

GLOBAL MILD SOLUTIONS TO MODIFIED NAVIER-STOKES EQUATIONS WITH SMALL INITIAL DATA IN CRITICAL BESOV-Q SPACES

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ABSTRACT. This article is devoted to establishing the global existence and uniqueness of a mild solution of the modified Navier-Stokes equations with a small initial data in the critical Besov-Q space.

1. STATEMENT OF THE MAIN RESULTS

For $\beta > 1/2$, the Cauchy problem of the modified Navier-Stokes equations on the half-space $\mathbb{R}_+^{1+n} = (0, \infty) \times \mathbb{R}^n$, $n \geq 2$, consists of studying the existence of a solution u to

$$\begin{aligned} \frac{\partial u}{\partial t} + (-\Delta)^\beta u + u \cdot \nabla u - \nabla p &= 0, \quad \text{in } \mathbb{R}_+^{1+n}; \\ \nabla \cdot u &= 0, \quad \text{in } \mathbb{R}_+^{1+n}; \\ u|_{t=0} &= a, \quad \text{in } \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where $(-\Delta)^\beta$ represents the β -order Laplace operator defined by the Fourier transform in the space variable:

$$\widehat{(-\Delta)^\beta u}(\cdot, \xi) = |\xi|^{2\beta} \hat{u}(\cdot, \xi).$$

Here, we point out that (1.1) is a generalization of the classical Navier-Stokes system and two-dimensional quasi-geostrophic equation which have continued to attract attention extensively, and that the dissipation $(-\Delta)^\beta u$ still retains the physical meaning of the nonlinearity $u \cdot \nabla u + \nabla p$ and the divergence-free condition $\nabla \cdot u = 0$.

Upon letting R_j , $j = 1, 2, \dots, n$, be the Riesz transforms, writing

$$\begin{aligned} \mathbb{P} &= \{\delta_{l,l'} + R_l R_{l'}\}, \quad l, l' = 1, \dots, n; \\ \mathbb{P} \nabla(u \otimes u) &= \sum_l \frac{\partial}{\partial x_l} (u_l u) - \sum_l \sum_{l'} R_l R_{l'} \nabla(u_l u_{l'}); \\ e^{-t \widehat{(-\Delta)^\beta}} f(\xi) &= e^{-t|\xi|^{2\beta}} \hat{f}(\xi), \end{aligned} \tag{1.2}$$

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and using $\nabla \cdot u = 0$, we can see that a solution of the above Cauchy problem is then obtained via the integral equation

$$\begin{aligned} u(t, x) &= e^{-t(-\Delta)^\beta} a(x) - B(u, u)(t, x); \\ B(u, u)(t, x) &\equiv \int_0^t e^{-(t-s)(-\Delta)^\beta} \mathbb{P}\nabla(u \otimes u) ds, \end{aligned} \quad (1.3)$$

which can be solved by a fixed-point method whenever the convergence is suitably defined in a function space. Solutions of (1.3) are called mild solutions of (1.1). The notion of such a mild solution was pioneered by Kato-Fujita [14] in 1960s. During the latest decades, many important results about the mild solutions to (1.1) have been established; see for example, Cannone [3, 4], Germin-Pavlovic-Staffilani [11], Giga-Miyakawa [12], Kato [13], Koch-Tataru [16], Wu [30, 31, 32, 33], and their references including Kato-Ponce [15] and Taylor [28].

The main purpose of this paper is to establish the following global existence and uniqueness of a mild solution to (1.1) with a small initial data in the critical Besov-Q space.

Theorem 1.1. *Assume that $\beta > 1/2$; $1 < p, q < \infty$; $\gamma_1 = \gamma_2 - 2\beta + 1$; $m > \max\{p, \frac{n}{2\beta}\}$; $0 < m' < \min\{1, \frac{p}{2\beta}\}$. If the index (β, p, γ_2) satisfies*

$$1 < p \leq 2 \text{ and } \frac{2\beta - 2}{p} < \gamma_2 \leq \frac{n}{p}$$

or

$$2 < p < \infty \text{ and } \beta - 1 < \gamma_2 \leq \frac{n}{p},$$

then (1.1) has a unique global mild solution in $(\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2})^n$ for any initial data a with $\|a\|_{(\dot{B}_{p,q}^{\gamma_1,\gamma_2})^n}$ being small. Here the symbols $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$, and $\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}$ stand for the so-called Besov-Q spaces and their induced tent spaces, and will be determined properly in Sections 3 and 4.

Needless to say, our current work grows from the already-known results. Lions [21] proved the global existence of the classical solutions of (1.1) when $\beta \geq \frac{5}{4}$ and $n = 3$. This existence result was extended to $\beta \geq \frac{1}{2} + \frac{n}{4}$ by Wu [30], and moreover, for the important case $\beta < \frac{1}{2} + \frac{n}{4}$. Wu [31, 32] established the global existence for (1.1) in the Besov spaces $\dot{B}_p^{1+\frac{n}{p}-2\beta,q}(\mathbb{R}^n)$ for $1 \leq q \leq \infty$ and for either $\frac{1}{2} < \beta$ and $p = 2$ or $\frac{1}{2} < \beta \leq 1$ and $2 < p < \infty$ and in $\dot{B}_2^{r,\infty}(\mathbb{R}^n)$ with $r > \max\{1, 1 + \frac{n}{p} - 2\beta\}$; see also [33] concerning the corresponding regularity. Importantly, Koch-Tataru [16] studied the global existence and uniqueness of (1.1) with $\beta = 1$ via introducing $BMO^{-1}(\mathbb{R}^n)$. Extending Koch-Tataru's work [16], Xiao [35, 36] introduced the Q-spaces $Q_{0 < \alpha < 1}^{-1}(\mathbb{R}^n)$ to investigate the global existence and uniqueness of the classical Navier-Stokes system. The ideas of [35] were developed by Li-Zhai [18] to study the global existence and uniqueness of (1.1) with small data in a class of Q-type spaces $Q_\alpha^{\beta,-1}(\mathbb{R}^n)$ under $\beta \in (\frac{1}{2}, 1)$. Recently, Lin-Yang [20] got the global existence and uniqueness of (1.1) with initial data being small in a diagonal Besov-Q space for $\beta \in (\frac{1}{2}, 1)$.

In fact, the above historical citations lead us to make a decisive two-fold observation. On the one hand, thanks to that (1.1) is invariant under the scaling

$$u_\lambda(t, x) = \lambda^{2\beta-1} u(\lambda^{2\beta} t, \lambda x);$$

$$p_\lambda(t, x) = \lambda^{4\beta-2} p(\lambda^{2\beta} t, \lambda x),$$

the initial data space $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$ is critical for (1.1) in the sense that the space is invariant under the scaling

$$f_\lambda(x) = \lambda^{2\beta-1} f(\lambda x). \tag{1.4}$$

A simple computation, along with letting $\beta = 1$ in (1.4), indicates that the function spaces

$$\begin{aligned} \dot{L}_{\frac{n}{2}-1}^2(\mathbb{R}^n) &= \dot{B}_2^{-1+\frac{n}{2}, 2}(\mathbb{R}^n); & L^n(\mathbb{R}^n); \\ \dot{B}_p^{-1+\frac{n}{p}, q}(\mathbb{R}^n); & BMO^{-1}(\mathbb{R}^n), \end{aligned}$$

are critical for (1.1) with $\beta = 1$. Moreover, (1.4) under $\beta > 1/2$ is valid for functions in the homogeneous Besov spaces $\dot{B}_2^{1+\frac{n}{2}-2\beta, 1}(\mathbb{R}^n)$ and $\dot{B}_2^{1+\frac{n}{2}-2\beta, \infty}(\mathbb{R}^n)$ attached to (1.1). On the other hand, it is suitable to mention the following relations:

$$\begin{aligned} \dot{B}_{p,q}^{\gamma_1, \frac{n}{p}} &= \dot{B}_p^{\gamma_1, q}(\mathbb{R}^n) \quad \text{for } 1 \leq p, q < \infty, -\infty < \gamma_1 < \infty; \\ \dot{B}_{p_0, q_0}^{1+\frac{n}{p}-2\beta, \frac{n}{p}} &\supseteq \dot{B}_p^{1+\frac{n}{p}-2\beta, q}(\mathbb{R}^n) \quad \text{for } 1 < p \leq p_0, 1 < q \leq q_0 < \infty, \beta > 0; \\ \dot{B}_{2,2}^{\alpha-\beta+1, \alpha+\beta-1} &= Q_\alpha^\beta(\mathbb{R}^n) \quad \text{for } \alpha \in (0, 1), \beta \in (1/2, 1), \alpha + \beta - 1 \geq 0. \end{aligned}$$

To briefly describe the argument for Theorem 1.1, we should point out that the function spaces used in [16, 35, 18] have a common trait in the structure; i.e., these spaces can be seen as the Q-spaces with L^2 norm, and the advantage of such spaces is that Fourier transform plays an important role in estimating the bilinear term on the corresponding solution spaces. Nevertheless, for the global existence and uniqueness of a mild solution to (1.1) with a small initial data in $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$, we have to seek a new approach. Generally speaking, a mild solution of (1.1) is obtained by using the following method. Assume that the initial data belongs to $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$. Via the iteration process:

$$\begin{aligned} u^{(0)}(t, x) &= e^{-t(-\Delta)^\beta} a(x); \\ u^{(j+1)}(t, x) &= u^{(0)}(t, x) - B(u^{(j)}, u^{(j)})(t, x) \quad \text{for } j = 0, 1, 2, \dots, \end{aligned}$$

we construct a contraction mapping on a space in \mathbb{R}_+^{1+n} , denoted by $X(\mathbb{R}_+^{1+n})$. With the initial data being small, the fixed point theorem implies that there exists a unique mild solution of (1.1) in $X(\mathbb{R}_+^{1+n})$. In this paper, we choose $X(\mathbb{R}_+^{1+n}) = \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$ associated with $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$. Owing to Theorem 4.5, we know that if $f \in \dot{B}_{p,q}^{\gamma_1, \gamma_2}$ then $e^{-t(-\Delta)^\beta} f(x) \in X(\mathbb{R}_+^{1+n})$. Hence the construction of contraction mapping comes down to prove the following assertion: The bilinear operator

$$B(u, v) = \int_0^t e^{-(t-s)(-\Delta)^\beta} \mathbb{P}\nabla(u \otimes v) ds$$

is bounded from $(X(\mathbb{R}_+^{1+n}))^n \times (X(\mathbb{R}_+^{1+n}))^n$ to $(X(\mathbb{R}_+^{1+n}))^n$.

For this purpose, using multi-resolution analysis, we decompose $B_l(u, v)$ into several parts based on the relation between t and $2^{-2j\beta}$, and expand every part in terms of $\{\Phi_{j,k}^\varepsilon\}$. More importantly, Lemmas 6.1 & 6.2 enable us to obtain an estimate from $(\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2})^n \times (\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2})^n$ to $(\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2})^n$.

Remark 1.2. (i) Our initial spaces in Theorem 1.1 include both $\dot{B}_p^{1+\frac{n}{p}-2\beta,q}(\mathbb{R}^n)$ in Wu [30, 31, 32, 33], $Q_\alpha^{\beta,-1}(\mathbb{R}^n)$ in Xiao [35] and Li-Zhai [18]. Moreover, in [18, 20], the scope of β is $(\frac{1}{2}, 1)$. Our method is valid for $\beta > 1/2$.

(ii) We point out that $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$ provide a lot of new critical initial spaces where the well-posedness of equations (1.1) holds. By Lemma 3.7, for $\beta = 1$, Theorem 1.1 holds for the initial spaces $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$ satisfying

$$\dot{B}_{2,q'}^{-1+w,w} \subset Q_\alpha^{-1}(\mathbb{R}^n), \quad q' < 2, w > 0$$

or

$$Q_\alpha^{-1}(\mathbb{R}^n) \subset \dot{B}_{2,q''}^{-1+w,w} \subset BMO^{-1}(\mathbb{R}^n), \quad 2 < q'' < \infty, w > 0.$$

In some sense, $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$ fill the gap between the spaces $Q_\alpha^{-1}(\mathbb{R}^n)$ and $BMO^{-1}(\mathbb{R}^n)$. See also Lemma 3.7, Corollary 3.8 and Remark 3.9.

(iii) For a initial data $a \in \dot{B}_{p,q}^{\gamma_1,\gamma_2}$, the index γ_1 represents the regularity of a . In Theorem 1.1, taking $\dot{B}_{p,q}^{\gamma_1,\gamma_2} = \dot{B}_{p,q}^{-1+w,w}$ with $w > 0$, $p > 2$ and $q > 2$ yields $\dot{B}_{p,q}^{-1+w,w} \subset BMO^{-1}(\mathbb{R}^n)$. Compared with the ones in $BMO^{-1}(\mathbb{R}^n)$, the elements of $\dot{B}_{p,q}^{-1+w,w}$ have higher regularity. Furthermore, Theorem 1.1 implies that the regularity of our solutions becomes higher along with the growth of γ_1 .

(iv) Interestingly, Federbush [8] employed the divergence-free wavelets to study the classical Navier-Stokes equations, while the wavelets used in this paper are classical Meyer wavelets. In addition, when constructing a contraction mapping, Federbush's method was based on the estimates of "long wavelength residues". Nevertheless, our wavelet approach based on Lemmas 5.1-5.2 and the Cauchy-Schwarz inequality is to convert the bilinear estimate of $B(u, v)$ into various efficient computations involved in the wavelet coefficients of u and v .

The remaining of this article is organized as follows. In Section 2, we list some preliminary knowledge on wavelets and give the wavelet characterization of the Besov-Q spaces. In Sections 3-4 we define the initial data spaces and the corresponding solution spaces. Section 5 carries out a necessary analysis of some non-linear terms and a priori estimates. In Section 6, we verify Theorem 1.1 via Lemmas 5.1-5.2 which will be demonstrated in Sections 7-8 respectively.

Notation. $U \approx V$ indicates that there is a constant $c > 0$ such that $c^{-1}V \leq U \leq cV$ whose right inequality is also written as $U \lesssim V$. Similarly, one writes $V \gtrsim U$ for $V \geq cU$.

2. PRELIMINARIES

First of all, we would like to say that we will always utilize tensorial orthogonal wavelets which may be regular Daubechies wavelets (only used for characterizing Besov and Besov-Q spaces) and classical Meyer wavelets, but also to recall that the regular Daubechies wavelets are such Daubechines wavelets that are smooth enough and have more sufficient vanishing moments than the relative spaces do; see Lemma 2.3 and the part before Lemma 3.2.

Next, we present some preliminaries on Meyer wavelets $\Phi^\epsilon(x)$ in detail and refer the reader to [22], [29] and [38] for further information. Let Ψ^0 be an even function in $C_0^\infty([-\frac{4\pi}{3}, \frac{4\pi}{3}])$ with

$$0 \leq \Psi^0(\xi) \leq 1;$$

$$\Psi^0(\xi) = 1 \quad \text{for } |\xi| \leq \frac{2\pi}{3}.$$

If

$$\Omega(\xi) = \sqrt{(\Psi^0(\frac{\xi}{2}))^2 - (\Psi^0(\xi))^2},$$

then Ω is an even function in $C_0^\infty([-\frac{8\pi}{3}, \frac{8\pi}{3}])$. Clearly,

$$\Omega(\xi) = 0 \quad \text{for } |\xi| \leq \frac{2\pi}{3};$$

$$\Omega^2(\xi) + \Omega^2(2\xi) = 1 = \Omega^2(\xi) + \Omega^2(2\pi - \xi) \quad \text{for } \xi \in [\frac{2\pi}{3}, \frac{4\pi}{3}].$$

Let $\Psi^1(\xi) = \Omega(\xi)e^{-\frac{i\xi}{2}}$. For any $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$, define $\Phi^\epsilon(x)$ via the Fourier transform $\hat{\Phi}^\epsilon(\xi) = \prod_{i=1}^n \Psi^{\epsilon_i}(\xi_i)$. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, set $\Phi_{j,k}^\epsilon(x) = 2^{\frac{nj}{2}} \Phi^\epsilon(2^j x - k)$. Furthermore, we put

$$\begin{aligned} E_n &= \{0, 1\}^n \setminus \{0\}; \\ F_n &= \{(\epsilon, k) : \epsilon \in E_n, k \in \mathbb{Z}^n\}; \\ \Lambda_n &= \{(\epsilon, j, k), \epsilon \in E_n, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}, \end{aligned}$$

and for any $\epsilon \in \{0, 1\}^n, k \in \mathbb{Z}^n$ and a function f on \mathbb{R}^n , we write $f_{j,k}^\epsilon = \langle f, \Phi_{j,k}^\epsilon \rangle$. The following result is well-known.

Lemma 2.1. *The Meyer wavelets $\{\Phi_{j,k}^\epsilon\}_{(\epsilon,j,k) \in \Lambda_n}$ form an orthogonal basis in the space $L^2(\mathbb{R}^n)$. Consequently, for any $f \in L^2(\mathbb{R}^n)$, the following wavelet decomposition holds in the L^2 convergence sense:*

$$f = \sum_{(\epsilon,j,k) \in \Lambda_n} f_{j,k}^\epsilon \Phi_{j,k}^\epsilon.$$

Moreover, for $j \in \mathbb{Z}$, let

$$P_j f = \sum_{k \in \mathbb{Z}^n} f_{j,k}^0 \Phi_{j,k}^0, \quad Q_j f = \sum_{(\epsilon,k) \in F_n} f_{j,k}^\epsilon \Phi_{j,k}^\epsilon.$$

For the above Meyer wavelets, by Lemma 2.1, the product of any two functions u and v can be decomposed as

$$\begin{aligned} uv &= \sum_{j \in \mathbb{Z}} P_{j-3} u Q_j v + \sum_{j \in \mathbb{Z}} Q_j u Q_j v + \sum_{0 < j-j' \leq 3} Q_j u Q_{j'} v \\ &+ \sum_{0 < j'-j \leq 3} Q_j u Q_{j'} v + \sum_{j \in \mathbb{Z}} Q_j u P_{j-3} v. \end{aligned} \tag{2.1}$$

Suppose that φ is a function on \mathbb{R}^n satisfying

$$\begin{aligned} \text{supp } \hat{\varphi} &\subset \{\xi \in \mathbb{R}^n : |\xi| \leq 1\}, \\ \hat{\varphi}(\xi) &= 1 \quad \text{for } \{\xi \in \mathbb{R}^n : |\xi| \leq \frac{1}{2}\}, \end{aligned}$$

and that

$$\varphi_v(x) = 2^{n(v+1)} \varphi(2^{v+1}x) - 2^{nv} \varphi(2^v x) \quad \forall v \in \mathbb{Z},$$

are the Littlewood-Paley functions; see [25].

Definition 2.2. Given constants $-\infty < \alpha < \infty$, $0 < p, q < \infty$. A function $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ belongs to $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$ if

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \left[\sum_{v \in \mathbb{Z}} 2^{qv\alpha} \|\varphi_v * f\|_p^q \right]^{1/q} < \infty.$$

The following lemma is essentially known.

Lemma 2.3 ([22]). Let $\{\Phi_{j,k}^{\epsilon,1}\}_{(\epsilon,j,k) \in \Lambda_n}$ and $\{\Phi_{j,k}^{\epsilon,2}\}_{(\epsilon,j,k) \in \Lambda_n}$ be different wavelet bases which are sufficiently regular. If

$$a_{j,k,j',k'}^{\epsilon,\epsilon'} = \langle \Phi_{j,k}^{\epsilon,1}, \Phi_{j',k'}^{\epsilon',2} \rangle,$$

then for any natural number N there exists a positive constant C_N such that for $j, j' \in \mathbb{Z}$ and $k, k' \in \mathbb{Z}^n$,

$$|a_{j,k,j',k'}^{\epsilon,\epsilon'}| \leq C_N 2^{-|j-j'|(\frac{n}{2}+N)} \left(\frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |2^{-j}k - 2^{-j'}k'|} \right)^{n+N}. \quad (2.2)$$

According to Lemma 2.3 and Peetre's paper [25], we see that this definition of $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$ is independent of the choice of $\{\varphi_v\}_{v \in \mathbb{Z}}$, whence reaching the following description of $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$.

Theorem 2.4. Given $s \in \mathbb{R}$ and $0 < p, q < \infty$. A function f belongs to $\dot{B}_p^{s,q}(\mathbb{R}^n)$ if and only if

$$\left[\sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left(\sum_{\epsilon,k} |f_{j,k}^\epsilon|^p \right)^{q/p} \right]^{1/q} < \infty.$$

3. BESOV-Q SPACES VIA WAVELETS

3.1. Definition and wavelet formulation. The forthcoming Besov-Q spaces cover many important function spaces, for example, Besov spaces, Morrey spaces and Q-spaces and so on. Such spaces were first introduced by wavelets in Yang [38] and were studied by several authors. For a related overview, we refer to Yuan-Sickel-Yang [40].

Let $\varphi \in C_0^\infty(B(0, n))$ and $\varphi(x) = 1$ for $x \in B(0, \sqrt{n})$. Let $Q(x_0, r)$ be a cube parallel to the coordinate axis, centered at x_0 and with side length r . For simplicity, sometimes, we denote by $Q = Q(r)$ the cube $Q(x_0, r)$ and let $\varphi_Q(x) = \varphi(\frac{x-x_Q}{r})$. For $1 < p, q < \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$, let $m_0 = m_{p,q}^{\gamma_1,\gamma_2}$ be a positive constant large enough. For arbitrary function f , let $S_{p,q,f}^{\gamma_1,\gamma_2}$ be the class of the polynomial functions $P_{Q,f}$ such that

$$\int x^\alpha \varphi_Q(x) (f(x) - P_{Q,f}(x)) dx = 0 \quad \forall |\alpha| \leq m_0.$$

Definition 3.1. Given $1 < p, q < \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. We say that f belongs to the Besov-Q space $\dot{B}_{p,q}^{\gamma_1,\gamma_2} := \dot{B}_{p,q}^{\gamma_1,\gamma_2}(\mathbb{R}^n)$ provided

$$\sup_Q |Q|^{\frac{\gamma_2}{n} - \frac{1}{p}} \inf_{P_{Q,f} \in S_{p,q,f}^{\gamma_1,\gamma_2}} \|\varphi_Q(f - P_{Q,f})\|_{\dot{B}_p^{\gamma_1,q}} < \infty, \quad (3.1)$$

where the supremum is taken over all cubes Q with center x_Q and length r .

As a generalization of the Morrey spaces, the forthcoming Besov-Q spaces cover many important function spaces, for example, Besov spaces, Morrey spaces and Q-spaces and so on. Such spaces were first introduced by wavelets in Yang [38].

On the other hand, our Lemma 3.2 as below and Yang-Yuan’s [37, Theorem 3.1] show that our Besov-Q spaces and their Besov type spaces coincide; see also Liang-Sawano-Ullrich-Yang-Yuan [19] and Yuan-Sickel-Yang [40] for more information on the so-called Yang-Yuan’s spaces.

Given $1 < p, q < \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. Let $m_0 = m_{p,q}^{\gamma_1, \gamma_2}$ be a sufficiently big integer. For the regular Daubechies wavelets $\Phi^\epsilon(x)$, there exist two integers $m \geq m_0 = m_{p,q}^{\gamma_1, \gamma_2}$ and M such that for $\epsilon \in E_n$, $\Phi^\epsilon(x) \in C_0^m([-2^M, 2^M]^n)$ and $\int x^\alpha \Phi^\epsilon(x) dx = 0 \ \forall |\alpha| \leq m$. By applying the regular Daubechies wavelets, we have the following wavelet characterization for $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$.

Lemma 3.2. (i) $f = \sum_{\epsilon, j, k} a_{j,k}^\epsilon \Phi_{j,k}^\epsilon \in \dot{B}_{p,q}^{\gamma_1, \gamma_2}$ if and only if

$$\sup_Q |Q|^{\frac{\gamma_2}{n} - \frac{1}{p}} \left[\sum_{nj \geq -\log_2 |Q|} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left(\sum_{(\epsilon, k): Q_{j,k} \subset Q} |a_{j,k}^\epsilon|^p \right)^{q/p} \right]^{1/q} < +\infty, \quad (3.2)$$

where the supremum is taken over all dyadic cubes in \mathbb{R}^n .

(ii) The wavelet characterization in (i) is also true for the Meyer wavelets.

A direct application of Lemma 3.2 gives the following assertion.

Corollary 3.3. Given $1 < p, q < \infty$, $\gamma_1, \gamma_2 \in \mathbb{R}$.

- (i) Each $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$ is a Banach space.
- (ii) The definition of $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$ is independent of the choice of ϕ .

Now we recall some preliminaries on the Calderón-Zygmund operators (cf. [22, 23]). For $x \neq y$, let $K(x, y)$ be a smooth function such that there exists a sufficiently large $N_0 \leq m$ satisfying

$$|\partial_x^\alpha \partial_y^\beta K(x, y)| \lesssim |x - y|^{-(n+|\alpha|+|\beta|)} \quad \forall |\alpha| + |\beta| \leq N_0. \quad (3.3)$$

A linear operator

$$Tf(x) = \int K(x, y)f(y)dy$$

is said to be a Calderón-Zygmund one if it is continuous from $C^1(\mathbb{R}^n)$ to $(C^1(\mathbb{R}^n))'$, where the kernel $K(\cdot, \cdot)$ satisfies (3.3) and

$$Tx^\alpha = T^*x^\alpha = 0 \quad \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq N_0.$$

For such an operator, we write $T \in CZO(N_0)$.

The kernel $K(\cdot, \cdot)$ may have a high singularity on the diagonal $x = y$, so according to the Schwartz kernel theorem, it is only a distribution in $S'(\mathbb{R}^{2n})$. For $(\epsilon, j, k), (\epsilon', j', k') \in \Lambda_n$, let

$$a_{j,k,j',k'}^{\epsilon, \epsilon'} = \langle K(x, y), \Phi_{j,k}^\epsilon(x) \Phi_{j',k'}^{\epsilon'}(y) \rangle.$$

If T is a Calderón-Zygmund operator, then its kernel $K(\cdot, \cdot)$ and the related coefficients satisfy the following relations (cf. [22, 23, 38]).

Lemma 3.4. (i) If $T \in CZO(N_0)$, then the coefficients $a_{j,k,j',k'}^{\epsilon, \epsilon'}$ satisfy

$$|a_{j,k,j',k'}^{\epsilon, \epsilon'}| \lesssim \frac{\left(\frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |k2^{-j} - k'2^{-j'}|} \right)^{n+N_0}}{2^{|j-j'|(\frac{n}{2} + N_0)}} \quad \forall (\epsilon, j, k), (\epsilon', j', k') \in \Lambda_n. \quad (3.4)$$

(ii) If $a_{j,k,j',k'}^{\epsilon,\epsilon'}$ satisfy (3.4), then $K(\cdot, \cdot)$, the kernel of the operator T , can be written as

$$K(x, y) = \sum_{(\epsilon,j,k),(\epsilon',j',k') \in \Lambda_n} a_{j,k,j',k'}^{\epsilon,\epsilon'} \Phi_{j,k}^\epsilon(x) \Phi_{j',k'}^{\epsilon'}(y)$$

in the distribution sense. Moreover, T belongs to $CZO(N_0 - \delta)$ for any small positive number δ .

By the above lemma, we can prove the following result.

Corollary 3.5. For any $1/2 \leq \lambda \leq 2$ we have $\|f(\lambda \cdot)\|_{\dot{B}_{p,q}^{\gamma_1, \gamma_2}} \approx \|f\|_{\dot{B}_{p,q}^{\gamma_1, \gamma_2}}$.

3.2. Critical spaces and their inclusions.

Definition 3.6. An initial data space is called critical for (1.1), if it is invariant under the scaling $f_\lambda(x) = \lambda^{2\beta-1} f(\lambda x)$.

Note that, if $u(t, x)$ is a solution of (1.1) and we replace $u(t, x), p(t, x), a(x)$ by $u_\lambda(t, x) = \lambda^{2\beta-1} u(\lambda^{2\beta} t, \lambda x), p_\lambda(t, x) = \lambda^{4\beta-2} u(\lambda^{2\beta} t, \lambda x), a_\lambda(x) = \lambda^{2\beta-1} a(\lambda x)$ respectively, $u_\lambda(t, x)$ is also a solution of (1.1). So, the critical spaces occupy a significant place for (1.1). For $\beta = 1$,

$$\dot{L}_{-1}^2(\mathbb{R}^n) = \dot{B}_2^{-1+\frac{n}{2}, 2}(\mathbb{R}^n); \quad L^n(\mathbb{R}^n);$$

$$\dot{B}_p^{-1+\frac{n}{p}, \infty}(\mathbb{R}^n), p < \infty; \quad BMO^{-1}(\mathbb{R}^n); \quad \dot{B}_{2,2}^{\alpha-1, \alpha}(\mathbb{R}^n),$$

are critical spaces. For the general β ,

$$\dot{B}_p^{1+\frac{n}{p}-2\beta, \infty}(\mathbb{R}^n), p < \infty; \quad \dot{B}_{2,2}^{\alpha-\beta+1, \alpha+\beta}(\mathbb{R}^n),$$

are critical spaces.

By Corollary 3.5, it is easy to see that each $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$ enjoys following dilation-invariance. For $\beta > \frac{1}{2}$ and $\gamma_1 - \gamma_2 = 1 - 2\beta$, each $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$ is a critical space; i.e.,

$$\|\lambda^{\gamma_2-\gamma_1} f(\lambda \cdot)\|_{\dot{B}_{p,q}^{\gamma_1, \gamma_2}} \approx \|f\|_{\dot{B}_{p,q}^{\gamma_1, \gamma_2}} \quad \forall \lambda > 0.$$

To better understand why the Besov-Q spaces are larger than many spaces cited in the introduction, we should observe the basic fact below.

Lemma 3.7. Given $1 < p, q < \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$.

- (i) If $q_1 \leq q_2$, then $\dot{B}_{p,q_1}^{\gamma_1, \gamma_2} \subset \dot{B}_{p,q_2}^{\gamma_1, \gamma_2}$.
- (ii) $\dot{B}_{p,q}^{\gamma_1, \gamma_2} \subset \dot{B}_\infty^{\gamma_1-\gamma_2, \infty}(\mathbb{R}^n)$.
- (iii) Given $p_1 \geq 1$. For $w = 0, q_1 = 1$ or $w > 0, 1 \leq q_1 \leq \infty$, one has $\dot{B}_{p,q}^{\gamma_1, \gamma_2+w} \subset \dot{B}_{\frac{p}{p_1}, \frac{q}{q_1}}^{\gamma_1-w, \gamma_2}$.

For $0 \leq \alpha - \beta + 1$ and $\alpha + \beta - 1 \leq n/2$, we say that f belongs to the Q-type space $Q_\alpha^\beta(\mathbb{R}^n)$ provided

$$\sup_Q r^{2(\alpha+\beta-1)-n} \int_Q \int_Q \frac{|f(x) - f(y)|^2}{|x - y|^{n+2(\alpha-\beta+1)}} dx dy < \infty,$$

where the supremum is taken over all cubes with sidelength r . This definition was used in [18] to extend the results in [35] which initiated a PDE-analysis of the original Q-spaces introduced in [7] (cf. [5, 6, 26, 34, 38] for more information). The following is a direct consequence of Lemmas 3.2 and 3.7.

Corollary 3.8. (i) If $0 \leq \alpha - \beta + 1 < 1, \alpha + \beta - 1 \leq \frac{n}{2}$, then $Q_\alpha^\beta(\mathbb{R}^n) = \dot{B}_{2,2}^{\alpha-\beta+1,\alpha+\beta-1}$.

(ii) If $p = \frac{n}{\gamma_2}$, then $\dot{B}_{p,q}^{\gamma_1,\gamma_2} = \dot{B}_{p,q}^{\gamma_1,q}(\mathbb{R}^n)$.

(iii) Given $w = 0, v = 1$ or $w > 0, 1 \leq v \leq \infty$. If $p = n/(\gamma_2 + w)$, then

$$\dot{B}_p^{\gamma_1,q}(\mathbb{R}^n) \subset \dot{B}_{\frac{n}{u(w+\gamma_2)},\frac{q}{v}}^{\gamma_1-w,\gamma_2}.$$

Remark 3.9. Wu [31] obtained the well-posedness of (1.1) with an initial data in the critical Besov space $\dot{B}_p^{1+\frac{n}{p}-2\beta,q}(\mathbb{R}^n)$. Given $1 < p_0, q_0 < \infty$. By Lemma 3.7, we can see that if $1 < p \leq p_0, 1 < q \leq q_0$ and $\beta > 0$,

$$\dot{B}_p^{1+\frac{n}{p}-2\beta,q}(\mathbb{R}^n) \subset \dot{B}_{p_0,q_0}^{1+\frac{n}{p}-2\beta,\frac{n}{p}}.$$

4. BESOV-Q SPACES VIA SEMIGROUPS

To establish a semigroup characterization of the Besov-Q spaces, recall the following semigroup characterization of $Q_\alpha^\beta(\mathbb{R}^n)$, see [18]: Given $\max\{\alpha, 1/2\} < \beta < 1$ and $\alpha + \beta - 1 \geq 0$. $f \in Q_\alpha^\beta(\mathbb{R}^n)$ if and only if

$$\sup_{x \in \mathbb{R}^n, r \in (0, \infty)} r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x|<r} |\nabla e^{-t(-\Delta)^\beta} f(y)|^2 t^{-\frac{\alpha}{\beta}} dy dt < \infty.$$

This characterization was used to derive the global existence and uniqueness of a mild solution to (1.1) with a small initial data in $\nabla \cdot (Q_\alpha^\beta(\mathbb{R}^n))^n$. Notice that

$$Q_\alpha^\beta(\mathbb{R}^n) = \dot{B}_{2,2}^{\alpha+\beta-1,\alpha-\beta+1}.$$

So, to obtain the corresponding result of (1.1) with a small initial data in $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$ under $|p - 2| + |q - 2| \neq 0$, we need a more meticulous relation among time, frequency and locality. For this purpose, by the Meyer wavelets and the fractional heat semigroups, we introduce some new tent spaces associated with $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$, and then establish some connections between these tent spaces and $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$.

4.1. Wavelets and semigroups. For $\beta > 0$, let $\hat{K}_t^\beta(\xi) = e^{-t|\xi|^{2\beta}}$. We have

$$f(t, x) = e^{-t(-\Delta)^\beta} f(x) = K_t^\beta * f(x).$$

For the Meyer wavelets $\{\Phi_{j,k}^\epsilon\}_{(\epsilon,j,k) \in \Lambda_n}$, let $a_{j,k}^\epsilon(t) = \langle f(t, \cdot), \Phi_{j,k}^\epsilon \rangle$ and $a_{j,k}^\epsilon = \langle f, \Phi_{j,k}^\epsilon \rangle$. By Lemma 2.1 we obtain

$$f(x) = \sum_{\epsilon,j,k} a_{j,k}^\epsilon \Phi_{j,k}^\epsilon(x) \quad \text{and} \quad f(t, x) = \sum_{\epsilon,j,k} a_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x).$$

If $f(t, x) = K_t^\beta * f(x)$, then

$$\begin{aligned} a_{j,k}^\epsilon(t) &= \sum_{\epsilon',|j-j'|\leq 1,k'} a_{j',k'}^{\epsilon'} \langle K_t^\beta \Phi_{j',k'}^{\epsilon'}, \Phi_{j,k}^\epsilon \rangle \\ &= \sum_{\epsilon',|j-j'|\leq 1,k'} a_{j',k'}^{\epsilon'} \int e^{-t2^{2j\beta}|\xi|^{2\beta}} \hat{\Phi}^{\epsilon'}(2^{j-j'}\xi) \hat{\Phi}^\epsilon(\xi) e^{-i(2^{j-j'}k'-k)\xi} d\xi. \end{aligned} \tag{4.1}$$

Lemma 4.1. *Let $\{\Phi_{j,k}^\epsilon\}_{(\epsilon,j,k)\in\Lambda_n}$ be Meyer wavelets. For $\beta > 0$ there exist a large constant $N_\beta > 0$ and a small constant $\tilde{c} > 0$ such that if $N > N_\beta$ then*

$$|a_{j,k}^\epsilon(t)| \lesssim e^{-\tilde{c}t2^{2j\beta}} \sum_{\epsilon', |j-j'|\leq 1, k'} |a_{j',k'}^{\epsilon'}| (1 + |2^{j-j'}k' - k|)^{-N} \quad \forall t2^{2j\beta} \geq 1 \quad (4.2)$$

and

$$|a_{j,k}^\epsilon(t)| \lesssim \sum_{|j-j'|\leq 1} \sum_{\epsilon', k'} |a_{j',k'}^{\epsilon'}| (1 + |2^{j-j'}k' - k|)^{-N} \quad \forall 0 < t2^{2j\beta} \leq 1. \quad (4.3)$$

Proof. Formally, we can write

$$\begin{aligned} a_{j,k}^\epsilon(t) &= \sum_{\epsilon', |j-j'|\leq 1, k'} a_{j',k'}^{\epsilon'} \langle K_t^\beta * \Phi_{j',k'}^{\epsilon'}, \Phi_{j,k}^\epsilon \rangle \\ &= \sum_{\epsilon', |j-j'|\leq 1, k'} a_{j',k'}^{\epsilon'} \langle e^{-t(-\Delta)^\beta} \Phi_{j',k'}^{\epsilon'}, \Phi_{j,k}^\epsilon \rangle \\ &= \sum_{\epsilon', |j-j'|\leq 1, k'} a_{j',k'}^{\epsilon'} \int e^{-t2^{2j\beta}|\xi|^{2\beta}} \widehat{\Phi}^{\epsilon'}(2^{j-j'}\xi) \widehat{\Phi}^\epsilon(\xi) e^{-i(2^{j-j'}k'-k)\xi} d\xi. \end{aligned}$$

We divide the rest of the argument into two cases.

Case 1: $|2^{j-j'}k' - k| \leq 2$. Notice that $\widehat{\Phi}^\epsilon$ is supported on a ring. By a direct computation, we obtain

$$|a_{j,k}^\epsilon(t)| \lesssim \sum_{\epsilon', |j-j'|\leq 1, k'} |a_{j',k'}^{\epsilon'}| e^{-t2^{2j\beta}} \lesssim \sum_{\epsilon', |j-j'|\leq 1, k'} \frac{|a_{j',k'}^{\epsilon'}|}{(1 + |2^{j-j'}k' - k|)^N} e^{-t2^{2j\beta}}.$$

Case 2: $|2^{j-j'}k' - k| \geq 2$. Denote by l_{i_0} the largest component of $2^{j-j'}k' - k$. Then $(1 + |l_{i_0}|)^N \sim (1 + |2^{j-j'}k' - k|)^N$. We have

$$\begin{aligned} a_{j,k}^\epsilon(t) &= \sum_{\epsilon', |j-j'|\leq 1, k'} \frac{a_{j',k'}^{\epsilon'}}{(l_{i_0})^N} \int e^{-t2^{2j\beta}|\xi|^{2\beta}} \widehat{\Phi}^{\epsilon'}(2^{j-j'}\xi) \widehat{\Phi}^\epsilon(\xi) \left[\left(\frac{-1}{i} \partial_{\xi_{i_0}} \right)^N e^{-i(2^{j-j'}k'-k)\xi} \right] d\xi. \end{aligned}$$

By an integration-by-parts, we can obtain that if C_N^l is the binomial coefficient indexed by N and l then

$$\begin{aligned} |a_{j,k}^\epsilon(t)| &= \left| \sum_{\epsilon', |j-j'|\leq 1, k'} (-1)^N \frac{a_{j',k'}^{\epsilon'}}{(l_{i_0})^N} \int \sum_{l=0}^N C_N^l \partial_{\xi_{i_0}}^l (e^{-t2^{2j\beta}|\xi|^{2\beta}}) \right. \\ &\quad \times \left. \partial_{\xi_{i_0}}^{N-l} (\widehat{\Phi}^{\epsilon'}(2^{j-j'}\xi) \widehat{\Phi}^\epsilon(\xi)) e^{-i(2^{j-j'}k'-k)\xi} d\xi \right| \\ &\lesssim \sum_{\epsilon', |j-j'|\leq 1, k'} \frac{|a_{j',k'}^{\epsilon'}|}{(1 + |2^{j-j'}k' - k|)^N} \left| \int \sum_{l=0}^N C_N^l (-t2^{2j\beta}|\xi|^{2\beta})^l |\xi|^{2\beta-2} \xi_{i_0} \right. \\ &\quad \times \left. e^{-t2^{2j\beta}|\xi|^{2\beta}} \partial_{\xi_{i_0}}^{N-l} (\widehat{\Phi}^{\epsilon'}(2^{j-j'}\xi) \widehat{\Phi}^\epsilon(\xi)) e^{-i(2^{j-j'}k'-k)\xi} d\xi \right|. \end{aligned}$$

If $t2^{2j\beta} \geq 1$, there exists a constant c such that $(t2^{2j\beta})^l e^{-t2^{2j\beta}} \lesssim e^{-ct2^{2j\beta}}$. Since $\Phi^{\epsilon'}$ is defined on a ring, we obtain

$$|a_{j,k}^\epsilon(t)| \lesssim \sum_{\epsilon', |j-j'|\leq 1, k'} \frac{|a_{j',k'}^{\epsilon'}|}{(1 + |2^{j-j'}k' - k|)^N} (t2^{2j\beta})^l e^{-t2^{2j\beta}}$$

$$\lesssim \sum_{\epsilon', |j-j'| \leq 1, k'} e^{-ct2^{2j\beta}} \frac{|a_{j',k'}^{\epsilon'}|}{(1 + |2^{j-j'}k' - k|)^N}.$$

If $0 < t2^{2j\beta} \leq 1$, then we can deduce directly that

$$|a_{j,k}^\epsilon(t)| \lesssim \sum_{\epsilon', |j-j'| \leq 1, k'} \frac{|a_{j',k'}^{\epsilon'}|}{(1 + |2^{j-j'}k' - k|)^N}.$$

□

4.2. Tent spaces generated by Besov-Q spaces. Over \mathbb{R}_+^{1+n} we introduce a new tent type space $\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}$ associated with $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$, and then establish a relation between $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$ and $\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}$ via the fractional heat semigroup $e^{-t(-\Delta)^\beta}$.

Definition 4.2. Let $\gamma_1, \gamma_2, m \in \mathbb{R}, m' > 0, 1 < p, q < \infty$ and

$$a(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x).$$

We say that: (i) $f \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I}$ if $\sup_{t \geq 0} \sup_{x_{0,r}} I_{p,q,Q_r,m}^{\gamma_1,\gamma_2}(t) < \infty$, where

$$I_{p,q,Q_r,m}^{\gamma_1,\gamma_2}(t) =: |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\sum_{(\epsilon, k): Q_{j,k} \subset Q_r} |a_{j,k}^\epsilon(t)|^p (t2^{2j\beta})^m \right]^{q/p};$$

(ii) $f \in \mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II}$ if $\sup_{t \geq 0} \sup_{x_{0,r}} II_{p,q,Q_r}^{\gamma_1,\gamma_2}(t) < \infty$, where

$$II_{p,q,Q_r}^{\gamma_1,\gamma_2}(t) =: |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\sum_{(\epsilon, k): Q_{j,k} \subset Q_r} |a_{j,k}^\epsilon(t)|^p \right]^{q/p};$$

(iii) $f \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III}$ if $\sup_{x_{0,r}} III_{p,q,Q_r,m}^{\gamma_1,\gamma_2} < \infty$, where

$$III_{p,q,Q_r,m}^{\gamma_1,\gamma_2} = |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \times \left(\int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon, k): Q_{j,k} \subset Q_r} |a_{j,k}^\epsilon(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right)^{q/p};$$

(iv) $f \in \mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}$ if $\sup_{x_{0,r}} IV_{p,q,Q_r,m'}^{\gamma_1,\gamma_2} < \infty$, where

$$IV_{p,q,Q_r,m'}^{\gamma_1,\gamma_2} =: |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \times \left(\int_0^{2^{-2j\beta}} \sum_{(\epsilon, k): Q_{j,k} \subset Q_r} |a_{j,k}^\epsilon(t)|^p (t2^{2j\beta})^{m'} \frac{dt}{t} \right)^{q/p}.$$

Moreover, the associated tent type spaces are defined as

$$\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2} = \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I} \cap \mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II} \cap \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III} \cap \mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}.$$

To continue our discussion, we need to introduce two more function spaces $\mathbb{B}_{\tau,\infty}^\gamma$ and $\mathbb{B}_{0,\infty}^{\gamma_1}$.

Definition 4.3. For $(\epsilon, j, k) \in \Lambda_n$, write $a_{j,k}^\epsilon(t) = \langle a(t, \cdot), \Phi_{j,k}^\epsilon(\cdot) \rangle$. Given $\tau > 0$ and $\gamma \in \mathbb{R}$. We say that

$$a(\cdot, \cdot) \in \mathbb{B}_{\tau, \infty}^\gamma \quad \text{if} \quad \sup_{t2^{2j\beta} \geq 1} (t2^{2j\beta})^\tau 2^{\frac{nj}{2}} 2^{j\gamma} |a_{j,k}^\epsilon(t)| + \sup_{0 < t2^{2j\beta} < 1} 2^{\frac{nj}{2}} 2^{j\gamma} |a_{j,k}^\epsilon(t)| < \infty;$$

$$a(\cdot, \cdot) \in \mathbb{B}_{0, \infty}^\gamma \quad \text{if} \quad t^{\frac{-\gamma}{2\beta}} 2^{\frac{nj}{2}} |\langle a(t, \cdot), \Phi_{j,k}^0 \rangle| < \infty.$$

It is easy to verify the following inclusions.

Lemma 4.4. Given $1 < p, q < \infty$, $\gamma_1, \gamma_2 \in \mathbb{R}$, $m > p$ and $m', \tau > 0$.

- (i) If $m > 0$, then $\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2} \subset \mathbb{B}_{\frac{m}{p}, \infty}^{\gamma_1 - \gamma_2}$.
- (ii) If $-2\beta\tau < \gamma < 0 < \beta$, then $\mathbb{B}_{\tau, \infty}^\gamma \subset \mathbb{B}_{0, \infty}^\gamma$.

For convenience, for any dyadic cube Q_{j_0, k_0} , we always use \tilde{Q}_{j_0, k_0} to denote the dyadic cube containing Q_{j_0, k_0} with side length 2^{8-j_0} . Given $(\epsilon, j, k) \in \Lambda_n$. If $\epsilon \in E_n$ and $Q_{j,k} \subset Q_{j_0, k_0}$, we write $(\epsilon, k) \in S_{j_0, k_0}^j$. For any $w \in \mathbb{Z}^n$, denote $\tilde{Q}_{j_0, k_0}^w = 2^{8-j_0}w + \tilde{Q}_{j_0, k_0}$. Denote $(\epsilon, k) \in S_{j_0, k_0}^{w, j}$ whenever $Q_{j,k} \subset \tilde{Q}_{j_0, k_0}^w$. Furthermore, we frequently utilize the so-called α -triangle inequality:

$$(a + b)^\alpha \leq a^\alpha + b^\alpha \quad \forall (\alpha, a, b) \in (0, 1] \times (0, \infty) \times (0, \infty).$$

Now we characterize the Besov-Q spaces by using a semigroup operator.

Theorem 4.5. Given $1 < p < m < \infty$, $m' > 0$, $1 < q < \infty$, $\gamma_1 - \gamma_2 < 0 < \beta$. If $f \in \dot{B}_{p,q}^{\gamma_1, \gamma_2}$, then $f * K_t^\beta \in \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$.

Proof. We will prove

$$f = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^\epsilon \Phi_{j,k}^\epsilon \in \dot{B}_{p,q}^{\gamma_1, \gamma_2} \implies f * K_t^\beta = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon \in \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$$

via handling four situations.

Situation 1: $K_t^\beta * f \in \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, I}$. For $t2^{2j\beta} > 1, m > 0$, by (4.2), there exists a constant N large enough such that

$$|a_{j,k}^\epsilon(t)| \lesssim e^{-\tilde{c}t2^{2j\beta}} \sum_{\epsilon', |j-j'| \leq 1, k'} \frac{|a_{j',k'}^{\epsilon'}|}{(1 + |2^{j-j'}k' - k|)^N} \lesssim 2^{\frac{-nj}{2}} 2^{j(\gamma_2 - \gamma_1)} e^{-\tilde{c}t2^{2j\beta}}.$$

Choosing a sufficiently large N' (depending on N) in the last estimate we have

$$I_{p,q,Q_r,m}^{\gamma_1, \gamma_2}(t) \lesssim |Q_r|^{\frac{q\gamma_2 - q}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})}$$

$$\times \left[\sum_{(\epsilon, k) \in S_r^j} e^{-\tilde{c}t2^{2j\beta}} \left(\sum_{\epsilon', |j-j'| \leq 1, k'} \frac{|a_{j',k'}^{\epsilon'}|}{(1 + |2^{j-j'}k' - k|)^N} \right)^p (t2^{2j\beta})^m \right]^{q/p},$$

where $p > 1$ has been used. In the sequel, denote by $S_r^{w, j'}$ the set of all indexes (ϵ', k') ensuring $Q_{j',k'} \subset Q_r^w$, and divide the proof into two cases.

Case 1.1: $q \leq p$. Because $|j - j'| \leq 1$ and $j > -\log_2 r$, one gets $2^{-(j'+1)n} \leq r^n$. This implies that $(2^{nj'}|Q|)^{-N'} \lesssim 1$. Hence

$$I_{p,q,Q_r,m}^{\gamma_1, \gamma_2}(t)$$

$$\begin{aligned}
 &\lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1 + |w|)^{Nq/p}} \sum_{j' \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{jq'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\sum_{(\epsilon', k') \in S_r^{w, j'}} |a_{j', k'}^{\epsilon'}|^p \right]^{q/p} \\
 &\lesssim \|f\|_{\dot{B}_{p, q}^{\gamma_1, \gamma_2}}. \\
 &\text{Case 1.2: } q > p. \text{ Applying Hölder's inequality to } w \in \mathbb{Z}^n, \text{ we similarly have} \\
 &II_{p, q, Q_r, m}^{\gamma_1, \gamma_2}(t) \\
 &\lesssim \sum_{w \in \mathbb{Z}^n} \left\{ |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j' \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{jq'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\sum_{(\epsilon', k') \in S_r^{w, j'}} |a_{j', k'}^{\epsilon'}|^p \right]^{q/p} \right\} \\
 &\lesssim \|f\|_{\dot{B}_{p, q}^{\gamma_1, \gamma_2}}.
 \end{aligned}$$

Situation 2: $K_t^\beta * f \in \mathbb{B}_{p, q}^{\gamma_1, \gamma_2, II}$. For $t2^{2\beta j} \leq 1$ and $m' > 0$, by (4.3), there exists a natural number N large enough such that $N > 2n$ and

$$|a_{j, k}^\epsilon(t)| \lesssim \sum_{\epsilon', |j-j'| \leq 1, k'} |a_{j', k'}^{\epsilon'}| (1 + |2^{j-j'} k' - k|)^{-N}.$$

Consequently,

$$\begin{aligned}
 II_{p, q, Q_r}^{\gamma_1, \gamma_2}(t) &\lesssim |Q|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r \leq j \leq -\frac{\log_2 t}{2\beta}} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
 &\quad \times \left[\sum_{(\epsilon, k) \in S_r^j} \left(\sum_{\epsilon', |j-j'| \leq 1, k'} \frac{|a_{j', k'}^{\epsilon'}|}{(1 + |2^{j-j'} k' - k|)^N} \right)^p \right]^{q/p}.
 \end{aligned}$$

Case 2.1: $q \leq p$. Notice that $(2^{nj'}|Q|)^{-N} \lesssim 1$. We have

$$\begin{aligned}
 &II_{p, q, Q_r}^{\gamma_1, \gamma_2}(t) \\
 &\lesssim \sum_{|w| \in \mathbb{Z}^n} \frac{|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1 + |w|)^{qN/p}} \sum_{-\log_2 r - 1 \leq j' \leq -\frac{\log_2 t}{2\beta} - 1} 2^{j'q(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left(\sum_{(\epsilon', k') \in S_r^{w, j'}} |a_{j', k'}^{\epsilon'}|^p \right)^{q/p} \\
 &\lesssim \|f\|_{\dot{B}_{p, q}^{\gamma_1, \gamma_2}}.
 \end{aligned}$$

Case 2.2: $q > p$. By Hölder's inequality, in a similar manner, we can obtain

$$\begin{aligned}
 &II_{p, q, Q_r}^{\gamma_1, \gamma_2}(t) \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r - 1 \leq j' \leq -\frac{\log_2 t}{2\beta} - 1} 2^{j'q(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
 &\quad \times \left[\sum_{(\epsilon', k') \in S_r^{w, j'}} |a_{j', k'}^{\epsilon'}|^p (1 + |2^{j-j'} k' - k|)^{-N'} \right]^{q/p} \\
 &\lesssim \|f\|_{\dot{B}_{p, q}^{\gamma_1, \gamma_2}}.
 \end{aligned}$$

Situation 3: $K_t^\beta * f \in \mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}$. For this case we have $2^{-2j\beta} < t < r^{2\beta}$ and thus

$$|a_{j, k}^\epsilon(t)| \lesssim e^{-ct2^{2j\beta}} \sum_{\epsilon', |j-j'| \leq 1, k'} \frac{|a_{j', k'}^{\epsilon'}|}{(1 + |2^{j-j'} k' - k|)^{N'}} \lesssim 2^{-\frac{nj}{2}} 2^{j(\gamma_2 - \gamma_1)} (t2^{2j\beta})^{-\tau}.$$

This yields

$$III_{p, q, Q_r, m}^{\gamma_1, \gamma_2} \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\int_{2^{-2j\beta}}^{r^{2\beta}} (t2^{2j\beta})^m \right]$$

$$\times \sum_{(\epsilon, k) \in S_r^j} e^{-cpt2^{2j\beta}} \left(\sum_{\epsilon', |j-j'| \leq 3, k'} |a_{j',k'}^{\epsilon'}| (1 + |2^{j-j'}k' - k|)^{-N} \right)^p \frac{dt}{t} \Big]^{q/p}$$

Notice that $j \sim j'$ and the number of ϵ' is finite. Applying Hölder's inequality on k' we obtain

$$\left[\sum_{\epsilon', |j-j'| \leq 3, k'} \frac{|a_{j',k'}^{\epsilon'}|}{(1 + |2^{j-j'}k' - k|)^N} \right]^p \lesssim \sum_{|j-j'| \leq 3} \sum_{\epsilon', k'} \frac{|a_{j',k'}^{\epsilon'}|^p}{(1 + |2^{j-j'}k' - k|)^N}.$$

Let $Q_{j,k}$ and $Q_{j',k'}$ be two dyadic cubes. Denote by $\tilde{Q}_{j,k}$ the dyadic cube containing $Q_{j,k}$ with side length 2^{8-j} . For $w \in \mathbb{Z}^n$, denote by $Q_{j,k}^w$ the cube $\tilde{Q}_{j,k} + 2^{8-j}w$. It is easy to see that if $Q_{j',k'} \subset Q_{j,k}^w$, then

$$(1 + |2^{j-j'}k' - k|)^{-N} \lesssim (1 + |w|)^{-N}$$

(see also (5.2)). We obtain that

$$\begin{aligned} III_{p,q,Q_r,m}^{\gamma_1,\gamma_2} &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j' \geq -\log_2 r} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\int_{2^{-2j\beta}}^{r^{2\beta}} (t2^{2j\beta})^m \sum_{(\epsilon,k) \in S_r^j} e^{-cpt2^{2j\beta}} \right. \\ &\quad \left. \times \sum_{|j-j'| \leq 3} \sum_{w \in \mathbb{Z}^n} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |a_{j',k'}^{\epsilon'}|^p (1 + |2^{j-j'}k' - k|)^{-N} \frac{dt}{t} \right]^{q/p}. \end{aligned}$$

The number of $Q_{j',k'}$ which are contained in the dyadic cube $Q_{j,k}^w = 2^{8-j}w + \tilde{Q}_{j,k}$ equals to $2^{n(8+j'-j)}$. On the other hand, for any dyadic cube Q_r with radius r , the number of $Q_{j,k} \subset Q_r$ equals to $(2^j r)^n$. Then the number of $Q_{j',k'}$, which are contained in the dyadic cube Q_r^w , equals $(2^{8+j'}r)^n$. Finally we have

$$\begin{aligned} III_{p,q,Q_r,m}^{\gamma_1,\gamma_2} &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j' \geq -\log_2 r - 3} 2^{j'q(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ \int_{2^{-2j'\beta}}^{r^{2\beta}} (t2^{2j'\beta}) \right. \\ &\quad \left. \times \left[\sum_{|w| \leq 2^n} + \sum_{|w| > 2^n} \right] (1 + |w|)^{-N} \sum_{(\epsilon',k') \in S_r^{w,j'}} e^{-ct2^{2j'\beta}} |a_{j',k'}^{\epsilon'}|^p \frac{dt}{t} \right\}^{q/p} \\ &=: M_1 + M_2. \end{aligned}$$

By the definition of $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$, it is easy to see that $M_1 \lesssim \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}$. For the term M_2 , we divide the estimate into two cases.

Case 3.1: $q \leq p$. For this case, $j' \geq -\log_2 r - 3$ implies $(2^{n j'} r^n)^{N'} \lesssim 1$ and

$$\begin{aligned} M_2 &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j' \geq -\log_2 r - 3} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN'}{p}} \\ &\quad \times \left[\int_{2^{-2j'\beta}}^{r^{2\beta}} (t2^{2j'\beta})^m \sum_{(\epsilon',k') \in S_r^{w,j'}} e^{-cpt2^{2j'\beta}} |a_{j',k'}^{\epsilon'}|^p (2^{nj'}|Q|)^{-N'} \frac{dt}{t} \right]^{q/p} \\ &\lesssim \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}. \end{aligned}$$

Case 3.2: $q > p$. For this case, by Hölder's inequality and $j \sim j'$ we obtain

$$\left[\int_{2^{-2j'\beta}}^{r^{2\beta}} (t2^{2j'\beta})^m \sum_{|w| > 2^n} \sum_{(\epsilon',k') \in S_r^{w,j'}} e^{-cpt2^{2j'\beta}} (1 + |w|)^{-N} |a_{j',k'}^{\epsilon'}|^p (2^{nj'}|Q|)^{-N'} \frac{dt}{t} \right]^{q/p}$$

$$\lesssim \sum_{|w|>2^n} (1+|w|)^{-\frac{qN'}{p}} \left(\int_{2^{-2j\beta}}^{r^{2\beta}} (t2^{2j'\beta})^m \sum_{(\epsilon',k') \in S_r^{w,j'}} e^{-cpt2^{2j'\beta}} |a_{j',k'}^{\epsilon'}|^p \frac{dt}{t} \right)^{q/p}.$$

The rest of the argument is similar to that of Case 3.1, and so omitted.

Situation 4: $K_t^\beta * f \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,IV}$. Because $|j - j'| \leq 1$ and $0 < t < 2^{-2j\beta}$, we can obtain

$$\begin{aligned} & IV_{p,q,Q_r,m'}^{\gamma_1,\gamma_2} \\ & \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\sum_{|w| \leq 2^n} \frac{1}{(1+|w|)^N} \sum_{(\epsilon',k') \in S_r^{w,j'}} |a_{j',k'}^{\epsilon'}|^p \right]^{q/p} \\ & + |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\sum_{|w| > 2^n} \frac{1}{(1+|w|)^N} \sum_{(\epsilon',k') \in S_r^{w,j'}} \frac{|a_{j',k'}^{\epsilon'}|^p}{(2^{nj'}|Q|)^{N'}} \right]^{q/p}. \end{aligned}$$

Case 4.1: $q \leq p$. For this case, by the α -triangle inequality we have

$$\begin{aligned} IV_{p,q,Q_r,m'}^{\gamma_1,\gamma_2} & \lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1+|w|)^{\frac{qN}{p}}} \sum_{j' \geq -\log_2 r - 1} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left(\sum_{(\epsilon',k') \in S_r^{w,j'}} |a_{j',k'}^{\epsilon'}|^p \right)^{q/p} \\ & \lesssim \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}. \end{aligned}$$

Case 4.2: $q > p$. Using Hölder's inequality we have

$$\begin{aligned} IV_{p,q,Q_r,m'}^{\gamma_1,\gamma_2} & \lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1+|w|)^N} \sum_{j' \geq -\log_2 r - 1} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left(\sum_{(\epsilon',k') \in S_r^{w,j'}} |a_{j',k'}^{\epsilon'}|^p \right)^{q/p} \\ & \lesssim \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}. \end{aligned}$$

This completes the proof. □

We close this section by showing the following continuity of the Riesz transforms acting on the Besov-Q spaces; see also [2] and [38] for some related results.

Theorem 4.6. For $1 < p, q < \infty$, $\gamma_1, \gamma_2 \in \mathbb{R}$, $m > p$, and $m' > 0$, the Riesz transforms R_1, R_2, \dots, R_n are continuous on $\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}$.

Proof. For convenience, we choose the classical Meyer wavelet basis $\{\Phi_{j,k}^\epsilon\}_{(\epsilon,j,k) \in \Lambda_n}$. For any $g(\cdot, \cdot) \in \mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}$ and $l = 1, 2, \dots, n$, we need to prove $(R_l g)(\cdot, \cdot) \in \mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}$. Write $g(t, x) = \sum_{(\epsilon,j,k) \in \Lambda_n} g_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x)$. Then

$$R_l g(t, x) = \sum_{(\epsilon,j,k) \in \Lambda_n} g_{j,k}^\epsilon(t) R_l \Phi_{j,k}^\epsilon(x) =: \sum_{(\epsilon,j,k) \in \Lambda_n} b_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x),$$

where $b_{j,k}^\epsilon(t)$ is defined by

$$b_{j,k}^\epsilon(t) = \sum_{(\epsilon',j',k') \in \Lambda_n} g_{j',k'}^{\epsilon'}(t) \langle R_l \Phi_{j',k'}^{\epsilon'}, \Phi_{j,k}^\epsilon \rangle =: \sum_{|j-j'| \leq 1} \sum_{\epsilon',k'} a_{j,k,j',k'}^{\epsilon,\epsilon'} g_{j',k'}^{\epsilon'}(t).$$

Because R_l is a Calderón-Zygmund operator, by (3.4) we obtain

$$|a_{j,k,j',k'}^{\epsilon,\epsilon'}| \lesssim 2^{-|j-j'|(\frac{n}{2} + N_0)} \left(\frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |2^{-j}k - 2^{-j'}k'|} \right)^{n+N_0}.$$

The rest of the proof is similar to Theorem 4.5. We omit the details. □

5. NONLINEAR TERMS AND THEIR A PRIORI ESTIMATES

5.1. **Decompositions of non-linear terms.** For the rest of this article, let

$$u(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} u_{j, k}^\epsilon(t) \Phi_{j, k}^\epsilon(x); \quad v(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} v_{j, k}^\epsilon(t) \Phi_{j, k}^\epsilon(x).$$

For $l = 1, \dots, n$, we will derive some inequalities about

$$B_l(u, v)(t, x) = \int_0^t e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (uv) ds.$$

Here, it is worth pointing out that (2.1) gives

$$e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (uv)(s, t, x) = \sum_{j' \in \mathbb{Z}} \sum_{i=1}^4 I_{j'}^{i, l}(s, t, x),$$

where

$$\begin{aligned} & I_{j'}^{1, l}(u, v)(s, t, x) \\ &= \sum_{\epsilon', k'} \sum_{k''} u_{j', k'}^{\epsilon'}(s) v_{j'-3, k''}^0(s) e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'-3, k''}^0(x)), \\ & I_{j'}^{2, l}(u, v)(s, t, x) \\ &= \sum_{\epsilon', k'} \sum_{\epsilon'', k''} u_{j', k'}^{\epsilon'}(s) v_{j', k''}^{\epsilon''}(s) e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j', k''}^{\epsilon''}(x)), \\ & I_{j'}^{3, l}(u, v)(s, t, x) \\ &= \sum_{0 < |j' - j''| \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)), \\ & I_{j'}^{4, l}(u, v)(s, t, x) \\ &= \sum_{\epsilon', k'} \sum_{k''} v_{j', k'}^{\epsilon'}(s) u_{j'-3, k''}^0(s) e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'-3, k''}^0(x)). \end{aligned}$$

Hence

$$B_l(u, v)(t, x) =: \int_0^t \sum_{j' \in \mathbb{Z}} \sum_{i=1}^4 I_{j'}^{i, l}(s, t, x) ds =: \sum_{i=1}^4 \int_0^t I_l^i(s, t, x) ds.$$

Therefore, we can write

$$B_l(u, v)(t, x) = \sum_{i=1}^4 I_l^i(u, v)(t, x), \quad (5.1)$$

where

$$I_l^i(u, v)(t, x) = \int_0^t I_l^i(s, t, x) ds.$$

To estimate the bilinear term $B(u, v)$ in some suitable function spaces on \mathbb{R}_+^{1+n} , we are required to decompose the terms $I_l^i(u, v)(t, x)$, $i = 1, 2, \dots, 4$, respectively.

Decomposition of $I_l^1(u, v)(t, x)$. The term $I_l^1(u, v)(t, x)$ is decomposed according to two cases.

Case $[I_l^1]_1$: $t \geq 2^{-2j\beta}$. For this case, we write $I_l^1(u, v)(t, x)$ as the sum of the following three terms:

$$\begin{aligned} I_l^1(u, v)(t, x) &= \sum_{\epsilon', j', k'} \sum_{k''} \left(\int_0^{2^{-1-2j'\beta}} + \int_{2^{-1-2j'\beta}}^{t/2} + \int_{t/2}^t \right) \\ &\quad \times \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'-3, k''}^0(s) e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'-3, k''}^0(x)) \right\} ds \\ &=: I_l^{1,1}(u, v)(t, x) + I_l^{1,2}(u, v)(t, x) + I_l^{1,3}(u, v)(t, x). \end{aligned}$$

For $i = 1, 2, 3$, denote

$$I_l^{1,i}(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, i}(t) \Phi_{j, k}^\epsilon(x).$$

Case $[I_l^1]_2$: $t < 2^{-2j\beta}$. For this case, we denote $a_{j, k}^{\epsilon, 4}(t) = a_{j, k}^\epsilon(t)$ and then have

$$I_l^1(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, 4}(t) \Phi_{j, k}^\epsilon(x).$$

Decomposition of $I_l^2(u, v)(t, x)$. The decomposition of $I_l^2(u, v)(t, x)$ is made according to two cases.

Case $[I_l^2]_1$: $t \geq 2^{-2j\beta}$. Naturally, $I_l^2(u, v)(t, x)$ can be divided into the following three terms:

$$\begin{aligned} I_l^2(u, v)(t, x) &= \sum_{j'} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \left(\int_0^{2^{-1-2j'\beta}} + \int_{2^{-1-2j'\beta}}^{t/2} + \int_{t/2}^t \right) \\ &\quad \times \left\{ u_{j', k'}^{\epsilon'}(s) v_{j', k''}^{\epsilon''}(s) e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j', k''}^{\epsilon''}(x)) \right\} ds \\ &=: I_l^{2,1}(u, v)(t, x) + I_l^{2,2}(u, v)(t, x) + I_l^{2,3}(u, v)(t, x). \end{aligned}$$

Case $[I_l^2]_2$: $t < 2^{-2j\beta}$. This $I_l^2(u, v)(t, x)$ can be decomposed into the sum of $II^4(u, v)(t, x)$ and $II^5(u, v)(t, x)$, where

$$\begin{aligned} I_l^2(u, v)(t, x) &= \sum_{j'} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \left(\int_0^{2^{-2j'\beta}} + \int_{2^{-2j'\beta}}^t \right) \\ &\quad \times \left\{ u_{j', k'}^{\epsilon'}(s) v_{j', k''}^{\epsilon''}(s) e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j', k''}^{\epsilon''}(x)) \right\} ds \\ &=: I_l^{2,4}(u, v)(t, x) + I_l^{2,5}(u, v)(t, x). \end{aligned}$$

For $i = 1, 2, 3, 4, 5$, set

$$I_l^{2,i}(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, i}(t) \Phi_{j, k}^\epsilon(x).$$

Decompositions of $I_l^3(u, v)(t, x)$. Similarly, we have the following two cases:

Case $[I_l^3]_1$: $t \geq 2^{-2j\beta}$. This $I_l^3(u, v)(t, x)$ can be divided into the following three terms:

$$\begin{aligned} I_l^3(u, v)(t, x) &= \sum_{0 < |j' - j''| \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \left(\int_0^{2^{-1-2j'\beta}} + \int_{2^{-1-2j'\beta}}^{t/2} + \int_{t/2}^t \right) \\ &\quad \times \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)) \right\} ds \\ &=: I_l^{3,1}(u, v)(t, x) + I_l^{3,2}(u, v)(t, x) + I_l^{3,3}(u, v)(t, x). \end{aligned}$$

Case $[I_l^3]_2$: $t < 2^{-2j\beta}$. This $I_l^3(u, v)(t, x)$ can be decomposed into the sum of $II^4(u, v)(t, x)$ and $II^5(u, v)(t, x)$, where

$$\begin{aligned} I_l^3(u, v)(t, x) &= \sum_{0 < |j' - j''| \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \left(\int_0^{2^{-2j'\beta}} + \int_{2^{-2j'\beta}}^t \right) \\ &\quad \times \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)) \right\} ds \\ &=: I_l^{3,4}(u, v)(t, x) + I_l^{3,5}(u, v)(t, x). \end{aligned}$$

For $i = 1, 2, 3, 4, 5$, denote

$$I_l^{3,i}(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, i}(t) \Phi_{j, k}^\epsilon(x).$$

Decomposition of $I_j^{4,l}(u, v)(t, x)$. It is easy to see that the terms $I_j^{1,l}(u, v)(t, x)$ and $I_j^{4,l}(u, v)(t, x)$ are symmetric associated with $u(t, x)$ and $v(t, x)$. Hence for $I_l^4(u, v)$ we have a similar decomposition.

Case $[I_l^4]_1$: $t \geq 2^{-2j\beta}$. For this case, we write $I_l^4(u, v)(t, x)$ as the sum of the following three terms:

$$\begin{aligned} I_l^4(u, v)(t, x) &= \sum_{\epsilon', j', k'} \sum_{k''} \left(\int_0^{2^{-1-2j'\beta}} + \int_{2^{-1-2j'\beta}}^{t/2} + \int_{t/2}^t \right) \\ &\quad \times \left\{ v_{j', k'}^{\epsilon'}(s) u_{j''-3, k''}^0(s) e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j''-3, k''}^0(x)) \right\} ds \\ &=: I_l^{4,1}(u, v)(t, x) + I_l^{4,2}(u, v)(t, x) + I_l^{4,3}(u, v)(t, x). \end{aligned}$$

For $i = 1, 2, 3$, denote

$$I_l^{4,i}(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, i}(t) \Phi_{j, k}^\epsilon(x).$$

Case $[I_l^4]_2$: $t < 2^{-2j\beta}$. For this case, we denote $a_{j, k}^{\epsilon, 4}(t) = a_{j, k}^\epsilon(t)$ and then have

$$I_l^1(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, 4}(t) \Phi_{j, k}^\epsilon(x).$$

5.2. Induced a priori estimates. In the sequel we are about to dominate the above-defined $a_{j,k}^{\epsilon,i}$, $b_{j,k}^{\epsilon,i}$ by $u_{j',k'}^{\epsilon'}$ and $v_{j',k''}^{\epsilon''}$.

Lemma 5.1. *There is a constant $\tilde{c} > 0$ such that:*

(i) For $i = 1, 2$,

$$|a_{j,k}^{\epsilon,i}(t)| \lesssim 2^{\frac{nj}{2}+j} \sum_{|j-j'| \leq 2} \sum_{\epsilon', k', k''} \int_{I_i} \frac{|u_{j',k'}^{\epsilon'}(s)|}{(1 + |2^{j-j'}k' - k|)^N} \frac{|v_{j'-3,k''}^0(s)|}{(1 + |2^{j-j'+3}k'' - k'|)^N} \times e^{-\tilde{c}t2^{2j\beta}} ds,$$

where $I_1 = [0, 2^{-1-2j'\beta}]$ and $I_2 = [2^{-1-2j'\beta}, \frac{t}{2}]$.

(ii) For $i = 3, 4$,

$$|a_{j,k}^{\epsilon,i}(t)| \lesssim 2^{\frac{nj}{2}+j} \sum_{|j-j'| \leq 2} \sum_{\epsilon', k', k''} \int_{I_i} \frac{|u_{j',k'}^{\epsilon'}(s)|}{(1 + |2^{j-j'}k' - k|)^N} \frac{|v_{j'-3,k''}^0(s)|}{(1 + |2^{j-j'+3}k'' - k'|)^N} \times e^{-\tilde{c}(t-s)2^{2j\beta}} ds,$$

where $I_3 = [t/2, t]$ and $I_4 = [0, t]$.

Proof.

$$\begin{aligned} a_{j,k}^{\epsilon,1}(t) &= \langle I_l^{1,1}(u, v), \Phi_{j,k}^\epsilon \rangle \\ &= \sum_{\epsilon', k', k''} \sum_{|j-j'| \leq 2} \int_0^{2^{-1-2j'\beta}} \left\{ u_{j',k'}^{\epsilon'}(s) v_{j'-3,k''}^0(s) \right. \\ &\quad \left. \times \left\langle e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0), \Phi_{j,k}^\epsilon \right\rangle \right\} ds. \end{aligned}$$

The Fourier transform gives

$$\begin{aligned} &\left\langle e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0), \Phi_{j,k}^\epsilon \right\rangle \\ &= \int e^{-(t-s)|\xi|^{2\beta}} \xi_l (\widehat{\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0})(\xi) 2^{-jn/2} e^{-i2^{-j}k\xi} \widehat{\Phi^\epsilon}(2^{-j}\xi) d\xi \\ &= \int e^{-(t-s)|\xi|^{2\beta}} \xi_l e^{-i2^{-j}k'\xi} \left[\int e^{ik'\eta} \widehat{\Phi^{\epsilon'}}(2^{-j'}\xi - \eta) e^{-8ik''\eta} \widehat{\Phi^0}(8\eta) d\eta \right] \\ &\quad \times 2^{-jn/2} e^{-i2^{-j}k\xi} \widehat{\Phi^\epsilon}(2^{-j}\xi) d\xi. \end{aligned}$$

Because $0 < s < 2^{-1-2j'\beta}$, we can see $(t-s) \sim t$. Hence

$$\begin{aligned} &\left\langle e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0), \Phi_{j,k}^\epsilon \right\rangle \\ &= 2^{jn/2+j} \int e^{-(t-s)2^{2j\beta}|\xi|^{2\beta}} \xi_l e^{-i(k-2^{j-j'}k')\xi} \\ &\quad \times \left[\int \widehat{\Phi^{\epsilon'}}(2^{j-j'}\xi - \eta) \widehat{\Phi^0}(8\eta) e^{-i(k-8k'')\eta} d\eta \right] \widehat{\Phi^\epsilon}(\xi) d\xi \\ &= 2^{jn/2+j} \int e^{-t2^{2j\beta}|\xi|^{2\beta}} \xi_l e^{-i(k-2^{j-j'}k')\xi} \\ &\quad \times \left[\int \widehat{\Phi^{\epsilon'}}(2^{j-j'}\xi - \eta) \widehat{\Phi^0}(8\eta) e^{-i(k-8k'')\eta} d\eta \right] \widehat{\Phi^\epsilon}(\xi) d\xi. \end{aligned}$$

Situation I. We first consider the case: $|2^{j-j'}k' - k| \leq 2$. We can see

$$(1 + |2^{j-j'}k' - k|)^{-N} \gtrsim 1.$$

Under this situation, we divide the argument into two cases.

Case 1: $|k' - 8k''| \leq 2$. For any positive integer N ,

$$(1 + |2^{j-j'+3}k' - k''|)^{-N} \gtrsim 1.$$

On the other hand, the support of $\widehat{\Phi}^\epsilon(\xi)$ is a ring. A direct computation derives

$$\begin{aligned} & \left| \left\langle e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0), \Phi_{j,k}^\epsilon \right\rangle \right| \\ & \lesssim e^{-\tilde{c}t2^{2j\beta}} 2^{jn/2+j} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j'+3}k'' - k'|)^{-N}. \end{aligned}$$

Case 2: $|k' - 8k''| \geq 2$. Denote by l_{j_0} the largest component of $k' - 8k''$. We have

$$\begin{aligned} & \left| \int \widehat{\Phi}^{\epsilon'}(2^{j-j'}\xi - \eta) \widehat{\Phi}^0(8\eta) e^{-i(k-8k'')\eta} d\eta \right| \\ & \lesssim \frac{1}{(1 + |k' - 8k''|)^N} \left| \int \widehat{\Phi}^{\epsilon'}(2^{j-j'}\xi - \eta) \widehat{\Phi}^0(8\eta) \left(\frac{1}{i} \partial_{\eta_{i_0}}\right)^N (e^{-i(k-8k'')\eta}) d\eta \right| \\ & \lesssim \frac{1}{(1 + |k' - 8k''|)^N} \left| \int \sum_{l=0}^N C_N^l \partial_{\eta_{i_0}}^l (\widehat{\Phi}^{\epsilon'}(2^{j-j'}\xi - \eta)) \partial_{\eta_{i_0}}^{N-l} (\widehat{\Phi}^0(8\eta)) e^{-i(k-8k'')\eta} d\eta \right| \\ & \lesssim \frac{1}{(1 + |k' - 8k''|)^N}, \end{aligned}$$

which gives

$$\begin{aligned} & \left| \left\langle e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0), \Phi_{j,k}^\epsilon \right\rangle \right| \\ & \lesssim 2^{jn/2+j} e^{-\tilde{c}t2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j'+3}k'' - k'|)^{-N}. \end{aligned}$$

Situation II. We then consider the case: $|2^{j-j'}k' - k| \geq 2$. We still divide the discussion into the following two cases.

Case 3: $|k' - 8k''| \leq 2$. Denote by k_{i_0} the largest component of $2^{j-j'}k' - k$. Then

$$(1 + |k_{i_0}|)^N \sim (1 + |2^{j-j'}k' - k|)^N.$$

This fact implies

$$\begin{aligned} & \left| \left\langle e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0), \Phi_{j,k}^\epsilon \right\rangle \right| \\ & \lesssim \frac{2^{jn/2+j}}{(1 + |2^{j-j'}k' - k|)^N} \left| \int \left(\frac{1}{i} \partial_{\xi_{i_0}}\right)^N (e^{-i(k-2^{j-j'}k')\xi}) e^{-t2^{2j\beta}|\xi|^{2\beta}} \xi_l \right. \\ & \quad \times \left. \left[\int \widehat{\Phi}^{\epsilon'}(2^{j-j'}\xi - \eta) \widehat{\Phi}^0(8\eta) e^{-i(k-8k'')\eta} d\eta \right] \widehat{\Phi}^\epsilon(\xi) d\xi \right| \\ & \lesssim \frac{2^{jn/2+j}}{(1 + |2^{j-j'}k' - k|)^N} \left| \int e^{-i(k-2^{j-j'}k')\xi} \sum_{l=0}^N C_N^l \partial_{\xi_{i_0}}^l (e^{-t2^{2j\beta}|\xi|^{2\beta}} \xi_l) \right. \\ & \quad \times \left. \partial_{\xi_{i_0}}^{N-l} \left(\int \widehat{\Phi}^{\epsilon'}(2^{j-j'}\xi - \eta) \widehat{\Phi}^0(8\eta) e^{-i(k-8k'')\eta} d\eta \right) \widehat{\Phi}^\epsilon(\xi) d\xi \right|. \end{aligned}$$

In the above and below, C_N^l stands for the binomial coefficient indexed by N and l . Because the support of $\widehat{\Phi}^\epsilon(\xi)$ is a ring, there exists a small constant $c > 0$ such that

$$|\partial_{\xi_{i_0}}^l (e^{-t2^{2j\beta}|\xi|^{2\beta}} \xi_l)| \lesssim e^{-ct2^{2j\beta}}.$$

Consequently, we have a constant $\tilde{c} > 0$ such that

$$\begin{aligned} & \left| \left\langle e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0), \Phi_{j,k}^\epsilon \right\rangle \right| \\ & \lesssim \frac{2^{jn/2+j}}{(1 + |2^{j-j'}k' - k|)^N} \left| \int e^{-i(k-2^{j-j'}k')\xi} \sum_{l=0}^N C_N^l \partial_{\xi_{i_0}}^l (e^{-t2^{2j\beta}|\xi|^{2\beta}} \xi_l) \right. \\ & \quad \times \partial_{\xi_{i_0}}^{N-l} \left(\int \widehat{\Phi}^{\epsilon'}(2^{j-j'}\xi - \eta) \widehat{\Phi}^0(8\eta) e^{-i(k-8k'')\eta} d\eta \right) \widehat{\Phi}^\epsilon(\xi) d\xi \Big| \\ & \lesssim e^{-\tilde{c}t2^{2j\beta}} 2^{jn/2+j} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j'+3}k'' - k'|)^{-N}. \end{aligned}$$

Case 4: $|k' - 8k''| \geq 2$. In a similar manner to treat Case 3, we denote by k_{i_0} the largest component of $2^{j-j'}k' - k$, and then obtain

$$\begin{aligned} & \left| \left\langle e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0), \Phi_{j,k}^\epsilon \right\rangle \right| \\ & \lesssim \frac{2^{jn/2+j}}{(1 + |2^{j-j'}k' - k|)^N} \int \sum_{l=0}^N C_N^l \left| \partial_{\xi_{i_0}}^l (e^{-t2^{2j\beta}|\xi|^{2\beta}} \xi_l) \right| \\ & \quad \times \left| \int \partial_{\xi_{i_0}}^{N-l} \left(\widehat{\Phi}^{\epsilon'}(2^{j-j'}\xi - \eta) \right) \widehat{\Phi}^0(8\eta) e^{-i(k-8k'')\eta} d\eta \right| \widehat{\Phi}^\epsilon(\xi) |d\xi|. \end{aligned}$$

As in Case 2, upon choosing l_{j_0} as the largest component of $k' - 8k$, applying an integration-by-parts, and utilizing the fact that $\widehat{\Phi}^\epsilon$ is supported on a ring, we can get a constant $\tilde{c} > 0$ such that

$$\begin{aligned} & \left| \left\langle e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0), \Phi_{j,k}^\epsilon \right\rangle \right| \\ & \lesssim \frac{2^{jn/2+j}}{(1 + |2^{j-j'}k' - k|)^N (1 + |2^{j-j'+3}k'' - k'|)^N} \int \sum_{l=0}^N C_N^l \left| \partial_{\xi_{i_0}}^l (e^{-t2^{2j\beta}|\xi|^{2\beta}} \xi_l) \right| \\ & \quad \times \left| \int \partial_{\xi_{i_0}}^{N-l} \left(\widehat{\Phi}^{\epsilon'}(2^{j-j'}\xi - \eta) \right) \widehat{\Phi}^0(8\eta) \left(\frac{1}{i} \partial_{\eta_{i_0}} \right)^N \left(e^{-i(k'-8k'')\eta} \right) d\eta \right| \widehat{\Phi}^\epsilon(\xi) |d\xi| \\ & \lesssim e^{-\tilde{c}t2^{2j\beta}} 2^{jn/2+j} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j'+3}k'' - k'|)^{-N}. \end{aligned}$$

This completes the estimate of $a_{j,k}^{\epsilon,1}(t)$. The estimate of $a_{j,k}^{\epsilon,i}(t)$, for $i = 1, 2, 3, 4$, can be obtained similarly. \square

Using the same method, we can obtain the following estimates for $b_{j,k}^{\epsilon,i}(t)$, $i = 1, 2, 3, 4, 5$.

Lemma 5.2. *There is a constant $\tilde{c} > 0$ such that:*

(i) For $i = 1, 2$,

$$\begin{aligned} & |b_{j,k}^{\epsilon,i}(t)| \\ & \lesssim 2^{\frac{nj}{2}+j} \sum_{j \leq j'+2} \sum_{\epsilon', k', \epsilon'', k''} \int_{I_i} \frac{|u_{j',k'}^{\epsilon'}(s)|}{(1 + |2^{j-j'}k' - k|)^N} \frac{|v_{j',k''}^{\epsilon''}(s)|}{(1 + |2^{j-j'}k'' - k'|)^N} e^{-\tilde{c}t2^{2j\beta}} ds, \end{aligned}$$

where $I_1 = [0, 2^{-1-2j'\beta}]$ and $I_2 = [2^{-1-2j'\beta}, \frac{t}{2}]$.
 (ii) For $i = 3, 4, 5$,

$$|b_{j,k}^{\epsilon,i}(t)| \lesssim 2^{\frac{nj}{2}+j} \sum_{j \leq j'+2} \sum_{\epsilon',k',\epsilon'',k''} \int_{t/2}^t \frac{|u_{j',k'}^{\epsilon'}(s)|}{(1 + |2^{j-j'}k' - k|)^N} \frac{|v_{j',k''}^{\epsilon''}(s)|}{(1 + |2^{j-j'}k'' - k'|)^N} \times e^{-\bar{c}(t-s)2^{2j\beta}} ds,$$

where $I_3 = [t/2, t]$, $I_4 = [0, 2^{-2j'\beta}]$ and $I_5 = [2^{-2j'\beta}, t]$.

Let $Q_{j,k}$ and $Q_{j',k'}$ be two dyadic cubes, and for $w \in \mathbb{Z}^n$ denote by $Q_{j,k}^w$ the dyadic cube $\tilde{Q}_{j,k} + 2^{8-j}w$, where $\tilde{Q}_{j,k}$ denotes the dyadic cube containing $Q_{j,k}$ with side length 2^{8-j} . The forthcoming lemmas can be deduced from the Cauchy-Schwartz inequality.

Lemma 5.3. (i) For $j, j' \in \mathbb{Z}$ and $w, k, k' \in \mathbb{Z}^n$, if $Q_{j',k'} \subset Q_{j,k}^w$, then

$$(1 + |2^{j-j'}k' - k|)^{-N} \lesssim (1 + |w|)^{-N}. \tag{5.2}$$

(ii) Let $0 < j' - j'' \leq 3$, $j \leq j' + 5$ and $|w - w'| > 2^n$. If $Q_{j',k'} \subset Q_{j,k}^w$ and $Q_{j'',k''} \subset Q_{j,k}^{w'}$, then

$$(1 + |2^{j'-j''}k'' - k'|)^{-N} \lesssim 2^{N(j-j')} (1 + |w - w'|)^{-N}. \tag{5.3}$$

Lemma 5.4. Let $Q_{j,k}$ be a dyadic cube with radius 2^{-j} . For $w \in \mathbb{Z}^n$, set $Q_{j,k}^w$ be the dyadic cube $2^{8-j}w + \tilde{Q}_{j,k}$. Then

$$\begin{aligned} & \sum_{\epsilon',k'} \sum_{\epsilon'',k''} |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)|^{p-1} (1 + |2^{j-j'}k' - k|)^{-8N} (1 + |k' - k''|)^{-8N} \\ & \lesssim \sum_{w \in \mathbb{Z}^n} \sum_{w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} \\ & \times \left(\sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{1/p} \left(\sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p \right)^{1/p'}. \end{aligned} \tag{5.4}$$

Lemma 5.5. Let $Q_{j,k}$ be a dyadic cube with radius 2^{-j} . For $w \in \mathbb{Z}^n$, denote by $Q_{j,k}^w$ the dyadic cube $2^{8-j}w + \tilde{Q}_{j,k}$. If $\delta > 0$ is small enough, then

$$\begin{aligned} & \sum_{(\epsilon,k) \in S_r^j} \left\{ \sum_{j \leq j'+5} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left(\sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |a_{j',k'}^\epsilon|^p \right)^{1/p} \right\}^p \\ & \lesssim \sum_{j \leq j'+5} 2^{\delta(j'-j)} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{(\epsilon',k') \in S_{j,k}^{w,j}} |a_{j',k'}^\epsilon|^p. \end{aligned} \tag{5.5}$$

Here, it is worth mentioning that the proof of Lemma 5.5 needs also the following fact: for fixed j , the number of $Q_{j',k'}$ which are contained in the dyadic cube $Q_{j,k}^w = 2^{8-j}w + \tilde{Q}_{j,k}$ equals $2^{n(8+j'-j)}$. On the other hand, for any dyadic cube Q_r with radius r , the number of $Q_{j,k} \subset Q_r$ equals to $(2^j r)^n$. Then the number of $Q_{j',k'}$ which are contained in the dyadic cube Q_r^w equals $(2^{8+j'} r)^n$. In the proof of the main lemmas in Sections 7 and 8, we will use this fact again.

Lemma 5.6. *Let $Q_{j,k}$ be a dyadic cube with radius 2^{-j} . For $w \in \mathbb{Z}^n$, denote by $Q_{j,k}^w$ the dyadic cube $2^{8-j}w + \tilde{Q}_{j,k}$. If $j < j' + 2$, then*

$$\begin{aligned} & \sum_{\epsilon', k'} \sum_{\epsilon'', k''} |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)| (1 + |2^{j'-j}k' - k|)^{-N} (1 + |k' - k''|)^{-N} \\ & \lesssim \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} 2^{n(j'-j)(1-\frac{2}{p})} \\ & \times \left(\sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{1/p} \left(\sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p \right)^{1/p}. \end{aligned} \tag{5.6}$$

6. PROOF OF MAIN THEOREM

By Picard’s contraction principle and Theorems 4.5 & 4.6, it is sufficient to verify that the bilinear operator

$$B(u, v) = \int_0^t e^{-(t-s)(-\Delta)^\beta} \mathbb{P}\nabla \cdot (u \otimes v) ds$$

is bounded from $(\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2})^n \times (\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2})^n$ to $(\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2})^n$. To do so, let

$$\begin{aligned} B_l(u, v) &= \int_0^t e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (uv) ds, \\ B_{l,l',l''}(u, v) &= R_l R_{l'} \int_0^t e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_{l''}} (uv) ds. \end{aligned}$$

We need to prove that all $B_l(u, v)$, $B_{l,l',l''}(u, v)$ are bounded from $\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2} \times \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$ to $\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$. Because $R_{l'}, l' = 1, \dots, n$ are bounded on $\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$, we only consider the boundedness of $B_l(u, v)$. By (5.1), if

$$u(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} u_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x) \quad \text{and} \quad v(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} v_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x),$$

then

$$B_l(u, v)(t, x) = \sum_{i=1}^4 I_l^i(u, v)(t, x),$$

where the terms $I_l^i(u, v)(t, x)$, $i = 1, 2, \dots, 4$ are defined in Subsection 4.1.

It is not hard to see that the argument for $I_l^4(u, v)(t, x)$ is similar to that for $I_l^1(u, v)(t, x)$. Also the treatment of $I_l^3(u, v)(t, x)$ is similar to that of $I_l^2(u, v)(t, x)$. So, we are only required to show that the following functions

$$\begin{aligned} (t, x) \mapsto I_l^1(u, v)(t, x) &= \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x), \\ (t, x) \mapsto I_l^2(u, v)(t, x) &= \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x) \end{aligned}$$

belong to $\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$. By the decompositions of non-linear terms obtained in Subsection 4.1, it amounts to verifying that the following functions

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^{\epsilon, i}(t) \Phi_{j,k}^\epsilon(x), \quad i = 1, 2, 3, 4;$$

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon, i}(t) \Phi_{j,k}^\epsilon(x), \quad i = 1, 2, 3, 4, 5$$

are members of $\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$. The demonstration will be concluded by proving the following two lemmas:

Lemma 6.1. *If $(\beta, p, q, \gamma_1, \gamma_2, m, m')$ satisfies the conditions of Theorem 1.1 and $u, v \in \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$, then*

- (i) *For $i = 1, 2, 3$, the function $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^{\epsilon, i}(t) \Phi_{j,k}^\epsilon(x)$ is in the space $\mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, I}$;*
- (ii) *For $i = 1, 2, 3$, the function $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^{\epsilon, i}(t) \Phi_{j,k}^\epsilon(x)$ is in the space $\mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, III}$;*
- (iii) *The function $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^{\epsilon, 4}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II} \cap \mathbb{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}$.*

Lemma 6.2. *If $(\beta, p, q, \gamma_1, \gamma_2, m, m')$ satisfies the conditions of Theorem 1.1 and $u, v \in \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$, then*

- (i) *For $i = 1, 2, 3$, the function $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon, i}(t) \Phi_{j,k}^\epsilon(x)$ is in the space $\mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, I}$;*
- (ii) *For $i = 1, 2, 3$, the function $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon, i}(t) \Phi_{j,k}^\epsilon(x)$ is in the space $\mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, III}$;*
- (iii) *For $i = 4, 5$, the function $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon, i}(t) \Phi_{j,k}^\epsilon(x)$ is in the space $\mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II}$;*
- (iv) *For $i = 4, 5$, the function $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon, i}(t) \Phi_{j,k}^\epsilon(x)$ is in the space $\mathbb{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}$.*

7. PROOF OF LEMMA 6.1

7.1. Setting (i). For $i = 1, 2, 3$, define $S_r^j = \{(\epsilon, k) : Q_{j,k} \subset Q_r\}$ and

$$\begin{aligned} & I_{a, Q_r}^{m, i}(t) \\ &= |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\sum_{(\epsilon, k) \in S_r^j} |a_{j,k}^{\epsilon, i}(t)|^p (t 2^{2j\beta})^m \right]^{q/p}. \end{aligned}$$

According to the relation between $2^{-2j\beta}$ and t , we divide the proof into three cases. The proofs of $\sum_{(\epsilon, j, k)} a_{j,k}^{\epsilon, i}(t) \Phi_{j,k}^\epsilon(x)$, $i = 1, 2, 3$ are similar. For simplicity, we prove only one case.

Case 7.1: $(t, x) \mapsto \sum_{(\epsilon, j, k)} a_{j,k}^{\epsilon, 1}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, I}$. Without loss of generality, we may assume $\|u\|_{\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}} = \|v\|_{\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}} = 1$. Because $v \in \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$, one has $v \in \mathbb{B}_{\frac{m}{p}, \infty}^{\gamma_1 - \gamma_2} \subset \mathbb{B}_{0, \infty}^{\gamma_1 - \gamma_2}$. Hence

$$|v_{j', -3, k'}^0(s)| \lesssim s^{-\frac{\gamma_2 - \gamma_1}{2\beta}} 2^{-\frac{nj'}{2}},$$

and consequently, by (i) of Lemma 5.1,

$$|a_{j,k}^{\epsilon, 1}(t)| \lesssim 2^j \sum_{|j-j'| \leq 2} \sum_{\epsilon', k'} \int_0^{2^{-1-2j'\beta}} |u_{j', k'}^{\epsilon'}(s)| (1 + |2^{j-j'} k' - k|)^{-N} e^{-\tilde{c}t 2^{2j\beta}} s^{\frac{1}{2\beta} - 1} ds.$$

Notice that $|j - j'| \leq 2$. So, applying Hölder's inequality to k' we obtain

$$\begin{aligned}
 I_{a,Q_r}^{m,1}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
 &\times \left[\sum_{(\epsilon,k) \in S_r^j} \sum_{|j-j'| \leq 2} \sum_{\epsilon',k'} 2^{pj} e^{-\tilde{c}pt2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} \right. \\
 &\times \left. \left(\int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p (t2^{2j\beta})^m \right]^{q/p}.
 \end{aligned}$$

By $p > 2m'\beta$, $j \sim j'$ and Hölder's inequality, we apply (5.2) to obtain

$$\begin{aligned}
 I_{a,Q_r}^{m,1}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
 &\times \left[\sum_{(\epsilon,k) \in S_r^j} e^{-cpt2^{2j\beta}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \right. \\
 &\times \left. \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} (t2^{2j\beta})^m \right]^{q/p}.
 \end{aligned}$$

If $q \leq p$, by the α -triangle inequality we obtain

$$\begin{aligned}
 I_{a,Q_r}^{m,1}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \\
 &\times \left[\sum_{(\epsilon',k') \in S_r^{w,j'}} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{q/p} \\
 &\lesssim \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}} \lesssim 1.
 \end{aligned}$$

If $q > p$, Hölder's inequality implies

$$\begin{aligned}
 I_{a,Q_r}^{m,1}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \\
 &\times \left[\sum_{(\epsilon',k') \in S_r^{w,j'}} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{q/p} \\
 &\lesssim \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}} \lesssim 1.
 \end{aligned}$$

In a similar manner, we can obtain the following two assertions.

Case 7.2: $(t, x) \mapsto \sum_{(\epsilon,j,k)} a_{j,k}^{\epsilon,2}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I}$;

Case 7.3: $(t, x) \mapsto \sum_{(\epsilon,j,k)} a_{j,k}^{\epsilon,3}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I}$.

7.2. **Setting (ii).** For $i = 1, 2, 3$, define

$$III_{a,Q_r}^{m,i} = |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} |a_{j,k}^{\epsilon,i}(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{q/p}.$$

We consider the following three cases:

Case 7.4: $(t, x) \mapsto \sum_{(\epsilon,j,k) \in \Lambda_n} a_{j,k}^{\epsilon,1}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III}$;

Case 7.5: $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^{\epsilon,2}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, III}$;

Case 7.6: $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^{\epsilon,3}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, III}$.

It is sufficient to check Case 7.4 since Cases 7.5 and 7.6 can be dealt with similarly. In fact, for $v \in \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$ we have, by (i) of Lemma 5.1,

$$|a_{j,k}^{\epsilon,1}(t)| \lesssim 2^j \sum_{|j-j'| \leq 2} \sum_{\epsilon', k'} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)| e^{-\tilde{c}t2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} s^{\frac{1}{2\beta}} \frac{ds}{s},$$

whence obtaining

$$\begin{aligned} III_{a,Q_r}^{m,1} &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{q}{p})} \left[\int_{2^{-2j\beta}}^{r^{2\beta}} 2^{pj} e^{-\tilde{c}pt2^{2j\beta}} \right. \\ &\quad \times \sum_{(\epsilon, k) \in S_r^j} \left(\sum_{|j-j'| \leq 2} \sum_{\epsilon', k'} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)| (1 + |2^{j-j'}k' - k|)^{-N} s^{\frac{1}{2\beta}} \frac{ds}{s} \right)^p \\ &\quad \left. \times (t2^{2j\beta})^m \frac{dt}{t} \right]^{q/p}. \end{aligned}$$

Applying Hölder's inequality to k' and s respectively, along with (5.2) and $|j-j'| \leq 2$, we find

$$\begin{aligned} III_{a,Q_r}^{m,1} &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{q}{p})} \left[\sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \int_{2^{-2j\beta}}^{r^{2\beta}} 2^{jp} e^{-\tilde{c}pt2^{2j\beta}} \right. \\ &\quad \times \sum_{(\epsilon, k) \in S_r^j} 2^{-j p} \left(\sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j\beta})^{m'} \frac{ds}{s} \right) (t2^{2j\beta})^m \frac{dt}{t} \Big]^{q/p}. \end{aligned}$$

If $q \leq p$, by changing variables we obtain

$$\begin{aligned} III_{a,Q_r}^{m,1} &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{q}{p})} \\ &\quad \times \left[\int_{2^{-2j\beta}}^{r^{2\beta}} e^{-\tilde{c}t2^{2j\beta}} (t2^{2j\beta})^m \left(\sum_{(\epsilon', k') \in S_r^{w, j'}} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j\beta})^{m'} \frac{ds}{s} \right) \frac{dt}{t} \right]^{q/p} \\ &\lesssim \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}. \end{aligned}$$

If $q > p$, via applying Hölder's inequality for w and $\frac{q}{p} > 1$, we similarly have

$$\begin{aligned} III_{a,Q_r}^{m,1} &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{q}{p})} \\ &\quad \left[\int_{2^{-2j\beta}}^{r^{2\beta}} e^{-\tilde{c}t2^{2j\beta}} (t2^{2j\beta})^m \left(\sum_{(\epsilon', k') \in S_r^{w, j'}} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j\beta})^{m'} \frac{ds}{s} \right) \frac{dt}{t} \right]^{q/p} \\ &\lesssim \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}. \end{aligned}$$

7.3. **Setting (iii).** The argument for that the function

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^{\epsilon,4}(t) \Phi_{j,k}^\epsilon(x) \text{ is in } \mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II} \cap \mathbb{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}$$

is divided into two cases.

Case 7.7: $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^{\epsilon,4}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II}$. Let

$$II_{a, Q_r}^4(t) = |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\sum_{(\epsilon, k) \in S_r^j} |a_{j,k}^{\epsilon,4}(t)|^p \right]^{q/p}.$$

Then, by (ii) of Lemma 5.1 we have

$$|a_{j,k}^{\epsilon,4}(t)| \lesssim 2^j \sum_{|j-j'| \leq 2} \sum_{\epsilon', k'} \int_0^t |u_{j',k'}^{\epsilon'}(s)| e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} s^{\frac{1}{2\beta}-1} ds,$$

whence via Hölder’s inequality and (5.2), we obtain

$$\begin{aligned} II_{a, Q_r}^4(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[2^{pj} \sum_{|j-j'| \leq 2} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \right. \\ &\quad \left. \times \sum_{(\epsilon', k') \in S_r^{w, j'}} t^{p-1-\mu p} \int_0^t |u_{j',k'}^{\epsilon'}(s)|^p s^{(\frac{1}{2\beta}-1+\mu)p} ds \right]^{q/p} \end{aligned}$$

If $q > p$, by Hölder’s inequality we have

$$\begin{aligned} II_{a, Q_r}^4(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \\ &\quad \times 2^{qj} \sum_{|j-j'| \leq 2} t^{\frac{(p-1-\mu p)q}{p}} \left[\int_0^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p s^{(\frac{1}{2\beta}-1+\mu)p} ds \right]^{q/p}. \end{aligned}$$

Because $2^{2j\beta}t \leq 1$, one has $2^{qj} \leq t^{-\frac{q}{2\beta}}$. This in turn gives

$$\begin{aligned} II_{a, Q_r}^4(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} 2^{qj} \\ &\quad \times \sum_{|j-j'| \leq 2} t^{\frac{q}{2\beta}} t^{-p(\frac{1}{2\beta}-1+\mu)-1} \left[\int_0^t \left(\sum_{\epsilon', k'} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{q/p} s^{(\frac{1}{2\beta}-1+\mu)p} ds \right] \\ &\lesssim \|u\|_{\mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II}}. \end{aligned}$$

If $q \leq p$, we have

$$\begin{aligned} II_{a, Q_r}^4(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ &\quad \times \left[\sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{|j-j'| \leq 2} 2^{jp} \sum_{(\epsilon', k') \in S_r^{w, j'}} \left(\int_0^t |u_{j',k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p \right]^{q/p}. \end{aligned}$$

Because $t \leq 2^{-2j\beta}$ and $0 < m' < \min\{1, \frac{p}{2\beta}\}$, Hölder's inequality implies

$$\begin{aligned} II_{a,Q_r}^4(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} \\ &\quad \times \sum_{|j-j'| \leq 2} \left[2^{pj} \int_0^{2^{-2j\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p s^{m'} 2^{-2j'\beta(\frac{p}{2\beta}-m')} \frac{ds}{s} \right]^{q/p} \\ &\lesssim \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}. \end{aligned}$$

Case 7.8: $(t, x) \mapsto \sum_{(\epsilon,j,k) \in \Lambda_n} a_{j,k}^{\epsilon,4}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}$. Similarly, let

$$\begin{aligned} IV_{a,Q_r}^{4,m'} &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[\int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} |a_{j,k}^{\epsilon,4}(t)|^p (t2^{2j\beta})^{m'} \frac{dt}{t} \right]^{q/p}. \end{aligned}$$

Choosing a constant μ such that $m' + p - 1 - \frac{p}{2\beta} \leq p\mu < p - 1$, we use (5.2) and Hölder's inequality to obtain

$$\begin{aligned} IV_{a,Q_r}^{4,m'} &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[\int_0^{2^{-2j\beta}} 2^{pj} \sum_{|j-j'| \leq 2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \right. \\ &\quad \left. \times \sum_{(\epsilon',k') \in S_r^{w,j'}} t^{p-1-p\mu} \left(\int_0^t |u_{j',k'}^{\epsilon'}(s)|^p s^{(\frac{1}{2\beta}-1+\mu)p} ds \right) (t2^{2j\beta})^{m'} \frac{dt}{t} \right]^{q/p}. \end{aligned}$$

If $q \leq p$, by the α -triangle inequality we have

$$\begin{aligned} IV_{a,Q_r}^{4,m'} &\lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \sum_{|j-j'| \leq 2} 2^{qj} \\ &\quad \times \left[\int_0^{2^{-2j\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p s^{\frac{p}{2\beta}-p+p\mu} \left(\int_s^{2^{-2j\beta}} 2^{2mj'\beta} t^{m'+p-p\mu} \frac{dt}{t^2} \right) ds \right]^{q/p}. \end{aligned}$$

Because $s2^{2j'\beta} \leq 1$ and $m' + p - 1 - \frac{p}{2\beta} \leq p\mu$, we obtain

$$\begin{aligned} IV_{a,Q_r}^{4,m'} &\lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j' \geq -\log_2 r_w} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\ &\quad \times \left[\int_0^{2^{-2(j'-2)\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{\frac{p}{2\beta}+1-p+p\mu} \frac{ds}{s} \right]^{q/p} \\ &\lesssim \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,III}} + \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}. \end{aligned}$$

If $q > p$, by Hölder's inequality, we have

$$\begin{aligned} IV_{a,Q_r}^{4,m'} &\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} (1+|w|)^{-N} \sum_{j \geq -\log_2 r_w} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\ &\quad \times \left[\int_0^{2^{-2(j'-2)\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{\frac{p}{2\beta}+1-p+p\mu} \frac{ds}{s} \right]^{q/p} \end{aligned}$$

$$\lesssim \|u\|_{\mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III}} + \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}.$$

8. PROOF OF LEMMA 6.2

8.1. **Setting (i).** For $i = 1, 2, 3$, define

$$\begin{aligned} & I_{b,Q_r}^{m,i}(t) \\ &= |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,i}(t)|^p (t2^{2j\beta})^m \right]^{q/p}. \end{aligned}$$

We divide the proof into three cases:

Case 8.1: $(t, x) \mapsto \sum_{(\epsilon,j,k) \in \Lambda_n} b_{j,k}^{\epsilon,1}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I}$;

Case 8.2: $(t, x) \mapsto \sum_{(\epsilon,j,k) \in \Lambda_n} b_{j,k}^{\epsilon,2}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I}$;

Case 8.3: $(t, x) \mapsto \sum_{(\epsilon,j,k) \in \Lambda_n} b_{j,k}^{\epsilon,3}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I}$.

But, we demonstrate only Case 8.1 and omit the proofs of Cases 8.2 and 8.3 due to their similarity. Assume first $1 < p \leq 2$. If $s2^{2j'\beta} \leq 1$, then

$$v \in \mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2} \subset \mathbb{B}_{\frac{m}{p},\infty}^{\gamma_1-\gamma_2} \Rightarrow |v_{j',k''}^{\epsilon''}(s)| \lesssim 2^{-(\frac{n}{2} + \gamma_1 - \gamma_2)j'}.$$

Because $v \in \mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II}$, we have

$$\sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p \lesssim 2^{-nj+p\gamma_2j} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})}.$$

By Lemma 5.2 (i), (5.4) and Hölder's inequality we obtain

$$\begin{aligned} |b_{j,k}^{\epsilon,1}(t)| &\lesssim 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} 2^{j(-n+p\gamma_2)(1-\frac{1}{p})} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} \\ &\times 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})(p-1)} 2^{-2j'\beta(1-\frac{1}{p})} \left(\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{1/p}. \end{aligned}$$

This in turn yields

$$\begin{aligned} & I_{b,Q_r}^{m,1}(t) \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{q}{p})} 2^{\frac{qnj}{2} + qj} 2^{qj(-n+p\gamma_2)(1-\frac{1}{p})} \\ &\times \left\{ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{(\epsilon,k) \in S_r^j} \left[\sum_{j < j'+2} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})(p-1)} \right. \right. \\ &\left. \left. \times 2^{-2j'\beta(1-\frac{1}{p})} \left(\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{1/p} \right]^p \right\}^{q/p}. \end{aligned}$$

Since $0 < s < 2^{-1-2j'\beta}$ and $m' < 1$, one has $(s2^{2j'\beta})^{1-m'} \lesssim 1$. This implies

$$\begin{aligned} & I_{b,Q_r}^{m,1}(t) \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{q}{p})} 2^{\frac{qnj}{2} + qj} 2^{qj(-n+p\gamma_2)(1-\frac{1}{p})} \\ &\times \left\{ \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \sum_{(\epsilon,k) \in S_r^j} \sum_{j < j'+2} 2^{\delta(j'-j)} 2^{-p(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} \right\} \end{aligned}$$

$$\begin{aligned} &\times 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})(p-1)} 2^{-2pj'\beta} \\ &\times \left(\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right)^{q/p}. \end{aligned}$$

If $q \leq p$, for $p\gamma_2 + 2 - 2\beta > 0$ take $0 < \delta < p(p\gamma_2 + 2 - 2\beta)$. By the α -triangle inequality, we obtain

$$\begin{aligned} I_{b, Q_r}^{m,1}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \geq \{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \sum_{j \leq j'+2} 2^{q(j'-j)[\frac{\delta}{p} - (p\gamma_2 + 2 - 2\beta)]} \\ &\times 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left(\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right)^{q/p} \\ &\lesssim \|u\|_{\mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, IV}}. \end{aligned}$$

If $q > p$, by Hölder's inequality we obtain

$$\begin{aligned} I_{b, Q_r}^{m,1}(t) &\lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1 + |w|)^N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left[\sum_{j \leq j'+2} 2^{(j'-j)[\delta - (p\gamma_2 + 2 - 2\beta)]} \right]^{\frac{q-p}{p}} \\ &\times \left\{ \sum_{j \leq j'+2} 2^{(j'-j)[\delta - (p\gamma_2 + 2 - 2\beta)]} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \right. \\ &\times \left. \left(\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right)^{q/p} \right\} \\ &\lesssim \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}. \end{aligned}$$

Then assume $2 < p < \infty$. Because $0 < s < 2^{-1-2j'\beta}$, $v \in \mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II}$ implies

$$\left(\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p ds \right)^{1/p} \lesssim 2^{\gamma_2 j - \frac{nj}{p} - j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{-\frac{2j'\beta}{p}}.$$

In view of Lemma 5.2 (i), Hölder's inequality and (5.6), we achieve

$$\begin{aligned} |b_{j,k}^{\epsilon,1}(t)| &\lesssim 2^{\frac{nj}{2} + j} e^{-ct2^{2j\beta}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j < j'+2} 2^{n(j'-j)(1 - \frac{2}{p})} 2^{-2j'\beta(1 - \frac{2}{p})} \\ &\times 2^{\gamma_2 j - \frac{nj}{p}} 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{-\frac{2j'\beta}{p}} \left(\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{1/p}. \end{aligned}$$

Using the above estimate we obtain

$$\begin{aligned} I_{b, Q_r}^{m,1}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{\frac{qnj}{2} + qj} 2^{qj(\gamma_2 - \frac{n}{p})} \\ &\times \left\{ \sum_{(\epsilon, k) \in S_r^j} e^{-cpt2^{2j\beta}} (t2^{2j\beta})^m \sum_{w \in \mathbb{Z}^n} \frac{1}{(1 + |w|)^N} \left[\sum_{j < j'+2} 2^{n(j'-j)(1 - \frac{2}{p})} 2^{-j'\beta(2 - \frac{4}{p})} \right. \right. \end{aligned}$$

$$\times 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{-\frac{2j'\beta}{p}} \left(\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{1/p}]^p \}^{q/p}.$$

Notice that $0 < s < 2^{-1-2j'\beta}$ and $m' < 1$. For any $0 < \delta < (\gamma_1 + \gamma_2 + 1)$ we have

$$\begin{aligned} I_{b, Q_r}^{m, 1}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \gamma_2 + 1 + n - \frac{2n}{p})} \\ &\times \left\{ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j < j'+2} 2^{(\delta + pn - 2n)(j' - j)} 2^{-2j'\beta(p-2)} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \right. \\ &\times \left. 2^{-4j'\beta} \left[\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{q/p} \right\}. \end{aligned}$$

If $q \leq p$, the α -triangle inequality implies

$$\begin{aligned} I_{b, Q_r}^{m, 1}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \sum_{j < j'+2} 2^{q(\frac{\delta}{p} - \gamma_1 - \gamma_2 - 1)(j' - j)} \\ &\times 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{q/p} \\ &\lesssim \|u\|_{\mathbb{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}. \end{aligned}$$

If $q > p$, by Hölder's inequality we obtain

$$\begin{aligned} I_{b, Q_r}^{m, 1}(t) &\lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1 + |w|)^N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left(\sum_{j < j'+2} 2^{p(j' - j)(\frac{\delta}{p} - \gamma_1 - \gamma_2 - 1)} \right)^{\frac{q-p}{p}} \\ &\times \left\{ \sum_{j < j'+2} 2^{(j' - j)[\delta - p(\gamma_1 + \gamma_2 + 1)]} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \right. \\ &\times \left. \left[\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{q/p} \right\} \lesssim \|u\|_{\mathbb{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}. \end{aligned}$$

8.2. **Setting (ii).** For $i = 1, 2, 3$, define

$$III_{b, Q_r}^{m, i} = |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon, k) \in S_r^j} |b_{j, k}^{\epsilon, i}(t)|^p (t 2^{2j\beta})^m \frac{dt}{t} \right]^{q/p}.$$

In a way similar to the setting (i) of the subsection 8.1, we may only handle the situation $1 < p \leq 2$. Under this the argument is split into three cases.

Case 8.4: $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 1}(t) \Phi_{j, k}^\epsilon(x)$ is in $\mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}$;

Case 8.5: $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 2}(t) \Phi_{j, k}^\epsilon(x)$ is in $\mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}$;

Case 8.6: $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 3}(t) \Phi_{j, k}^\epsilon(x)$ is in $\mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}$.

It suffices to treat Case 8.4 since Cases 8.5 and 8.6 can be verified similarly. Because u and v both belong to $\mathbb{B}_{\frac{m}{p}, \infty}^{\gamma_1 - \gamma_2}$, for $s 2^{2j'\beta} \leq 1$ we have $|v_{j', k''}^{\epsilon''}(s)| \lesssim$

$2^{-(\frac{n}{2}+\gamma_1-\gamma_2)j'}$. Owing to $v \in \mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II}$, we also have

$$\sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p \lesssim 2^{p\gamma_2 j - nj} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \quad \text{for a fixed } j'.$$

This, along with Hölder's inequality, (5.4) and Lemma 5.2 (i), implies

$$\begin{aligned} |b_{j,k}^{\epsilon,1}(t)| &\lesssim 2^{[\frac{n}{p} - \frac{n}{2} + (p-1)\gamma_2 + 1]j} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \\ &\times 2^{[\frac{n}{2} - \frac{n}{p} + \frac{2\beta}{p} - (p-1)\gamma_2 - 1]j'} \left(\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{1/p}. \end{aligned}$$

The last estimate for $|b_{j,k}^{\epsilon,1}(t)|$ and (5.5) are used to derive

$$\begin{aligned} III_{b,Q_r}^{m,1} &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(p\gamma_2 + 2 - 2\beta)} \left\{ \sum_{j \leq j'+2} 2^{\delta(j-j')} 2^{-pj'(p\gamma_2 + 2 - 2\beta)} \right. \\ &\times \left. 2^{pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right\}^{q/p}. \end{aligned}$$

If $q \leq p$, by the α -triangle inequality we have

$$\begin{aligned} III_{b,Q_r}^{m,1} &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \sum_{j \leq j'+2} 2^{q(j-j')(p\gamma_2 + 2 - 2\beta - \frac{\delta}{p})} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ &\times \left[\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{q/p}. \end{aligned}$$

Changing the order of j and j' , we find $I_{Q_r}^5 \lesssim \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}$.

If $q > p$, by Hölder's inequality we obtain

$$\begin{aligned} III_{b,Q_r}^{m,1} &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \left(\sum_{j \leq j'+2} 2^{p(j-j')(p\gamma_2 + 2 - 2\beta - \delta)} \right)^{\frac{q-p}{p}} \\ &\times \left\{ \sum_{j \leq j'+2} 2^{q(j-j')(p\gamma_2 + 2 - 2\beta - \delta)} \left[\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right. \right. \\ &\times \left. \left. (s2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{q/p} \right\}. \end{aligned}$$

Upon taking $0 < \delta < p\gamma_2 + 2 - 2\beta$ and changing the order of j and j' , we reach

$$III_{b,Q_r}^{m,1} \lesssim \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}.$$

8.3. Setting (iii). Like in subsection 8.2, it is sufficient to deal with the situation $1 < p \leq 2$ below. For $i = 4, 5$, define

$$II_{b,Q_r}^i(t) = |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left(\sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,i}(t)|^p \right)^{q/p}.$$

We divide the argument into two cases.

Case 8.7: $(t, x) \mapsto \sum_{(\epsilon,j,k) \in \Lambda_n} b_{j,k}^{\epsilon,4}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II}$;

Case 8.8: $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon,5}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II}$.

In view of the settings (i) and (ii) above, we only give a proof of Case 8.7 since the proof of Case 8.8 is similar. Due to $v \in \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$, one gets $|v_{j',k''}^{\epsilon''}(s)| \lesssim 2^{-(\frac{n}{2} + \gamma_1 - \gamma_2)j'}$. For $0 < s < 2^{-2j'\beta}$, one has that $v \in \mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II}$ implies

$$\int_0^{2^{-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p ds \lesssim 2^{-nj+p\gamma_2j} 2^{-p(\gamma_1 + \frac{n}{2} - \frac{n}{p})j'} 2^{-2j'\beta}.$$

By (5.4), Hölder’s inequality and Lemma 5.2 (ii) we obtain

$$\begin{aligned} |b_{j,k}^{\epsilon,4}(t)| &\lesssim 2^{-\frac{nj}{2} + j + \frac{nj}{p} + (p-1)\gamma_2j} \sum_{j < j'+2} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} 2^{-(\frac{n}{2} + \gamma_1)j'} 2^{\frac{(p-1)nj'}{p}} \\ &\times 2^{(2-p)\gamma_2j'} 2^{-\frac{2\beta(p-1)j'}{p}} \left(\int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{1/p}. \end{aligned}$$

By (5.5), $s2^{2j'\beta} \leq 1$ and $m' < 1$, we can similarly achieve that for $\delta > 0$,

$$\begin{aligned} II_{b, Q_r}^4(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} 2^{qj(2-2\beta+p\gamma_2)} \left[\sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j < j'+2} 2^{\delta(j'-j)} \right. \\ &\left. 2^{-pj'(2-2\beta+p\gamma_2)} 2^{pj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{q/p}. \end{aligned}$$

If $q \leq p$, we apply the α -triangle inequality to obtain

$$\begin{aligned} II_{b, Q_r}^4(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} \sum_{j < j'+2} 2^{q(j-j')(2-2\beta+p\gamma_2 - \frac{\delta}{p})} \\ &\times 2^{qj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} \left[\int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{q/p}. \end{aligned}$$

If $0 < \frac{\delta}{p} < 2(1 - \beta) + p\gamma_2$, then $II_{b, Q_r}^4(t) \lesssim \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}$ follows from changing the order of j and j' .

If $q > p$, by Hölder’s inequality we obtain

$$\begin{aligned} II_{b, Q_r}^4(t) &\lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1 + |w|)^N} \sum_{j \geq -\log_2 r} \left(\sum_{j < j'+2} 2^{(j-j')[2p(1-\beta)+p^2\gamma_2-\delta]} \right)^{\frac{q-p}{p}} \\ &\times \left\{ \sum_{j < j'+2} 2^{(j-j')[2p(1-\beta)+p^2\gamma_2-\delta]} 2^{qj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} \right. \\ &\times \left. \left[\int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{q/p} \right\} \\ &\lesssim \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}. \end{aligned}$$

8.4. **Setting (iv).** Again, we consider only the situation $1 < p \leq 2$ in the sequel. For $i = 4, 5$, define

$$IV_{b,Q_r}^{m',i} = |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,i}(t)|^p (t2^{2j\beta})^{m'} \frac{dt}{t} \right]^{q/p}.$$

Due to their similarity, we only check the first one of the following two cases:

Case 8.9: $(t, x) \mapsto \sum_{(\epsilon,j,k) \in \Lambda_n} b_{j,k}^{\epsilon,4}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}$;

Case 8.10: $(t, x) \mapsto \sum_{(\epsilon,j,k) \in \Lambda_n} b_{j,k}^{\epsilon,5}(t) \Phi_{j,k}^\epsilon(x)$ is in $\mathbb{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}$.

Because $0 < s < 2^{-2j'\beta}$, one gets $|v_{j',k''}^{\epsilon''}(s)| \lesssim 2^{-\frac{nj'}{2}} 2^{-j'(\gamma_1 - \gamma_2)}$ and

$$\int_0^{2^{-2j'\beta}} \sum_{(\epsilon'',k'') \in S_{j,k}^{w,j'}} |v_{j',k''}^{\epsilon''}(s)|^p ds \lesssim 2^{-nj + p\gamma_2 j} 2^{-p(\frac{n}{2} + \gamma_1 - \frac{n}{p})j'} 2^{-2j'\beta}.$$

By using (5.4), Lemma 5.2 (ii) and Hölder's inequality we obtain

$$\begin{aligned} |b_{j,k}^{\epsilon,4}(t)| &\lesssim 2^{-\frac{nj}{2} + j + \frac{nj}{p} + (p-1)\gamma_2 j} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j < j' + 2} 2^{-(\frac{n}{2} + \gamma_1)j'} \\ &\times 2^{\frac{nj'(p-1)}{p} + (2-p)\gamma_2 j' - \frac{2j'\beta(p-1)}{p}} \left(\int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{1/p}. \end{aligned}$$

For $t2^{2j\beta} < 1$, we obtain

$$\begin{aligned} IV_{b,Q_r}^{m',4} &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ \int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} 2^{-\frac{pnj}{2} + pj + nj + p(p-1)\gamma_2 j} \right. \\ &\times \left[\sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j < j' + 2} 2^{-(\frac{n}{2} + \gamma_1)j'} 2^{\frac{nj'(p-1)}{p} + (2-p)\gamma_2 j'} \right. \\ &\times \left. \left. 2^{-\frac{2j'\beta(p-1)}{p}} \left(\int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{1/p} \right]^p (t2^{2j\beta})^{m'} \frac{dt}{t} \right\}^{q/p}. \end{aligned}$$

Furthermore, we first apply Hölder's inequality to w , and then employ (5.5) to obtain that for $\delta > 0$,

$$\begin{aligned} IV_{b,Q_r}^{m',4} &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + 1 + (p-1)\gamma_2)} 2^{-\frac{q\delta j}{p}} \\ &\times \left\{ \int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} \sum_{j < j' + 2} 2^{\delta j'} 2^{-p(\frac{n}{2} + \gamma_1)j'} 2^{(p-1)nj' + p(2-p)\gamma_2 j' - 2\beta(p-1)j'} \right. \\ &\times \left. \sum_{w \in \mathbb{Z}^n} \frac{1}{(1 + |w|)^N} \left(\int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right) (t2^{2j'\beta})^{m'} \frac{dt}{t} \right\}^{q/p}. \end{aligned}$$

Because $\int_0^{2^{-2j\beta}} (t2^{2j\beta})^m \frac{dt}{t} \lesssim 1$, we obtain

$$IV_{b,Q_r}^{m',4} \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + 1 + (p-1)\gamma_2)} 2^{-\frac{q\delta j}{p}}$$

$$\begin{aligned} & \times \left[\sum_{j < j'+2} 2^{\delta j'} 2^{pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{2\beta j'} 2^{-pj'(2-2\beta+p\gamma_2)} \right. \\ & \left. \times \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \left(\int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right) \right]^{q/p}. \end{aligned}$$

If $q \leq p$, noticing $p\gamma_2 + 2 - 2\beta > 0$ and taking $0 < \delta < p(\gamma_1 + 1 + (p-1)\gamma_2)$ we obtain

$$\begin{aligned} IV_{b, Q_r}^{m', 4} & \lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1+|w|)^{-\frac{qN}{p}} \sum_{j \geq -\log_2 r} \sum_{j < j'+2} 2^{q(\frac{\delta}{p} - p\gamma_2 - 2 + 2\beta)(j'-j)} \\ & \times 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[\int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{q/p} \\ & \lesssim \|u\|_{\mathbb{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}, \end{aligned}$$

where we have used the inequality

$$(s2^{2j'\beta})^{1-m'} \leq 1 \quad \text{for } s \leq 2^{-2j'\beta}, \quad 1 - m' > 0.$$

and then changed the order of j and j' .

If $q > p$, by Hölder's inequality we have

$$\begin{aligned} IV_{b, Q_r}^{m', 4} & \lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} \left[\sum_{j < j'+2} 2^{p(2-2\beta+p\gamma_2 - \frac{\delta}{p})(j-j')} \right. \\ & \left. \times 2^{pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{2\beta j'} \left(\int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right) \right]^{q/p} \\ & \lesssim \|u\|_{\mathbb{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}. \end{aligned}$$

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