by recurrence. By Lemma 5.3 the claim holds for $n = 0$.

Let α_{n-1} is a lower solution of (1.6) and (1.7) and $\alpha_n \leq \alpha_{n-1}$. Let $y = \alpha_n - \alpha_{n-1}$. Then we have

$$-\alpha_n'' - f(t, \alpha_n, \alpha_n') \leq (\lambda - M)y + N(\operatorname{sign} y')y'.$$

Let $(\lambda - M)y + N(\operatorname{sign} y')y' = g$. Now to show α_n is a lower solution we have to show that $g \leq 0$. Since y is given by Lemma 2.2 with $h(t) = \alpha_{n-1}'' + f(t, \alpha_{n-1}, \alpha_{n-1}') \geq 0$. Thus to show $g \leq 0$, it is enough to prove that

$$(\lambda - M)\cos\sqrt{\lambda}t - N\sqrt{\lambda}\sin\sqrt{\lambda}t \geq 0,$$

$$(\lambda - M)G(t,s) + N\frac{\partial G(t,s)}{\partial t} \geq 0$$

for all $t \in [0,1]$. Lemma 3.1 and Lemma 3.3 verify the existence of above two inequalities. Thus $\alpha_{n+1} \geq \alpha_n$. □

Similarly we can prove the following result.

Proposition 5.5. *Let* (H0)–(H4) *be true and* $(\lambda - M)\cos\sqrt{\lambda} - N\sqrt{\lambda}\sin\sqrt{\lambda} \geq 0$. *Then the function* β_n *defined by* (5.3)-(5.4) *are such that for all* $n \in N$

(i) β_n *is an upper solution of* (1.6)-(1.7);
(ii) $\beta_{n+1} \geq \beta_n$.

Proposition 5.6. *Let* (H0)–(H4) *hold,* $(\lambda - M)\cos\sqrt{\lambda} - N\sqrt{\lambda}\sin\sqrt{\lambda} \geq 0$ *and for all* $t \in [0,1]$,

$$f(t, \beta(t), \beta'(t)) - f(t, \alpha(t), \alpha'(t)) - \lambda(\beta - \alpha) \geq 0\,.$$

Then for all $n \in N$, *the functions* α_n *and* β_n *defined by* (5.1)-(5.2) *and* (5.3)-(5.4) *satisfy* $\alpha_n \geq \beta_n$.

Proof. We define

$$h_i(t) = f(t, \beta_i, \beta_i') - f(t, \alpha_i, \alpha_i') - \lambda(\beta_i - \alpha_i), \quad \text{for all } i \in N.$$

Now, for all $i \in N$, $y_i := \beta_i - \alpha_i$ satisfies

$$-y_i'' - \lambda y_i = f(t, \beta_{i-1}, \beta_{i-1}') - f(t, \alpha_{i-1}, \alpha_{i-1}') - \lambda(\beta_{i-1} - \alpha_{i-1}) = h_{i-1}.$$

Claim 1. $\alpha_1 \geq \beta_1$. The function $y_1 = \beta_1 - \alpha_1$ is a solution of (2.1)-(2.2) with $h(t) = h_0(t) \geq 0$ and $b = 0$, by Proposition 4.1, $y_1(t) \leq 0$; i.e., $\alpha_1 \geq \beta_1$.

Claim 2. Let $n \geq 2$. If $h_{n-2} \geq 0$ and $\alpha_{n-1} \geq \beta_{n-1}$, then $h_{n-1} \geq 0$ and $\alpha_n \geq \beta_n$. First we will prove that, for all $t \in [0,1]$, the function h_{n-1} is non-negative, as we have

$$h_{n-1} \geq -[(\lambda - M)y_{n-1} + N(\operatorname{sign} y_{n-1}')y_{n-1}'].$$

Since y_{n-1} is a solution of (2.1)-(2.2) with $h(t) = h_{n-2}(t) \geq 0$, $b = 0$. Hence we can proceed similar to the proof of Proposition 5.4 to show that $h_{n-1} \geq 0$. Now $y_n'(0) = 0$ and $y_n'(1) = \delta y'(\eta)$; i.e., $b = 0$, we deduce from Proposition 4.1 that $y_n \leq 0$; i.e., $\alpha_n \geq \beta_n$. □

Lemma 5.7. *If* $f(t, y, y')$ *satisfies* (H5), *then there exists* $R > 0$ *such that any solution of the differential inequality*

$$-y''(t) \geq f(t, y, y'), \quad 0 < t < 1, \tag{5.5}$$

$$y'(0) = 0, \quad y'(1) \geq \delta y'(\eta) \tag{5.6}$$

with $y \in [\beta(t), \alpha(t)]$ *for all* $t \in [0,1]$ *satisfies* $\|y'\|_\infty \leq R$.

Lemma 5.8. *If $f(t,y,y')$ satisfies* (H5)*, then there exists $R>0$ such that any solution of the differential inequality*

$$-y''(t)\le f(t,y,y'),\quad 0<t<1, \tag{5.7}$$

$$y'(0)=0,\quad y'(1)\le \delta y'(\eta) \tag{5.8}$$

with $y\in[\beta(t),\alpha(t)]$ for all $t\in[0,1]$ satisfies $\|y'\|_\infty\le R$.

The proof of above two Lemmas are similar to the proof of [12, Lemma 3.2] (Priory bound). Now we can complete the proof of Theorem 5.2.

Proof of Theorem 5.2. Consider $(\alpha_n)_n$ and $(\beta_n)_n$ defined, respectively by (5.1)-(5.2) and (5.3)-(5.4). By using the Propositions 5.4, 5.5 and 5.6 we deduce the following inequality

$$\alpha=\alpha_0\ge\alpha_1\ge\dots\ge\alpha_n\ge\dots\ge\beta_n\ge\dots\ge\beta_1\ge\beta_0=\beta. \tag{5.9}$$

Since $(\alpha_n)_n$ and $(\beta_n)_n$ are monotonic in nature and bounded, they converge pointwise to functions

$$v(t)=\lim_{n\to\infty}\alpha_n(t)\quad\text{and}\quad u(t)=\lim_{n\to\infty}\beta_n(t)$$

which are such that for all n, $\alpha_n\ge u\ge v\ge\beta_n$. Using (5.1), (5.2) and (5.3), (5.4) along with (5.9) and Lemma 5.8 we verify that $(\alpha_n)_n$ is equibounded and equicontinuous in $C^1([0,1])$. Hence any subsequence $(\alpha_{n_k})_k$ of $(\alpha_n)_n$ is also equibounded and equicontinuous in $C^1([0,1])$ and from Arzela-Ascoli theorem we deduce that there exists a subsequence $(\alpha_{n_{k_j}})_j$ of $(\alpha_{n_k})_k$ which converge in $C^1([0,1])$. By monotonicity and uniqueness of the limit of the sequence $(\alpha_n)_n$, we have $\alpha_n\to v$ in $C^1([0,1])$. As any subsequence of $(\alpha_n)_n$ contains a subsequence $(\alpha_{n_{k_j}})_j$ which converge in $C^1([0,1])$ to v it follows that $\alpha_n\to v$ in $C^1([0,1])$. In a similar way, using Proposition 5.5 and Lemma 5.7 it can be proved that $(\beta_n)_n$ converge to u in $C^1([0,1])$.

The solution of the iterative schemes (5.1)-(5.2) and (5.3)-(5.4) can be written by equations given in Lemmas 2.2 and 2.4 for corresponding $h(t)$ given in terms of nonlinear f. Now by using the uniform convergence of α_n and β_n and taking Limit both sides we get solution of the nonlinear three point BVPs (1.6)-(1.7). This proves existence of solution of the nonlinear three point BVPs (1.6)-(1.7). □

5.2. **Well ordered case.** We state our main result as Theorem 5.9. Proof here again can be written as we did earlier for non well ordered case. We skip the proof for brevity.

Theorem 5.9. *Assume* (H0′) *and the following hypothesis hold:*

- (H1′) *there exist $\alpha,\beta\in C^2[0,1]$ lower and upper solutions of* (1.6)-(1.7) *such that $\alpha\le\beta$;*
- (H2′) *the function $f:\widetilde{D}\to R$ is continuous on $\widetilde{D}:=\{(t,u,v)\in[0,1]\times\mathbb{R}^2: \alpha(t)\le u\le\beta(t)\}$;*
- (H3′) *there exist $M\ge0$ such that for all $(t,u_1,v),(t,u_2,v)\in\widetilde{D}$,*
$$u_1\le u_2\to f(t,u_2,v)-f(t,u_1,v)\ge -M(u_2-u_1);$$
- (H4′) *there exist $N\ge0$ such that for all $(t,u,v_1),(t,u,v_2)\in\widetilde{D}$,*
$$|f(t,u,v_2)-f(t,u,v_1)|\le N|v_2-v_1|;$$

(H5′) *for all* $(t,u,v)\in\widetilde{D}, |f(t,u,v)|\le\varphi(|v|)$; *where* $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ *is continuous and satisfies*
$$\max_{t\in[0,1]}\beta-\min_{t\in[0,1]}\alpha\le\int_{l_0}^{\infty}\frac{s\ ds}{\varphi(s)}$$
where $l_0=\sup_{[0,1]}[2|\beta(t)|]$.

Let $\lambda<0$ *be such that* $\lambda\le\min\{-M,-\frac{M}{1-N},-M-\frac{N^2}{2}-\frac{N}{2}\sqrt{N^2+4M}\}$, *and for all* $t\in[0,1]$,
$$f(t,\beta(t),\beta'(t))-f(t,\alpha(t),\alpha'(t))-\lambda(\beta-\alpha)\ge 0$$
then the sequences (α_n) *and* (β_n) *defined by* (5.1)-(5.2) *and* (5.3)-(5.4) *converges monotonically in* $C^1([0,1])$ *to solution* v *and* u *of* (1.6)-(1.7), *such that for all* $t\in[0,1]$, $\alpha\le v\le u\le\beta$.

6. Numerical illustrations

In this section we consider two examples and verify that conditions derived in this paper can actually be verified and existence of solutions can be proved.

Example 6.1. Consider the nonlinear three-point BVPs given by
$$-y''(t)=\frac{10y^3-9e^{y'}}{90},\quad 0<t<1, \tag{6.1}$$
$$y'(0)=0,\quad y'(1)=0.6y'(0.9). \tag{6.2}$$
where $f(t,y,y')=\frac{10y^3-9e^{y'}}{90}$, $\delta=0.6$, $\eta=0.9$. Here $\alpha=1$ and $\beta=-1$ are lower and upper solutions. It is a non well ordered case.

The priory bound can be computed as follows. $\varphi=(10+9e^{|v|})/90$. $|y'|\le\sqrt{\frac{1}{5}}$; i.e., $R=\sqrt{1/5}$. The Lipschitz constants are computed as $M=1/3$ and $N=e^R/10=0.156395$.

The inequality $f(t,\beta(t),\beta'(t))-f(t,\alpha(t),\alpha'(t))-\lambda(\beta-\alpha)\ge 0$ is satisfied when $\lambda\ge 0.11111$. Now we can find out a subinterval $(0.44,1.8)\subset(0.11111,2.4674)$ (approx) such that $(\lambda-M)\cos\sqrt{\lambda}-N\sqrt{\lambda}\sin\sqrt{\lambda}\ge 0$ and inequalities in (H0) are satisfied (cf. Figure 1).

Thus it is guaranteed that $\exists$ at least one $\lambda\in(0.44,1.8)$ such that sequences generated by iterative scheme converge uniformly to a solution of the nonlinear three point boundary value problem (6.1) and (6.2).

Example 6.2. Consider the nonlinear three-point BVPs
$$-y''(t)=\frac{[y'(t)]^2}{60}-5y(t)-\frac{e^2}{18}, \tag{6.3}$$
$$y'(0)=0,\quad y'(1)=0.7y'(0.5). \tag{6.4}$$
where $f(t,y,y')=\frac{[y'(t)]^2}{60}-5y(t)-\frac{e^2}{18}$, $\delta=7/10$, $\eta=\frac{1}{2}$.

This example has $\alpha=-(t^2+\frac{1}{2})$ and $\beta=(t^2+\frac{1}{2})$ as lower and upper solutions; i.e., we are in well ordered case. The priory bound can be computed as follows. $\varphi=\frac{15}{2}+\frac{e^2}{18}+\frac{|v|^2}{60}$, $|y'|\le 2e^{\frac{3}{32}}$; i.e., $R=3e^{\frac{1}{20}}$. The Lipschitz constants are give by $M=5$ and $N=R/30=0.105127$.

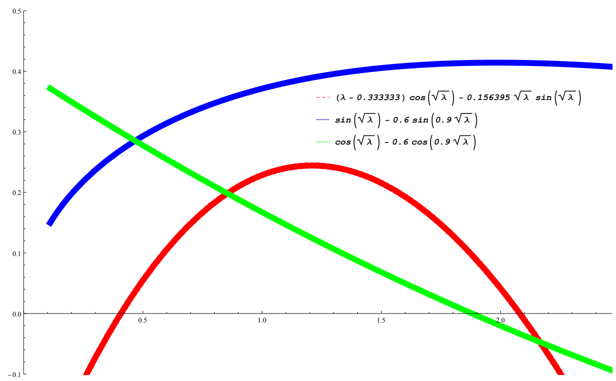

FIGURE 1. Plots of inequalities (H0) and $(\lambda - M)\cos\sqrt{\lambda} - N\sin\sqrt{\lambda} \geq 0$ for $\lambda \in (0, \pi^2/4)$

Now we can find out at least one $\lambda < 0$ such that when

$$\lambda \leq \min\big\{ -M, -\frac{M}{1-N}, -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2+4M}\big\},$$

Assumption (H0$'$) is satisfied (cf. Figure 2), we get two monotonic sequences.

Thus for any $\lambda < -5.58739$ the solutions of the iterative scheme converge uniformly to the solution of the nonlinear three point boundar- value problem (6.3) and (6.4).

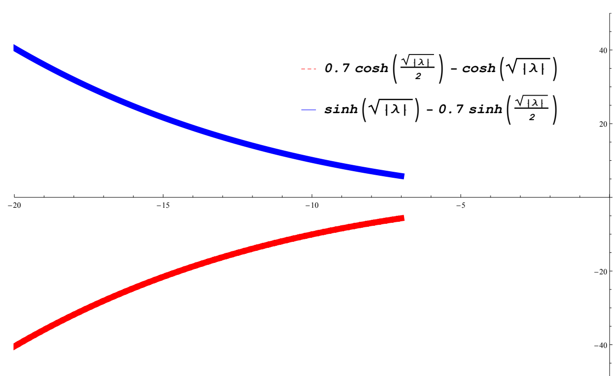

FIGURE 2. Plots of inequalities (H0$'$) for $\lambda < 0$

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