Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 183, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXACT BOUNDARY BEHAVIOR OF SOLUTIONS TO SINGULAR NONLINEAR DIRICHLET PROBLEMS

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ABSTRACT. In this article we analyze the exact boundary behavior of solutions to the singular nonlinear Dirichlet problem

$$-\Delta u = b(x)g(u) + \lambda a(x)f(u), \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0,$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $\lambda > 0$, $g \in C^1((0,\infty),(0,\infty))$, $\lim_{s\to 0^+} g(s) = \infty$, $b, a \in C^{\alpha}_{\rm loc}(\Omega)$, are positive, but may vanish or be singular on the boundary, and $f \in C([0,\infty),[0,\infty))$.

1. Introduction and statement of results

In this article, we consider the boundary behavior of solutions to the singular boundary-value problem

$$-\Delta u = b(x)g(u) + \lambda a(x)f(u), \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0, \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $\lambda > 0$, a, b, and following conditions are satisfied:

- (S1) $b, a \in C^{\alpha}_{\rm loc}(\Omega)$, for some $\alpha \in (0,1)$, are positive in Ω;
- (F1) $f \in C([0,\infty),[0,\infty))$;
- (G1) $g \in C^1((0,\infty),(0,\infty))$, $\lim_{s\to 0^+} g(s) = \infty$;
- (G2) there exists $s_0 > 0$ such that $g'(s) < 0$, for all $s \in (0, s_0)$;
- (G3) there exists $C_g \geq 0$ such that

$$\lim_{s\to 0^+} g'(s)\int_0^s \frac{d\tau}{g(\tau)} = -C_g.$$

For convenience, we denote by ψ the solution to the problem

$$\int_0^{\psi(t)} \frac{ds}{g(s)} = t, \quad \forall t > 0. \tag{1.2}$$

Problem (1.1) arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrical materials; see [9, 13, 15, 26, 32] and the references therein.

2000 *Mathematics Subject Classification.* 35J65, 35B05, 35J25, 60J50.
Key words and phrases. Semilinear elliptic equation; singular Dirichlet problem; positive solution; boundary behavior.

Submitted July 11, 2014. Published August 29, 2014.
Supported by grant 11301301 from the NNSF of China.

First, let us review the results for the problem

$$-\Delta u = b(x)g(u), \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0. \tag{1.3}$$

For $b \equiv 1$ in Ω, g satisfies (G1) and g is decreasing on $(0, \infty)$, Fulks and Maybee [13], Stuart [32], Crandall, Rabinowitz and Tartar [9] showed that problem (1.3) has a unique solution $u_0 \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$. Moreover, [9, Theorems 2.2 and 2.5] established the following result: if $\phi_1 \in C[0, \delta_0] \cap C^2(0, \delta_0]$ ($\delta_0 > 0$) is the local solution of the problem

$$-\phi_1''(t) = g(\phi_1(t)), \quad \phi_1(t) > 0, \quad 0 < t < \delta_0, \quad \phi_1(0) = 0, \tag{1.4}$$

then there exist positive constants c_1 and c_2 such that

$$c_1\phi_1(d(x)) \le u_0(x) \le c_2\phi_1(d(x)) \quad \text{near } \partial\Omega, \tag{1.5}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$, $x \in \Omega$. In particular, when $g(u) = u^{-\gamma}$, $\gamma > 1$, u_0 satisfies

$$c_1(d(x))^{2/(1+\gamma)} \le u_0(x) \le c_2(d(x))^{2/(1+\gamma)} \quad \text{near } \partial\Omega. \tag{1.6}$$

By constructing a pair of subsolution and supersolution on $\bar{\Omega}$, Lazer and McKenna [23] showed that (1.6) still holds on $\bar{\Omega}$ and u_0 has the properties:

(i) if $\gamma > 1$, then u_0 is not in $C^1(\bar{\Omega})$;
(ii) $u_0 \in H_0^1(\Omega)$ if and only if $\gamma < 3$.

The following are some basic results about the exact boundary behaviour of the solution to (1.3). When $b \equiv 1$ in Ω and $g(u) = u^{-\gamma}$ with $\gamma > 1$, Berhanu, Gladiali and Porru [3] showed that there exists $c_0 > 0$ such that

$$\Big|\frac{u_0(x)}{(d(x))^{2/(1+\gamma)}} - \Big(\frac{(1+\gamma)^2}{2(\gamma-1)}\Big)^{1/(1+\gamma)}\Big| < c_0(d(x))^{(\gamma-1)/(1+\gamma)}, \quad \forall x \in \Omega.$$

When $b \equiv 1$ in Ω and the function $g : (0, \infty) \to (0, \infty)$ is locally Lipschitz continuous and decreasing, Giarrusso and Porru [16] showed that if g satisfies the conditions

(G01) $\int_0^1 g(s)ds = \infty$, $\int_1^\infty g(s)ds < \infty$;
(G02) there exist positive constants δ and M with $M > 1$ such that

$$G_1(s) < MG_1(2s), \quad \forall s \in (0, \delta), \quad G_1(s) := \int_s^\infty g(\tau)d\tau, \quad s > 0,$$

then the unique solution u_0 of (1.3) satisfies

(I1) $|u_0(x) - \phi_2(d(x))| < C_0 d(x), \quad \forall x \in \Omega$,

where C_0 is a suitable positive constant and $\phi_2 \in C[0, \infty) \cap C^2(0, \infty)$ is unique solution of

$$\int_0^{\phi_2(t)} \frac{ds}{\sqrt{2G_1(s)}} = t, \quad \forall t > 0. \tag{1.7}$$

When $b \in C^\alpha(\bar{\Omega})$ satisfies the following assumptions: there exist $\delta_0 > 0$ and a positive non-decreasing function $h \in C(0, \delta_0)$ such that

(B01) $\lim_{d(x)\to 0} \frac{b(x)}{h(d(x))} = b_0 \in (0, \infty)$,
(B02) $\lim_{s\to 0^+} h(s)g(s) = \infty$;

and, g satisfies (G1) and the conditions that

(G03) g is non-increasing on $(0, \infty)$;
(G04) there exist positive constants c_0, η_0 and $\gamma \in (0, 1)$ such that $g(s) \le c_0 s^{-\gamma}$, for all $s \in (0, \eta_0)$;

(G05) there exist $q > 0$ and $s_0 \geq 1$ such that $g(\xi s) \geq \xi^{-q} g(s)$ for all $\xi \in (0,1)$ and $0 < s \leq s_0 \xi$;

(G06) $T(\xi) = \lim_{s \to 0^+} \frac{g(\xi s)}{\xi g(s)}$ is continuous in $(0, \infty)$;

then, Ghergu and Rădulescu [14] showed that the unique solution u_0 of problem (1.3) satisfies $u_0 \in C^{1,1-\alpha}(\bar{\Omega}) \cap C^2(\Omega)$ and

$$\lim_{d(x) \to 0} \frac{u_0(x)}{\phi_3(d(x))} = \xi_0, \tag{1.8}$$

where $T(\xi_0) = b_0^{-1}$ and $\phi_3 \in C^1[0, \eta] \cap C^2(0, \eta]$ $(\eta \in (0, \delta_0))$ is the local solution to the problem

$$-\phi_3''(t) = h(t) g(\phi_3(t)), \quad \phi_3(t) > 0, \quad 0 < t < \eta, \quad \phi_3(0) = 0. \tag{1.9}$$

Now let us return to problem (1.1). As a special model of (1.1), Stuart [32] established the following result for an arbitrary $\gamma > 0$:

(i) if $p \in (0,1)$, then the problem

$$-\Delta u = u^{-\gamma} + \lambda u^p, \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0, \tag{1.10}$$

has at least one classical solution for all $\lambda > 0$.

Subsequently, Coclite and Palmieri [8] proved that

(ii) if $p \geq 1$, then there exists $\bar{\lambda} \in (0, \infty)$ such that problem (1.10) has at least one classical solution for $\lambda \in [0, \bar{\lambda})$, and the problem has no classical solutions for $\lambda > \bar{\lambda}$.

There are a number of works which extended the above results, for instance:

(1) For the asymptotic behavior of the unique solution near the boundary to problem (1.1) in the case of $\lambda = 0$, see, for instance, [1, 2, 5, 7, 10, 15, 16, 17, 19, 20, 28], [37]-[43] and the references therein;

(2) del Pino [12], Gui and Lin [21] studied the regularity of the unique solution to problem (1.10) in the case of $\lambda = 0$; Shi and Yao [31] analyzed the regularity and uniqueness of solutions to problem (1.10) and showed that problem (1.10) has one unique solution $u_\lambda \in E := \{u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega}) : u^{-\gamma} \in L^1(\Omega)\}$ for fixed $\lambda > 0$ provided that $p, \gamma \in (0,1)$. For further works, see, Cîrstea, Ghergu, and Rădulescu [7], Rădulescu [29] and the references therein;

(3) for the multiplicity of positive weak solutions to problem (1.10), see, for instance, [27, 33, 35, 36] and the references therein;

(4) for the existence of solutions to problem (1.1), see, for instance, [11, 19, 22, 30, 34, 39] and the references therein.

For convenience, we define the assumption

(B1) there exists $\theta \in \Lambda$ such that

$$0 < b_1 := \lim_{d(x) \to 0} \inf \frac{b(x)}{\theta^2(d(x))} \leq b_2 := \lim_{d(x) \to 0} \sup \frac{b(x)}{\theta^2(d(x))} < \infty,$$

where Λ denotes the set of all positive monotonic functions θ in $C^1(0, \delta_0) \cap L^1(0, \delta_0)$ $(\delta_0 > 0)$ which satisfy

$$\lim_{t \to 0^+} \frac{d}{dt} \Big(\frac{\Theta(t)}{\theta(t)} \Big) := C_\theta \in [0, \infty), \quad \Theta(t) := \int_0^t \theta(s) ds. \tag{1.11}$$

The set Λ was first introduced by Cîrstea and Rădulescu [6] for non-decreasing functions and by Mohammed [25] for non-increasing functions to study the boundary behavior of solutions to boundary blow-up elliptic problems.

Recently, the authors in [43] established a local comparison principle of solutions near the boundary to problem (1.1). More precisely, by using Karamata regularly varying theory and constructing comparison functions near the boundary, they obtained the following results.

Lemma 1.1 ([43, Theorem 1.1]). *For fixed $\lambda > 0$, let f satisfy* (F1), *g satisfy* (G1)–(G3), *b, a satisfy* (S1), *and let b satisfy* (B1). *If*

$$C_\theta + 2C_g > 2, \tag{1.12}$$

and one of the following conditions holds

(1) *$a \equiv 1$ in Ω, $\lim_{d(x)\to 0} b(x) := b|_{\partial\Omega} \in (0, \infty]$;*

(2) *$C_g < 1$, $a \equiv 1$ in Ω, $b|_{\partial\Omega} = 0$, $f(0) = 0$, and there exist $q > 0$ and $\hat{L}_1 \in K$ such that*

$$\limsup_{s\to 0^+} \frac{f(s)}{s^q \hat{L}_1(s)} < \infty; \tag{1.13}$$

(3) *$C_g < 1$, $f(0) = 0$,* (1.13) *holds with $q = 1$, and there exists $\sigma \in \mathbb{R}$ which satisfies*

$$\sigma(C_\theta - 1) < 2C_\theta + 2C_g - 2, \tag{1.14}$$

such that a satisfies the condition that

(A1) *$\limsup_{d(x)\to 0} \frac{a(x)}{\theta^\sigma(d(x))} < \infty$;*

(4) *$C_g > 0$ and a satisfies* (A1) *with $\sigma = 2$;*

(5) *$C_g = 1$, $f(0) = 0$,* (1.14) *and* (A1) *hold, and there exist $q \geq 0$ and $\hat{L}_2 \in K$ such that*

$$\limsup_{s\to 0^+} \frac{s^q f(s)}{\hat{L}_2(s) g(s) \int_0^s \frac{d\tau}{g(\tau)}} < \infty; \tag{1.15}$$

(6) *$C_g < 1$, $f(0) > 0$,* (1.14) *and* (A1) *hold;*

(7) *$C_g = 1$, $f(0) > 0$,* (1.14) *and* (A1) *hold, and there exist $q > 0$ and $\hat{L}_3 \in K$ such that*

$$\limsup_{s\to 0^+} \frac{s^{1+q} \hat{L}_3(s)}{g(s) \int_0^s \frac{d\tau}{g(\tau)}} < \infty; \tag{1.16}$$

then for any classical solution u_λ of (1.1), *we have*

$$\xi_1^{1-C_g} \leq \liminf_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(\Theta^2(d(x)))} \leq \limsup_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(\Theta^2(d(x)))} \leq \xi_2^{1-C_g}, \tag{1.17}$$

where ψ is the solution of (1.2), *and*

$$\xi_1 = \frac{b_1}{2\big(C_\theta + 2C_g - 2\big)}, \quad \xi_2 = \frac{b_2}{2\big(C_\theta + 2C_g - 2\big)}. \tag{1.18}$$

In particular,

(i) *when $C_g = 1$, u_λ satisfies*

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(\Theta^2(d(x)))} = 1;$$

(ii) *when* $C_g < 1$ *and* $b_1 = b_2 = b_0$ *in* (B1), u_λ *satisifes*

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(d^2(x)\theta^2(d(x)))} = (\xi_{01} C_\theta^2)^{1-C_g},$$

where

$$\xi_{01} = \frac{b_0}{2(C_\theta + 2C_g - 2)}.$$

In this article, by recalculating the following limit for $\xi > 0$,

$$\lim_{d(x)\to 0} \frac{a(x)}{\theta^2(d(x))} \frac{f(\psi(\xi\Theta^2(d(x))))}{g(\psi(\xi\Theta^2(d(x))))} = 0,$$

we omit the additional conditions on f and a in Lemma 1.1 and reveal further that the nonlinear term $\lambda a(x)f(u)$ does not affect the first expansion of classical solutions near the boundary for problem (1.1). Our main results are summarized as follows.

Theorem 1.2. *For fixed* $\lambda > 0$*, let* f *satisfy* (F1)*,* g *satisfy* (G1)–(G3)*,* b, a *satisfy* (S1)*, and let* b *satisfy* (B1) *and* (1.12) *hold. If* (1.14) *holds and* a *satisfies* (A1)*, then the results of Lemma 1.1 hold.*

Remark 1.3. One can see in the following Lemma 2.13 that $C_g \in [0, 1]$. Then (1.12) implies $C_\theta > 0$.

Remark 1.4. When $\sigma = 2$ in (A1), (1.14) is precisely $C_g > 0$.

Corollary 1.5. For fixed $\lambda > 0$, let b, a satisfy (S1), $f(s) = s^p$ with $p > 0$, $g(s) = s^{-\gamma}$ with $\gamma > 0$, and let b satisfies (B1) with $b_1 = b_2 = b_0$. If

$$C_\theta(1+\gamma) > 2 \quad \text{and} \quad \sigma(C_\theta - 1) < 2C_\theta - \frac{2}{1+\gamma}, \tag{1.19}$$

and a satisfies (A1), then for any solution u_λ of (1.1), there holds

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{(d(x)\theta(d(x)))^{2/(1+\gamma)}} = (\xi_{01} C_\theta^2 (1+\gamma))^{1/(1+\gamma)}. \tag{1.20}$$

The outline of this paper is as follows. In section 2, we present some basic facts from Karamata regular variation theory and some preliminaries. In section 3, we prove Theorem 1.2.

2. Basic facts from Karamata regular variation theory

Our approach relies on Karamata regular variation theory established by Karamata in 1930 which is a basic tool in stochastic processes (see Bingham, Goldie and Teugels' book [4], Maric's book [24] and the references therein).

In this section, we present some basic facts from Karamata regular variation theory and some preliminaries.

Definition 2.1. A positive continuous function Z defined on $(0, \eta]$, for some $\eta > 0$, is called *regularly varying at zero* with index ρ, written as $Z \in RVZ_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{s\to 0^+} \frac{Z(\xi s)}{Z(s)} = \xi^\rho. \tag{2.1}$$

In particular, when $\rho = 0$, Z is called *slowly varying at zero.*

Clearly, if $Z \in RVZ_\rho$, then $L(s) := Z(s)/s^\rho$ is slowly varying at zero.

Definition 2.2. A positive function $Z \in C(0,\eta]$ for some $\eta > 0$, is called *rapidly varying to infinity at zero* if for each $\xi \in (0,1)$

$$\lim_{s\to 0^+} \frac{Z(\xi s)}{Z(s)} = \infty. \tag{2.2}$$

Definition 2.3. A positive function $Z \in C(0,\eta]$ with $\lim_{s\to 0^+} Z(s) = 0$, for some $\eta > 0$, is called *rapidly varying to zero at zero* if for each $\xi \in (0,1)$

$$\lim_{s\to 0^+} \frac{Z(\xi s)}{Z(s)} = 0. \tag{2.3}$$

Proposition 2.4 (Uniform convergence theorem)**.** *If $Z \in RVZ_\rho$, then (2.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.*

Proposition 2.5 (Representation theorem)**.** *A function L is slowly varying at zero if and only if it may be written in the form*

$$L(s) = l(s)\exp\Big(\int_s^\eta \frac{y(\tau)}{\tau}d\tau\Big), \quad s \in (0,\eta], \tag{2.4}$$

where the functions l and y are continuous and for $s \to 0^+$, $y(s) \to 0$ and $l(s) \to c_0$, with $c_0 > 0$.

We call that

$$\hat{L}(s) = c_0\exp\Big(\int_s^\eta \frac{y(\tau)}{\tau}d\tau\Big), \quad s \in (0,\eta], \tag{2.5}$$

is *normalized* slowly varying at zero and

$$Z(s) = s^\rho \hat{L}(s), \quad s \in (0,\eta], \tag{2.6}$$

is *normalized* regularly varying at zero with index ρ (denoted by $Z \in NRVZ_\rho$), respectively.

A function $Z \in RVZ_\rho$ belongs to $NRVZ_\rho$ if and only if

$$Z \in C^1(0,\eta], \quad \text{for some } \eta > 0 \text{ and } \lim_{s\to 0^+} \frac{sZ'(s)}{Z(s)} = \rho. \tag{2.7}$$

Proposition 2.6. *If functions L, L_1 are slowly varying at zero, then*

(i) *L^ρ for every $\rho \in \mathbb{R}$, $c_1L + c_2L_1$ ($c_1 \ge 0, c_2 \ge 0$ with $c_1 + c_2 > 0$), and $L \cdot L_1$, $L \circ L_1$ (if $L_1(s) \to 0$ as $s \to 0^+$) are also slowly varying at zero.*

(ii) *For every $\rho > 0$ and $s \to 0^+$,*

$$s^\rho L(s) \to 0, \quad s^{-\rho}L(s) \to \infty.$$

(iii) *For $\rho \in \mathbb{R}$ and $s \to 0^+$, $\ln(L(s))/\ln s \to 0$ and $\ln(s^\rho L(s))/\ln s \to \rho$.*

Proposition 2.7. *If $Z_1 \in RVZ_{\rho_1}$, $Z_2 \in RVZ_{\rho_2}$ with $\lim_{s\to 0} Z_2(s) = 0$, then $Z_1 \circ Z_2 \in RVZ_{\rho_1\rho_2}$.*

Proposition 2.8 (Asymptotic behavior)**.** *If a function L is slowly varying at zero, then for $\eta > 0$ and $t \to 0^+$,*

(i) *$\int_0^t s^\rho L(s)ds \cong (1+\rho)^{-1}t^{1+\rho}L(t)$, for $\rho > -1$;*

(ii) *$\int_t^\eta s^\rho L(s)ds \cong (-\rho-1)^{-1}t^{1+\rho}L(t)$, for $\rho < -1$.*

Proposition 2.9. *Let $Z \in C^1(0,\eta]$ be positive and*

$$\lim_{s\to 0^+} \frac{sZ'(s)}{Z(s)} = +\infty.$$

Then Z is rapidly varying to zero at zero.

Proposition 2.10. *Let $Z \in C^1(0,\eta)$ be positive and*

$$\lim_{s\to 0^+} \frac{sZ'(s)}{Z(s)} = -\infty.$$

Then Z is rapidly varying to infinity at zero.

Proposition 2.11 ([37, Lemma 2.3]). *Let $\hat{L} \in NRVZ_0$ be defined on $(0,\eta]$. Then we have*

$$\lim_{t\to 0^+} \frac{\hat{L}(t)}{\int_t^\eta \frac{\hat{L}(\tau)}{\tau} d\tau} = 0.$$

If further $\int_0^\eta \frac{\hat{L}(\tau)}{\tau} d\tau$ converges, then we have

$$\lim_{t\to 0^+} \frac{\hat{L}(t)}{\int_0^t \frac{\hat{L}(\tau)}{\tau} d\tau} = 0.$$

Lemma 2.12 ([44, Lemma 2.1]). *Let $\theta \in \Lambda$.*

- (i) *When θ is non-decreasing, $C_\theta \in [0,1]$; and, when θ is non-increasing, $C_\theta \geq 1$;*
- (ii) $\lim_{t\to 0^+} \frac{\Theta(t)}{\theta(t)} = 0$ *and* $\lim_{t\to 0^+} \frac{\Theta(t)\theta'(t)}{\theta^2(t)} = 1 - \lim_{t\to 0^+} \frac{d}{dt}\big(\frac{\Theta(t)}{\theta(t)}\big) = 1 - C_\theta$;
- (iii) *when $C_\theta > 0$, $\theta \in NRVZ_{(1-C_\theta)/C_\theta}$. In particular, when $C_\theta = 1$, θ is normalized slowly varying at zero;*
- (iv) *when $C_\theta = 0$, θ is rapidly varying to zero at zero.*

Lemma 2.13 ([43, Lemma 2.2]). *Let g satisfy* (G1)–(G2).

- (i) *If g satisfies* (G3), *then $C_g \leq 1$;*
- (ii) (G3) *holds with $C_g \in (0,1)$ if and only if $g \in NRVZ_{-C_g/(1-C_g)}$;*
- (iii) (G3) *holds with $C_g = 0$ if and only if g is normalized slowly varying at zero;*
- (iv) *if* (G3) *holds with $C_g = 1$, then g is rapidly varying to infinity at zero.*

Lemma 2.14 ([42, Lemma 2.3]). *Let g satisfy* (G1)–(G3) *and let ψ be the unique solution to*

$$\int_0^{\psi(t)} \frac{d\tau}{g(\tau)} = t, \ t \in [0,\infty),$$

then

- (i) $\psi'(t) = g(\psi(t))$, $\psi(t) > 0$, $t > 0$, $\psi(0) = 0$, $\psi'(0) := \lim_{t\to 0} \psi'(t) = \lim_{t\to 0} g(\psi(t)) = \infty$, *and* $\psi''(t) = g(\psi(t))g'(\psi(t))$, $t > 0$;
- (ii) $\lim_{t\to 0^+} tg(\psi(t)) = 0$ *and* $\lim_{t\to 0^+} tg'(\psi(t)) = -C_g$;
- (iii) $\psi \in NRVZ_{1-C_g}$ *and* $\psi' \in NRVZ_{-C_g}$.

3. Boundary behaviors of solutions

In this section we prove Theorem 1.2. First, for any $\delta > 0$, we define

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\}.$$

Since $\partial\Omega \in C^2$, there exists a constant $\delta \in (0, \delta_0)$ which only depends on Ω such that (see, [18, Lemmas 14.16 and 14.17])

$$d \in C^2(\Omega_\delta), \quad |\nabla d(x)| = 1, \qquad \Delta d(x) = -(N-1)H(\bar{x}) + o(1), \quad \forall\, x \in \Omega_\delta, \tag{3.1}$$

where δ_0 in the definition of the set Λ, $\bar{x}$ is the nearest point to x on $\partial\Omega$, and $H(\bar{x})$ denotes the mean curvature of $\partial\Omega$ at $\bar{x}$.

Secondly, for a satisfying (S1), let $Va \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$ be the unique solution to the Poisson problem

$$-\Delta v = a(x), \quad v > 0, \quad x \in \Omega, \quad v|_{\partial\Omega} = 0. \tag{3.2}$$

Now we have a local comparison principle.

Lemma 3.1 ([43, Lemma 3.1])**.** *For fixed $\lambda > 0$, let f satisfy* (F1)*, g satisfy* (G1)*,* (G2)*, b, a satisfy* (S1)*, and let $u_\lambda \in C^2(\Omega) \cap C(\bar{\Omega})$ be an arbitrary solution to problem* (1.1)*, $\bar{u}_\lambda \in C^2(\Omega_\delta) \cap C(\bar{\Omega}_\delta)$ satisfy*

$$-\Delta \bar{u}_\lambda \ge b(x)g(\bar{u}_\lambda) + \lambda a(x) f(\bar{u}_\lambda), \quad \bar{u}_\lambda > 0, \quad x \in \Omega_\delta, \quad \bar{u}_\lambda|_{\partial\Omega} = 0, \tag{3.3}$$

and $\underline{u}_\lambda \in C^2(\Omega_\delta) \cap C(\bar{\Omega}_\delta)$ satisfy

$$-\Delta \underline{u}_\lambda \le b(x)g(\underline{u}_\lambda) + \lambda a(x) f(\underline{u}_\lambda), \quad \underline{u}_\lambda > 0, \quad x \in \Omega_\delta, \quad \underline{u}_\lambda|_{\partial\Omega} = 0, \tag{3.4}$$

where $\delta > 0$ sufficiently small such that

$$\underline{u}_\lambda(x), \quad \bar{u}_\lambda(x), \quad u_\lambda(x) \in (0, s_0), \quad x \in \Omega_\delta,$$

where s_0 is given as in (g_2)*. Then there exists a positive constant M_0 such that*

$$\underline{u}_\lambda(x) \le u_\lambda(x) + \lambda M_0 Va(x), \quad x \in \Omega_\delta; \tag{3.5}$$

$$u_\lambda(x) \le \bar{u}_\lambda(x) + \lambda M_0 Va(x), \quad x \in \Omega_\delta. \tag{3.6}$$

Lemma 3.2 ([43, Lemma 3.3])**.** *Let g satisfy* (G1)–(G3) *and $C_\theta + 2C_g > 2$. If* (1.14) *holds and a satisfies* (A1)*, then*

$$\lim_{d(x)\to 0} \frac{Va(x)}{\psi(\Theta^2(d(x)))} = 0. \tag{3.7}$$

Lemma 3.3. *Let g satisfy* (G1)–(G3) *and $C_\theta + 2C_g > 2$. If* (1.14) *holds and a satisfies* (A1)*, then there holds*

$$\lim_{d(x)\to 0} \frac{a(x)}{\theta^2(d(x))} \frac{f(\psi(\xi\Theta^2(d(x))))}{g(\psi(\xi\Theta^2(d(x))))} = 0, \tag{3.8}$$

uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$, where θ is as determined in (B1)*.*

Proof. First, by Proposition 2.4, (F1), Lemmas 2.13 and 2.14, we can obtain the above limits of uniform convergence for $\xi \in [c_1, c_2]$.

Secondly, (1.11) and the l'Hospital's rule imply that

$$\lim_{t\to 0^+} \frac{\Theta(t)}{t\theta(t)} = \lim_{t\to 0^+} \frac{\frac{\Theta(t)}{\theta(t)}}{t} = \lim_{t\to 0^+} \frac{d}{dt}\Big(\frac{\Theta(t)}{\theta(t)}\Big) = C_\theta. \tag{3.9}$$

Since $C_\theta > 0$ (Remark 1.3), we see that $\Theta \in NRVZ_{C_\theta^{-1}}$.

Next, one can see by Lemma 2.14 that $\psi'(t) = g(\psi(t))$ belongs to $NRVZ_{-C_g}$. So we obtain by Proposition 2.7 that $g(\psi(\Theta^2(t)))$ belongs to $NRVZ_{-2C_g/C_\theta}$. In succession, by Lemma 2.12 and Proposition 2.7, we see that

$$\theta^{\sigma-2} \in NRVZ_{(1-C_\theta)(\sigma-2)/C_\theta}.$$

Thus

$$\frac{\theta^{\sigma-2}(t)}{g(\psi(\Theta^2(t)))} \quad \text{belongs to } NRVZ_\rho,$$

with

$$\rho = \frac{2C_g + 2C_\theta - 2 - \sigma(C_\theta - 1))}{C_\theta} > 0.$$

Consequently, by Proposition 2.6 (ii),

$$\begin{aligned}
&\lim_{d(x)\to 0} \frac{a(x)}{\theta^2(d(x))} \frac{f(\psi(\xi\Theta^2(d(x))))}{g(\psi(\xi\Theta^2(d(x))))} \\
&= \lim_{d(x)\to 0} f(\psi(\xi\Theta^2(d(x)))) \lim_{d(x)\to 0} \frac{a(x)}{\theta^\sigma(d(x))} \frac{\theta^{\sigma-2}(d(x))}{g(\psi(\xi\Theta^2(d(x))))} = 0.
\end{aligned}$$

□

Proof of Theorem 1.2. Let $\varepsilon \in (0, b_1/4)$ and let

$$\tau_1 = \xi_1 - 2\varepsilon\xi_1/b_1, \quad \tau_2 = \xi_2 + 2\varepsilon\xi_2/b_2,$$

where ξ_1 and ξ_2 are given as in (1.18). It follows that

$$\xi_1/2 < \tau_1 < \tau_2 < 2\xi_2; \quad \lim_{\varepsilon\to 0} \tau_1 = \xi_1; \quad \lim_{\varepsilon\to 0} \tau_2 = \xi_2$$

and

$$-4\tau_2 C_g + 2\tau_2(2 - C_\theta) + b_2 = -2\varepsilon; \quad -4\tau_1 C_g + 2\tau_1(2 - C_\theta) + b_1 = 2\varepsilon. \tag{3.10}$$

By (B1), (3.1), Lemmas 2.12, 2.14 and 3.3, we see that

$$\begin{gathered}
\lim_{d(x)\to 0} \tau_2\Theta^2(d(x))g'(\psi(\tau_2\Theta^2(d(x)))) = -C_g; \\
\lim_{d(x)\to 0} \Big(\frac{\theta'(d(x))\Theta(d(x))}{\theta^2(d(x))} + 1 + \frac{\Theta(d(x))}{\theta(d(x))}\Delta d(x)\Big) = 2 - C_\theta; \\
\limsup_{d(x)\to 0} \frac{b(x)}{\theta^2(d(x))} \le b_2; \quad \lim_{d(x)\to 0} \frac{a(x)}{\theta^2(d(x))} \frac{f(\psi(\tau_2\Theta^2(d(x))))}{g(\psi(\tau_2\Theta^2(d(x))))} = 0.
\end{gathered}$$

Thus, corresponding to ε, s_0 and δ, where s_0 is given as in (G2) and δ in Lemma 3.1, respectively, there is $\delta_\varepsilon \in (0, \delta)$ sufficiently small such that for $x \in \Omega_{\delta_\varepsilon}$

$$\bar{u}_\varepsilon = \psi(\tau_2\Theta^2(d(x)))$$

satisfies

$$\bar{u}_\varepsilon(x) \in (0, s_0), \quad x \in \Omega_{\delta_\varepsilon}, \tag{3.11}$$

and

$$\begin{aligned}
&\Delta\bar{u}_\varepsilon(x) + b(x)g(\bar{u}_\varepsilon(x)) + \lambda a(x)f(\bar{u}_\varepsilon(x)) \\
&= \psi''(\tau_2\Theta^2(d(x)))(2\tau_2\Theta(d(x))\theta(d(x)))^2 + 2\tau_2\psi'(\tau_2\Theta^2(d(x))) \\
&\quad \times \big(\theta^2(d(x)) + \Theta(d(x))\theta'(d(x)) + \Theta(d(x))\theta(d(x))\Delta d(x)\big) \\
&\quad + b(x)g(\psi(\tau_2\Theta^2(d(x)))) + \lambda a(x)f(\psi(\tau_2\Theta^2(d(x)))) \\
&= g(\psi(\tau_2\Theta^2(d(x))))\theta^2(d(x))\Big(4\tau_2\tau_2\Theta^2(d(x))g'(\psi(\tau_2\Theta^2(d(x))))
\end{aligned}$$

$$
\begin{aligned}
&+ 2\tau_2\Big(\frac{\theta'(d(x))\Theta(d(x))}{\theta^2(d(x))} + 1 + \frac{\Theta(d(x))}{\theta(d(x))}\Delta d(x)\Big) \\
&+ \frac{b(x)}{\theta^2(d(x))} + \lambda\frac{a(x)}{\theta^2(d(x))}\frac{f(\psi(\tau_2\Theta^2(d(x))))}{g(\psi(\tau_2\Theta^2(d(x))))}\Big) \le 0;
\end{aligned}
$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of equation (1.1) in $\Omega_{\delta_\varepsilon}$. In a similar way, we can show that

$$\underline{u}_\varepsilon = \psi(\tau_1\Theta^2(d(x))), \quad x \in \Omega_{\delta_\varepsilon},$$

is a subsolution of equation (1.1) in $\Omega_{\delta_\varepsilon}$.

Now let $u_\lambda \in C(\bar{\Omega}) \cap C^{2+\alpha}(\Omega)$ be an arbitrary classical solution to problem (1.1). By Lemma 3.1, we see that there exists $M_0 > 0$ such that for $x \in \Omega_{\delta_\varepsilon}$

$$\underline{u}_\lambda(x) \le u_\lambda(x) + \lambda M_0 Va(x) \quad \text{and} \quad u_\lambda(x) \le \bar{u}_\lambda(x) + \lambda M_0 Va(x);$$

i.e.,

$$1 - \lambda M_0\frac{Va(x)}{\psi(\tau_1\Theta^2(d(x)))} \le \frac{u_\lambda(x)}{\psi(\tau_1\Theta^2(d(x)))}, \quad x \in \Omega_{\delta_\varepsilon},$$

and

$$\frac{u_\lambda(x)}{\psi(\tau_2\Theta^2(d(x)))} \le 1 + \lambda M_0\frac{Va(x)}{\psi(\tau_2\Theta^2(d(x)))}, \quad x \in \Omega_{\delta_\varepsilon}.$$

It follows by Lemma 3.2 that

$$1 \le \liminf_{d(x)\to 0}\frac{u_\lambda(x)}{\psi(\tau_1\Theta^2(d(x)))} \quad \text{and} \quad \limsup_{d(x)\to 0}\frac{u_\lambda(x)}{\psi(\tau_2\Theta^2(d(x)))} \le 1.$$

Using Lemma 2.14, we have

$$\lim_{d(x)\to 0}\frac{\psi(\xi_1\Theta^2(d(x)))}{\psi(\Theta^2(d(x)))} = {\xi_1}^{1-C_g}; \quad \lim_{d(x)\to 0}\frac{\psi(\xi_2\Theta^2(d(x)))}{\psi(\Theta^2(d(x)))} = {\xi_2}^{1-C_g}.$$

Moreover, since $C_\theta > 0$, by (3.9) and Lemma 2.14, we obtain that

$$\lim_{d(x)\to 0}\frac{\Theta(d(x))}{d(x)\theta(d(x))} = C_\theta, \quad \lim_{d(x)\to 0}\frac{\psi(\Theta^2(d(x)))}{\psi(d^2(x)\theta^2(d(x)))} = C_\theta^{2(1-C_g)}.$$

Thus letting $\varepsilon \to 0$, we have

$$\xi_1^{1-C_g} \le \liminf_{d(x)\to 0}\frac{u_\lambda(x)}{\psi(\Theta^2(d(x)))} \le \limsup_{d(x)\to 0}\frac{u_\lambda(x)}{\psi(\Theta^2(d(x)))} \le \xi_2^{1-C_g}.$$

In particular, when $C_g = 1$, u_λ satisfies

$$\lim_{d(x)\to 0}\frac{u_\lambda(x)}{\psi(\Theta^2(d(x)))} = 1;$$

and, when $C_g < 1$ and $b_1 = b_2 = b_0$ in (B1), u_λ satisfies

$$\lim_{d(x)\to 0}\frac{u_\lambda(x)}{\psi(d^2(x)\theta^2(d(x)))} = (\xi_{01}C_\theta^2)^{1-C_g}.$$

This completes the proof. $\square$

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