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CHAOTIC OSCILLATIONS OF THE KLEIN-GORDON EQUATION WITH DISTRIBUTED ENERGY PUMPING AND VAN DER POL BOUNDARY REGULATION AND DISTRIBUTED TIME-VARYING COEFFICIENTS

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ABSTRACT. Consider the Klein-Gordon equation with variable coefficients, a van der Pol cubic nonlinearity in one of the boundary conditions and a spatially distributed *antidamping* term, we use a *variable-substitution* technique together with the analogy with the 1-dimensional wave equation to prove that for the Klein-Gordon equation chaos occurs for a class of equations and boundary conditions when system parameters enter a certain regime. Chaotic and nonchaotic profiles of solutions are illustrated by computer graphics.

1. INTRODUCTION

During the past decade, progress has been made in dynamical system theory in proving the onset of chaos in the 1D wave equation and the Klein-Gordon equation with a van der Pol cubic nonlinearity in one of the boundary conditions and a spatially distributed *antidamping* term, see [1, 2, 3, 4, 5, 6]. The basic method is *characteristic reflections*, by which discrete dynamical systems are extracted. We first give a quick review of the work mentioned above, where the main motivating interest was the significance in *nonlinear feedback boundary control*. For the wave equation

$$w_{tt}(x,t) - c^2 w_{xx}(x,t) = 0, \quad 0 < x < 1,\ t > 0,\ c > 0, \tag{1.1}$$

we assume that at the left-end $x = 0$, the boundary condition is

$$w_t(0,t) = -\eta w_x(0,t), \quad t > 0,\ \eta > 0,\ \eta \neq c, \tag{1.2}$$

and at the right-end $x = 1$, the boundary condition is of the van der Pol type:

$$w_x(1,t) = \alpha w_t(1,t) - \beta w_t^3(1,t), \quad t > 0,\ 0 < \alpha < c,\ \beta > 0. \tag{1.3}$$

Then the energy functional

$$E(t) = \frac{1}{2}\int_0^1 \Big[w_x^2(x,t) + \frac{1}{c^2} w_t^2(x,t)\Big] dx \tag{1.4}$$

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rises if $|w_t(1,t)|$ is small, and *falls* if $|w_t(1,t)|$ is large. Thus, the van der Pol boundary condition (1.3) has a *self-regulating effect*. This can cause chaos to occur in w_x and w_t if the parameters α, c and η enter a certain regime.

The treatment in [3] relies heavily on the *method of characteristics* for linear hyperbolic systems and simple wave-reflecting relations. Let $c = 1$, and

$$u(x,t) = \frac{1}{2}[w_x(x,t) + w_t(x,t)], \quad v(x,t) = \frac{1}{2}[w_x(x,t) - w_t(x,t)]. \tag{1.5}$$

Then

$$v(0,t) = G_\eta(u(0,t)) \equiv \frac{1+\eta}{1-\eta} u(0,t), \tag{1.6}$$

$$u(1,t) = F_{\alpha,\beta}(v(1,t)), \tag{1.7}$$

where $F_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ is a nonlinear mapping such that for each given $v \in \mathbb{R}$, $u = F_{\alpha,\beta}(v)$ is the unique solution of the cubic equation

$$\beta(u - v)^3 + (1 - \alpha)(u - v) + 2v = 0. \tag{1.8}$$

Therefore, $u(x,t)$ and $v(x,t)$ for $x \in [0,1]$ and $t \in (0,\infty)$ are determined by the initial data $u(x,0)$, $v(x,0)$ and the iterative composition of $F_{\alpha,\beta} \circ G_\eta$ or $G_\eta \circ F_{\alpha,\beta}$. Finally, the chaotic dynamics in the 1D wave equation is reduced to the discrete dynamical system generated by the interval map $F_{\alpha,\beta} \circ G_\eta$ or $G_\eta \circ F_{\alpha,\beta}$.

A generalization of the 1D wave equation is the Klein-Gordon equation described as

$$w_{tt}(x,t) + \eta w_t(x,t) - w_{xx}(x,t) + k^2 w(x,t) = 0, \quad 0 < x < 1,\ t > 0, \tag{1.9}$$

where $k \neq 0$, $\eta > 0$. A special case is

$$w_{tt} + 2k w_t - w_{xx} + k^2 w = 0, \quad \text{for } (x,t) \in (0,1) \times (0,\infty). \tag{1.10}$$

We consider, for (1.10), the following boundary condition,

$$w_t(0,t) + k w(0,t) = -\lambda w_x(0,t), \quad t > 0, \text{ at } x = 0, \text{ for given } \lambda \in \mathbb{R}; \tag{1.11}$$

$$w_x(1,t) = \alpha[w_t(1,t) + k w(1,t)] - \beta[w_t(1,t) + k w(1,t)]^3, \quad t > 0, \text{ at } x = 1, \tag{1.12}$$

where $0 < \alpha < 1$, $\beta > 0$; and the energy function

$$\widetilde{E}(t) = \frac{1}{2}\int_0^1 [w_x^2 + (w_t + kw)^2] dx. \tag{1.13}$$

Then $\frac{d}{dt}\widetilde{E}(t)$ is indefinite, which is the sign of chaos.

The simple change of variable

$$w(x,t) = e^{-kt} W(x,t) \tag{1.14}$$

leads to

$$\frac{\partial^2}{\partial x^2} W(x,t) - \frac{\partial^2}{\partial t^2} W(x,t) = 0. \tag{1.15}$$

Define

$$u = \frac{1}{2}(w_x + w_t + kw), \quad v = \frac{1}{2}(w_x - w_t - kw). \tag{1.16}$$

Then

$$v(0,t) = \frac{1+\lambda}{1-\lambda} u(0,t) \equiv G_\lambda(u(0,t)), \tag{1.17}$$

where G_λ is defined to be the linear operator of multiplication by $(1+\lambda)/(1-\lambda)$. Also we have

$$\beta[u(1,t)-v(1,t)]^3+(1-\alpha)[u(1,t)-v(1,t)]+2v(1,t)=0. \tag{1.18}$$

For any $v\in\mathbb{R}$, define $g(v)$ to be the *unique real* solution to the cubic equation

$$\beta g(v)^3+(1-\alpha)g(v)+2v=0, \tag{1.19}$$

and

$$F(v)\equiv F_{\alpha,\beta}(v)\equiv v+g_{\alpha,\beta}(v). \tag{1.20}$$

Then for each given $v(1,t)$, equation (1.18) has a unique solution $u(1,t)$ given by

$$u(1,t)=F_{\alpha,\beta}(v(1,t)).$$

It is easy to check that $e^{kt}u(x,t)$ keeps constant along each characteristics $x+t=c$, and $e^{kt}v(x,t)$ keeps constant along each characteristics $x-t=c$. Therefore, we have

$$\begin{gathered} u(1,t+2)=F_{\alpha,\beta}(G_\lambda(e^{-2k}u(1,t))),\\ v(0,t+2)=G_\lambda(e^{-k}F_{\alpha,\beta}(e^{-k}v(0,t))), \end{gathered} \tag{1.21}$$

for any $t>0$. Finally, the dynamics of

$$u=\frac{1}{2}(w_x+w_t+kw),\quad v=\frac{1}{2}(w_x-w_t-kw) \tag{1.22}$$

are determined by the iterative compositions of the functions $F_{\alpha,\beta}(G_\lambda(e^{-2k}\cdot))$ or $G_\lambda(e^{-k}F_{\alpha,\beta}(e^{-k}\cdot))$.

One can imagine that there may be more chaos in the Klein-Gordon equation if its constant coefficients are replaced by variable coefficients. So in this paper we consider more general situations and problems below:

$$[\frac{\partial}{\partial t}-a(x)\frac{\partial}{\partial x}+k_1][\frac{\partial}{\partial t}+b(x)\frac{\partial}{\partial x}+k_2]w(x,t)=0,\quad\text{for}\quad x\in(0,1),\quad t>0 \tag{1.23}$$

with some linear and cubic nonlinear boundary condition, where $a(x)>0$ and $b(x)>0$ are continuous real functions defined on $[0,1]$.

The organization of this article is as follows: In Section 2, we consider a simple case with $a(x)\equiv Kb(x)$, and $k_1=k_2=0$, where K is a positive constant. In Section 3, we consider some more cases with $a(x)\equiv Kb(x)$, and $k_1,\ k_2\in\mathbb{R}$. In Section 4, we prove the bifurcation from a stable fixed point to chaos. In Section 5, we consider some more general cases.

2. Klein-Gordon Equation with variable coefficients but no state term

Let us consider a simple case of (1.23) with $k_1=k_2=0$; i.e.,

$$[\frac{\partial}{\partial t}-a(x)\frac{\partial}{\partial x}][\frac{\partial}{\partial t}+b(x)\frac{\partial}{\partial x}]w=0,\quad\text{for } x\in(0,1),\ t>0. \tag{2.1}$$

Let

$$\xi=\int_0^x\frac{dx}{a(x)},\quad \eta=\int_0^x\frac{dx}{b(x)}, \tag{2.2}$$

then it follows from $a=Kb$ that $\eta=K\xi$, and thus

$$\frac{\partial}{\partial\xi}=K\frac{\partial}{\partial\eta}.$$

Let $\tilde{w}(\eta,t)=w(x,t)$, then (2.1) is equivalent to

$$[\frac{\partial}{\partial t}-K\frac{\partial}{\partial \eta}][\frac{\partial}{\partial t}+\frac{\partial}{\partial \eta}]\tilde{w}=0, \tag{2.3}$$

or

$$[\frac{\partial}{\partial t}+\frac{\partial}{\partial \eta}][\frac{\partial}{\partial t}-K\frac{\partial}{\partial \eta}]\tilde{w}=0. \tag{2.4}$$

It follows immediately that

$$\tilde{w}_\eta+\tilde{w}_t=f(\eta+Kt),$$
$$K\tilde{w}_\eta-\tilde{w}_t=g(t-\eta)$$

for some functions f and g depending on the initial data. Let

$$\tilde{u}(\eta,t)=\frac{1}{2}[\tilde{w}_\eta+\tilde{w}_t],$$
$$\tilde{v}(\eta,t)=\frac{1}{2}[K\tilde{w}_\eta-\tilde{w}_t].$$

Then $\tilde{u}$ keeps constant along lines $\eta+Kt=c$, and $\tilde{v}$ keeps constant along lines $\eta-t=c$.

We impose nonlinear boundary conditions as

$$\tilde{w}_t(0,t)=-\lambda\tilde{w}_\eta(0,t), \tag{2.5}$$
$$\tilde{w}_\eta(L,t)=\alpha\tilde{w}_t(L,t)-\beta\tilde{w}_t^3(L,t), \tag{2.6}$$

where $L=\int_0^1\frac{dx}{b(x)}$. Then

$$\tilde{v}(0,t)=G_{K,\lambda}(\tilde{u}(0,t))=\frac{K+\lambda}{1-\lambda}\tilde{u}(0,t), \tag{2.7}$$
$$\tilde{u}(L,t)=F_{K,\alpha,\beta}(\tilde{v}(L,t)), \tag{2.8}$$

where $u=F_{K,\alpha,\beta}(v)$ is the unique real solution of the cubic equation

$$\frac{4\beta}{(K+1)^2}(Ku-v)^3+(\frac{1}{K}-\alpha)(Ku-v)+(\frac{1}{K}+1)v=0. \tag{2.9}$$

It follows from (2.2) that

$$d\xi=\frac{dx}{a(x)},\quad d\eta=\frac{dx}{b(x)}, \tag{2.10}$$

and thus

$$\frac{\partial w}{\partial x}=\frac{\partial \tilde{w}}{\partial \xi}\frac{d\xi}{dx}=\frac{1}{a(x)}\frac{\partial \tilde{w}}{\partial \xi}, \tag{2.11}$$
$$\frac{\partial w}{\partial x}=\frac{1}{b(x)}\frac{\partial \tilde{w}}{\partial \eta}, \tag{2.12}$$

or

$$\frac{\partial \tilde{w}}{\partial \xi}=a(x)\frac{\partial w}{\partial x}, \tag{2.13}$$
$$\frac{\partial \tilde{w}}{\partial \eta}=b(x)\frac{\partial w}{\partial x}. \tag{2.14}$$

So the boundary condition (2.5)-(2.6) is equivalent to

$$w_t(0,t)=-\lambda b(0)w_x(0,t), \tag{2.15}$$
$$b(1)w_x(1,t)=\alpha w_t(1,t)-\beta w_t^3(1,t). \tag{2.16}$$

Let

$$u(x,t) = \tilde{u}(\eta,t) = \frac{1}{2}[b(x)w_x + w_t],$$
$$v(x,t) = \tilde{v}(\eta,t) = \frac{1}{2}[a(x)w_x - w_t],$$

then

$$v(0,t) = G_{K,\lambda}(u(0,t)) = \frac{K+\lambda}{1-\lambda}u(0,t), \tag{2.17}$$
$$u(1,t) = F_{K,\alpha,\beta}(v(1,t)). \tag{2.18}$$

Therefore

$$u(1,t) = F_{K,\alpha,\beta}(G_{K,\lambda}(u(1,t-L-\frac{L}{K})), \tag{2.19}$$
$$v(0,t) = G_{K,\lambda}(F_{K,\alpha,\beta}(v(0,t-L-\frac{L}{K})). \tag{2.20}$$

The dynamics of u and v are reduced to the properties of $G\circ F$ and $F\circ G$. Define a function $\psi(x) = \int_0^x \frac{dx}{b(x)}$ on $[0,1]$, then ψ is one to one.

Lemma 2.1 (Solution representations for $u(x,t)$ and $v(x,t)$)**.** *Assume* (2.1), (2.15) *and* (2.16). *Then for any x: $0 < x < 1$, and $t > 0$, we have, for t: $t = (1+\frac{1}{K})jL + \tau$, $j = 0,1,2,\dots$, $\tau > 0$,*

$$u(x,t) = \begin{cases} (F_{\alpha,\beta}\circ G_{\lambda,K})^j(F_{\alpha,\beta}(v_0(\psi^{-1}(1+\frac{1}{K})L - \tau - \frac{\eta}{K}))), \\ \quad \text{if } L \le K\tau + \eta \le (K+1)L; \\ (F_{\alpha,\beta}\circ G_{\lambda,K})^{j+1}(u_0(\psi^{-1}(K\tau + \eta - (K+1)L)))), \\ \quad \text{if } (K+1)L \le K\tau + \eta \le (K+2)L; \end{cases}$$

$$v(x,t) = \begin{cases} (G_{\lambda,K}\circ F_{\alpha,\beta}))^j(G(u_0(\psi^{-1}(K(\tau-\eta)))))), \\ \quad \text{if } t = (1+\frac{1}{K})jL + \tau, \quad 0 \le \tau - \eta \le \frac{L}{K}; \\ (G_{\lambda,K}\circ F_{\alpha,\beta}))^{j+1}(v_0(\psi^{-1}(1+\frac{1}{K})L - \tau + \eta)), \\ \quad \text{if } \frac{L}{K} \le \tau - \eta \le (1+\frac{1}{K})L. \end{cases}$$

Lemma 2.2 (Derivative Formulas)**.** *Let $0 < \alpha \le \frac{1}{K}$, $\beta > 0$ and $\eta > 0$, $\eta \ne 1$, where α and β are given and fixed, but η is a varying parameter. Define*

$$f_1(v,\eta) = G\circ F(v) = \frac{K+\eta}{1-\eta}F(v), \quad f_2(v,\eta) = F\circ G(v) = F(\frac{K+\eta}{1-\eta}v), \quad v \in \mathbb{R}.$$

Let $g(v)$ be the unique real solution of the cubic equation

$$\frac{4\beta}{(K+1)^2}g(v)^3 + (\frac{1}{K}-\alpha)g(v) + (\frac{1}{K}+1)v = 0, \tag{2.21}$$

for a given $v \in \mathbb{R}$. Then

$$\frac{\partial}{\partial v}f_1(v,\eta) = \frac{K+\eta}{K(1-\eta)}[1 - \frac{K+1}{\frac{12K\beta}{(K+1)^2}g(v)^2 + 1 - K\alpha}], \tag{2.22}$$
$$\frac{\partial}{\partial v}f_2(v,\eta) = \frac{K+\eta}{K(1-\eta)}[1 - \frac{K+1}{\frac{12K\beta}{(K+1)^2}g(\frac{K+\eta}{1-\eta}v)^2 + 1 - K\alpha}],$$
$$\frac{\partial}{\partial\eta}f_1(v,\eta) = \frac{1+K}{K(1-\eta)^2}[v + g(v)],$$

$$
\begin{aligned}
\frac{\partial}{\partial\eta} f_2(v,\eta) &= \frac{1+K}{K(1-\eta)^2}[1-\frac{K+1}{\frac{12K\beta}{(K+1)^2} g(\frac{K+\eta}{1-\eta}v)^2+1-K\alpha}]v,\\
\frac{\partial^2}{\partial\eta\partial v} f_1(v,\eta) &= \frac{K+1}{K(1-\eta)^2}[1-\frac{K+1}{\frac{12K\beta}{(K+1)^2} g(v)^2+1-K\alpha}],\\
\frac{\partial^2}{\partial v^2} f_1(v,\eta) &= \frac{K+\eta}{(1-\eta)}(-24)\beta\cdot\frac{g(v)}{[\frac{12K\beta}{(K+1)^2} g(v)^2+1-K\alpha]^3},\\
\frac{\partial^3}{\partial v^3} f_1(v,\eta) &= \frac{K+\eta}{1-\eta}24(K+1)\beta\frac{1-K\alpha-\frac{60K\beta}{(K+1)^2}g(v)^2}{[\frac{12K\beta}{(K+1)^2} g(v)^2+1-K\alpha]^5}.
\end{aligned} \tag{2.23}
$$

Lemma 2.3 (Intersections with the Lines $u-v=0$ and $u+v=0$)**.** *Let $0<\alpha\le 1/K$, $\beta>0$, $\eta>0$, $\eta\neq 1$ be given. Then*

(i) *$u=G\circ F(v)$ intersects the line $u=v$ at the points*

$$
\begin{aligned}
(u,v) = (&-\frac{K+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}, -\frac{K+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}),\\
&(0,0),\\
&(\frac{K+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}, \frac{K+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}});
\end{aligned}
$$

(ii) *$u=G\circ F(v)$ intersects the line $u=-v$ at the points*

$$
\begin{aligned}
(u,v) = (&-\frac{(K+1)(K+\eta)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{[2K+(1-K)\eta]\beta}},\\
&\frac{(K+1)(K+\eta)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{[2K+(1-K)\eta]\beta}}),\\
&(0,0),\\
&(\frac{(K+1)(K+\eta)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{[2K+(1-K)\eta]\beta}},\\
&-\frac{(K+1)(K+\eta)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{[2K+(1-K)\eta]\beta}});
\end{aligned}
$$

(iii) *$u=F\circ G(v)$ intersects the line $u=v$ at the points*

$$
\begin{aligned}
(u,v) = (&-\frac{1-\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}, -\frac{1-\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}),\\
&(0,0),\\
&(\frac{1-\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}, \frac{1-\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}});
\end{aligned}
$$

(iv) *$u=F\circ G(v)$ intersects the line $u=-v$ at the points*

$$
\begin{aligned}
(u,v) = (&-\frac{(1-\eta)(K+1)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{\beta[2K+(1-K)\eta]}},\\
&\frac{(1-\eta)(K+1)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{\beta[2K+(1-K)\eta]}}),
\end{aligned}
$$

$$
\begin{aligned}
&(0,0),\\
&\Big(\frac{(1-\eta)(K+1)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{\beta[2K+(1-K)\eta]}},\\
&-\frac{(1-\eta)(K+1)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{\beta[2K+(1-K)\eta]}}\Big).
\end{aligned}
$$

Remark 2.4. Conclusions (ii) and (iv) are based on the assumption that

$$\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{\beta[2K+(1-K)\eta]}\geq 0. \tag{2.24}$$

So we assume $K \leq 1$ and $(1+2\alpha)K \geq 1$ for conclusions (ii) and (iv). We also make this assumption for some related results below, e.g., (ii) and (iv) in Lemma 2.7.

Lemma 2.5 (v-axis Intercepts)**.** *Let $0<\alpha\leq\frac{1}{K}$, $\beta>0$, $\eta>0$, $\eta\neq 1$ be given. Then*

(i) *$u=G\circ F(v)$ has v-axis intercepts $v=-\frac{K+1}{2}\sqrt{\frac{1+\alpha}{\beta}}$, 0, $\frac{K+1}{2}\sqrt{\frac{1+\alpha}{\beta}}$;*

(ii) *$u=F\circ G(v)$ has v-axis intercepts*

$$v=-\frac{(K+1)(1-\eta)}{2(K+\eta)}\sqrt{\frac{1+\alpha}{\beta}},\quad 0,\quad \frac{(K+1)(1-\eta)}{2(K+\eta)}\sqrt{\frac{1+\alpha}{\beta}}.$$

Lemma 2.6 (Local Maximum, Minimum and Piecewise Monotonicity)**.** *Let $0<\alpha\leq\frac{1}{K}$, $\beta>0$, $\eta>0$, $\eta\neq 1$ be given. Then*

(i) *If $0<\eta<1$, then $G\circ F$ has local extremal values*

$$
\begin{aligned}
M=G\circ F(-v_c)&=\frac{K+\eta}{1-\eta}\,\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}},\\
m=G\circ F(v_c)&=-\frac{K+\eta}{1-\eta}\,\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}},
\end{aligned}
$$

where $v_c=\frac{3+(1-2\alpha)K}{6}\sqrt{\frac{1+\alpha}{3\beta}}$, and M, m are, respectively, the local maximum and minimum of $G\circ F$. The function $G\circ F$ is strictly increasing on $(-\infty,-v_c)$ and (v_c,∞), but strictly decreasing on $(-v_c,v_c)$.

On the other hand, if $\eta>1$, then $G\circ F$ has local minimum (m) and maximum (M) values

$$
\begin{aligned}
m=G\circ F(-v_c)&=\frac{K+\eta}{1-\eta}\,\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}},\\
M=G\circ F(v_c)&=-\frac{K+\eta}{1-\eta}\,\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}},
\end{aligned}
$$

where v_c is the same as above. The function $G\circ F$ is strictly decreasing on $(-\infty,-v_c)$ and (v_c,∞), but strictly increasing on $(-v_c,v_c)$.

(ii) *If $0<\eta<1$, then $F\circ G$ has local extremal values*

$$
\begin{aligned}
M=F\circ G(-\tilde v_c)&=\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}},\\
m=F\circ G(\tilde v_c)&=-\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}},
\end{aligned}
$$

where $\tilde{v}_c = \frac{1-\eta}{K+\eta}\frac{3+(1-2\alpha)K}{6}\sqrt{\frac{1+\alpha}{3\beta}}$, *and* M, m *are, respectively, the local maximum and minimum of* $F\circ G$. *The function* $F\circ G$ *is strictly increasing on* $(-\infty,-\tilde{v}_c)$ *and* $(\tilde{v}_c,\infty)$, *but strictly decreasing on* $(-\tilde{v}_c,\tilde{v}_c)$.

On the other hand, if $\eta>1$, *then* $F\circ G$ *has local extremal values*

$$m = F\circ G(-\tilde{v}_c) = -\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}},$$
$$M = F\circ G(\tilde{v}_c) = \frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}}.$$

The function $F\circ G$ *is strictly decreasing on* $(-\infty,-\tilde{v}_c)$ *and* $(\tilde{v}_c,\infty)$, *but strictly increasing on* $(-\tilde{v}_c,\tilde{v}_c)$.

Lemma 2.7 (Bounded Invariant Intervals)**.** *Let* $0<\alpha\le\frac{1}{K}$, $\beta>0$, $\eta>0$, $\eta\neq 1$.

(i) *If* $0<\eta<1$ *and*

$$M = \frac{K+\eta}{1-\eta}\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}} \le \frac{K+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}},$$

then the iterates of every point in the set

$$U \equiv (-\infty, -\frac{K+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}) \cup (\frac{K+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}, \infty)$$

escape to $\pm\infty$, *while those of any point in* $\mathbb{R}\backslash\bar{U}$ *are attracted to the bounded invariant interval* $\mathcal{I}\equiv[-M,M]$ *of* $G\circ F$.

(ii) *If* $\eta>1$ *and*

$$\begin{aligned} M &= -\frac{K+\eta}{1-\eta}\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}} \\ &\le \frac{(K+1)(K+\eta)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{[2K+(1-K)\eta]\beta}}, \end{aligned}$$

then the same conclusion as in (i) holds, with

$$\begin{aligned} U \equiv\ & (-\infty, -\frac{(K+1)(K+\eta)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{[2K+(1-K)\eta]\beta}}) \\ & \cup (\frac{(K+1)(K+\eta)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{[2K+(1-K)\eta]\beta}}, \infty) \end{aligned}$$

and $\mathcal{I}\equiv[-M,M]$ *for* $G\circ F$.

(iii) *If* $0<\eta<1$ *and* $M=\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}}\le\frac{1-\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}$, *then the same conclusion holds, with*

$$U \equiv (-\infty, -\frac{1-\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}) \cup (\frac{1-\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}, \infty)$$

and $\mathcal{I}=[-M,M]$ *for* $F\circ G$.

(iv) *If* $\eta>1$ *and*

$$M = \frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}} \le -\frac{(1-\eta)(K+1)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{\beta[2K+(1-K)\eta]}}),$$

then the same conclusion holds, with

$$U \equiv (-\infty, \frac{(1-\eta)(K+1)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{\beta[2K+(1-K)\eta]}}) \cup (-\frac{(1-\eta)(K+1)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{\beta[2K+(1-K)\eta]}}), \infty)$$

and $\mathcal{I} \equiv [-M, M]$ *for* $F \circ G$.

Now we try to set a period-doubling bifurcation theorem similar to our earlier work.

Theorem 2.8 (Period-Doubling Bifurcation Theorem). *Let* α: $0 < \alpha \le \frac{1}{K}$, $\beta > 0$ *be fixed, and let* η: $0 < \eta \le \underline{\eta}$ *be a varying parameter. Let* $h_1(v, \eta) = -G \circ F(v)$. *Then*

(i) $v_0 = \frac{(K+1)(K+\eta)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{[2K+(1-K)\eta]\beta}}$ *is a curve of fixed points of* h_1.

(ii) *The algebraic equation*

$$\begin{aligned} &\frac{K-2K\alpha\eta+3\eta}{6\eta}\sqrt{\frac{1+\alpha\eta}{3\beta\eta}} \\ &= \frac{(K+1)(K+\eta)}{2[2K+(1-K)\eta]}\sqrt{\frac{(1+2\alpha)K-1+[2+\alpha(1-K)]\eta}{[2K+(1-K)\eta]\beta}} \end{aligned} \tag{2.25}$$

has a solution $\eta = \eta_0$ *in* $(0, 1)$ *for any given* α: $0 < \alpha \le \frac{1}{K}$ *and* $\beta > 0$. *We have*

$$\frac{\partial}{\partial v} h_1(v_0, \eta_0) = -1.$$

(iii) *For* $\eta = \eta_0$ *satisfying* (2.25), *we have*

$$\begin{aligned} A &= \frac{\partial^2}{\partial\eta\partial v} h_1 + \frac{1}{2}\frac{\partial h_1}{\partial\eta}\frac{\partial^2 h_1}{\partial v^2} \\ &= -\Big((K+1)\{[(K+1)\alpha(2\alpha+3)+3]\eta_0^3 + (6K+\alpha K+\alpha-3)\eta_0^2 \\ &\quad - (7K-2)\eta_0 + 3K\}\Big)/\Big(3(1-\eta_0)^3(K+\eta_0)^2\Big). \end{aligned}$$

for $\eta = \eta_0$ *and* $v = v_0(\eta_0)$.

(iv)

$$\begin{aligned} B &= \frac{1}{3}\frac{\partial^3 h_1}{\partial v^3} + \frac{1}{2}(\frac{\partial^2 h_1}{\partial v^2})^2 \\ &= \frac{8(K+1)\beta\eta^4}{(1-\eta_0)^2(K+\eta_0)^5}\Big[(3\alpha-3K\alpha+1)\eta_0^3 + (9K\alpha-3K^2\alpha-K+2)\eta_0^2 \\ &\quad + K(6K\alpha-2K+7)\eta_0 + 5K^2\Big]. \end{aligned}$$

Proof. (i) This is an immediate consequence of Lemma 2.3.

(ii) We first determine the point(s) $v > 0$ such that $\frac{\partial h_1}{\partial v} = -1$. By (2.22), with a change of sign for f_1, we obtain

$$\frac{K+\eta}{K(1-\eta)}[1 - \frac{K+1}{\frac{12K\beta}{(K+1)^2}g(v)^2 + 1 - K\alpha}] = 1.$$

Therefore,

$$\begin{aligned} \frac{12K\beta}{(K+1)^2} g(v)^2 &= \frac{K}{\eta} + K\alpha, \\ g(v) &= \pm\frac{K+1}{2}\sqrt{\frac{1+\alpha\eta}{3\beta\eta}}. \end{aligned} \tag{2.26}$$

We choose positive v and thus the "$-$" sign in (2.26). Hence

$$g(v) = -\frac{K+1}{2}\sqrt{\frac{1+\alpha\eta}{3\beta\eta}}. \tag{2.27}$$

Since $g(v)$ satisfies (2.21), from (2.27) we obtain

$$\begin{aligned} v &= -\frac{K}{K+1}[\frac{4\beta}{(K+1)^2} g(v)^3 + (\frac{1}{K} - \alpha) g(v)] \\ &= \frac{K - 2K\alpha\eta + 3\eta}{6\eta}\sqrt{\frac{1+\alpha\eta}{3\beta\eta}} \\ &= \text{LHS of (2.25)}. \end{aligned} \tag{2.28}$$

Further setting (2.28) equal to $v_0(\eta)$ in (i), we obtain the RHS of (2.25).

Now we show that (2.25) has a solution. It is easy to see that the LHS tends to $+\infty$, but the RHS keeps bounded as $\eta \to 0^+$. So the LHS is greater than the RHS for some η close to 0. On the other hand, it is easy to verify that the LHS is smaller than the RHS for some η close to 1. It follows from the Mean Value Theorem of continuous functions that (2.25) has a solution in $(0, 1)$. (iii) and (iv) are also immediate consequences of Lemma 2.3. However, it is hard to judge whether $A \neq 0$ in (iii) and $B \neq 0$ in (iv) without knowledge of η_0. So we can not conclude the period-doubling bifurcation so far. We will try other methods in next section. ☐

Theorem 2.9 (Homoclinic Orbits for the Case $0 < \eta < 1$)**.** *Let $K > 0$, α: $0 < \alpha \leq \frac{1}{K}$ and $\beta > 0$ be fixed, and let $\eta \in (0, 1)$ be a varying parameter such that*

$$\frac{K+1}{2}\sqrt{\frac{1+\alpha}{\beta}} < \frac{K+\eta}{1-\eta}\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}}, \tag{2.29}$$

then the repelling fixed point 0 *of $G \circ F$ and $F \circ G$ has homoclinic orbits.*

Proof. By (2.22) and (2.29), we easily obtain

$$\begin{aligned} \frac{\partial}{\partial v} f_i(v, \eta)|_{v=0} &= \frac{K+\eta}{K(1-\eta)}(1 - \frac{K+1}{1-K\alpha}) \\ &= -\frac{(\alpha+1)(K+\eta)}{(1-\eta)(1-K\alpha)} < -1, \quad i = 1, 2. \end{aligned}$$

Therefore 0 is a repelling fixed point of $G \circ F$ and $F \circ G$. For a homoclinic to exist, the local maximum of $G \circ F$ (resp., $F \circ G$) must be larger than the positive v-axis intercept of $G \circ F$ (resp., $F \circ G$); i.e.,

$$\frac{K+1}{2}\sqrt{\frac{1+\alpha}{\beta}} < \frac{K+\eta}{1-\eta}\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}}. \tag{2.30}$$

This is exactly (2.29). ☐

3. Klein-Gordon equation with variable coefficients

In this section, we consider (1.23) with $a(x) \equiv Kb(x)$, where $K > 0$ is a constant. Let $W = e^{-ct-d\eta}w$, where w satisfies (2.1). Then $w = e^{ct+d\eta}W$, and

$$w_t = e^{ct+d\eta}(W_t + cW),$$
$$w_\eta = e^{ct+d\eta}(W_\eta + dW).$$

Then it follows immediately that

$$[\frac{\partial}{\partial t} - K\frac{\partial}{\partial \eta} + c - Kd][\frac{\partial}{\partial t} + \frac{\partial}{\partial \eta} + c + d]W = 0. \tag{3.1}$$

Let $k_1 = c - Kd$, $k_2 = c + d$, then (3.1) becomes

$$[\frac{\partial}{\partial t} - a(x)\frac{\partial}{\partial x} + k_1][\frac{\partial}{\partial t} + b(x)\frac{\partial}{\partial x} + k_2]W = 0. \tag{3.2}$$

Reversely, for any k_1 and k_2, there exist a unique pair of c and d such that $k_1 = c - Kd$, $k_2 = c + d$. Note that (3.2) is just our original problem (1.23).

It follows immediately that

$$\frac{1}{2}(w_\eta + w_t) = e^{ct+d\eta}\frac{W_\eta + W_t + k_2W}{2} = c_1',$$

along each characteristics $\eta + Kt = c_1$; and

$$\frac{1}{2}(Kw_\eta - w_t) = e^{ct+d\eta}\frac{KW_\eta - W_t - k_1W}{2} = c_2',$$

along each characteristics $\eta - t = c_2$. Let

$$u = \frac{W_\eta + W_t + k_2W}{2} = \frac{b(x)W_x + W_t + k_2W}{2},$$
$$v = \frac{KW_\eta - W_t - k_1W}{2} = \frac{a(x)W_x - W_t - k_1W}{2}.$$

Then $e^{ct+d\eta}u$ keeps constant along lines $\eta + Kt = c$, and $e^{ct+d\eta}v$ keeps constant along lines $\eta - t = c$.

We impose boundary conditions such that

$$v(0,t) = G_{K,\lambda}(u(0,t)) = \frac{K+\lambda}{1-\lambda}u(0,t),$$
$$u(1,t) = F_{K,\alpha,\beta}(v(1,t)).$$

Then it is easy to deduce the corresponding boundary conditions:

$$W_t(0,t) = -\lambda b(0)W_x(0,t) - \frac{(1-\lambda)k_1 + (K+\lambda)k_2}{K+1}W(0,t), \tag{3.3}$$

$$\begin{aligned} &\frac{(K+1)b(1)W_x(1,t) + (k_2 - k_1)W(1,t)}{2} \\ &= \alpha\frac{(K+1)W_t(1,t) + (k_1 + Kk_2)W(1,t)}{2} \\ &\quad - \frac{4\beta}{(K+1)^2}[\frac{(K+1)W_t(1,t) + (k_1 + Kk_2)W(1,t)}{2}]^3 . \end{aligned} \tag{3.4}$$

It is easy to verify the following reflective iterations:

$$u(1,t) = F(G(e^{-(1+\frac{1}{K})cL}u(1, t - L - \frac{L}{K})), \tag{3.5}$$

$$v(0,t) = G(e^{(d-\frac{c}{K})L}F(e^{-(c+d)L}v(0,t-L-\frac{L}{K})). \tag{3.6}$$

Lemma 3.1 (Solution representations for $u(x,t)$ and $v(x,t)$)**.** *Assume* (3.1), (3.3) *and* (3.4). *Then for any x: $0<x<1$, and $t>0$, we have, for $t=(1+\frac{1}{K})jL+\tau$, $j=0,1,2,\dots$, $\tau>0$,*

$$u(x,t)=\begin{cases}\big(F(G(e^{-(1+\frac{1}{K})cL}\cdot))\big)^j\Big(F(e^{-(c+d)(\tau-\frac{L-\eta}{K})}v_0(\psi^{-1}(1+\frac{1}{K})L-\tau-\frac{\eta}{K}))\Big),\\ \quad \text{if } L\le K\tau+\eta\le (K+1)L;\\ \big(F(G(e^{-(1+\frac{1}{K})cL}\cdot))\big)^j\Big(F_{\alpha,\beta}\Big(e^{-(c+d)L}G(e^{(Kd-c)(\tau+\frac{\eta}{K}-(1+\frac{1}{K})L)}\\ \times u_0(\psi^{-1}(K\tau+\eta-(K+1)L)))\Big)\Big),\\ \quad \text{if } (K+1)L\le K\tau+\eta\le (K+2)L;\end{cases}$$

$$v(x,t)=\begin{cases}\big(G(e^{(d-\frac{c}{K})L}F(e^{-(c+d)L}\cdot))\big)^j\Big(G\Big(e^{(Kd-c)(\tau-\eta)}u_0(\psi^{-1}(K(\tau-\eta)))\Big)\Big),\\ \quad \text{if } 0\le\tau-\eta\le\frac{L}{K};\\ \big(G(e^{(d-\frac{c}{K})L}F(e^{-(c+d)L}\cdot))\big)^j\Big(Ge^{(d-\frac{c}{K})L}F\Big(e^{-(c+d)(\tau-\eta-\frac{L}{K})}\\ \times v_0(\psi^{-1}((1+\frac{1}{K})L-\tau+\eta))\Big)\Big),\\ \quad \text{if } \frac{L}{K}\le\tau-\eta\le(1+\frac{1}{K})L.\end{cases}$$

Over all, the dynamics of u and v are determined by iterative compositions of functions f_1 and f_2 as:

$$f_1(v,\eta)=G_\eta(e^{(d-\frac{c}{K})L}F(e^{-(c+d)L}v))=\frac{K+\eta}{1-\eta}e^{(d-\frac{c}{K})L}F(e^{-(c+d)L}v)),$$
$$f_2(v,\eta)=F(G(e^{-(1+\frac{1}{K})cL}v))=F(\frac{K+\eta}{1-\eta}(e^{-(1+\frac{1}{K})cL}v)),$$

where $F=F_{K,\alpha,\beta}$ is as defined in previous section.

The proof of bifurcations depend on the analysis of the derivatives of f_1 or f_2 with respect to v and η. One can imagine that it is a hard work, since the formulations of f_1 and f_2 are so complicated.

Chaos and bifurcations are determined by the reflection maps $F(G(e^{-(1+\frac{1}{K})cL}\cdot))$ or $G(e^{(d-\frac{c}{K})L}F(e^{-(c+d)L}\cdot))$. Since the two maps are conjugate to each other, it suffices to consider either one of them.

Let us look at $F(G(e^{-(1+\frac{1}{K})cL}\cdot))$. Given α, β and K, F is fixed, then the map varies with $G(e^{-(1+\frac{1}{K})cL}\cdot)$. In a word, the dynamics depend on the value of $\frac{K+\lambda}{1-\lambda}e^{-(1+\frac{1}{K})cL}$. So it may be reasonable to define the factor as a parameter. Let $\eta=\frac{K+\lambda}{1-\lambda}e^{-(1+\frac{1}{K})cL}$, then the reflective map f_2 can be simplified as

$$f_2(v)=F_{\alpha,\beta,K}(\eta v). \tag{3.7}$$

Of course, it should be much easier to calculate the derivatives of f_2 with respect to v and the new parameter η, so we will easily get the regime of η where f_2 is chaotic or bifurcated. Then the corresponding regime of λ can be obtained by simple calculations. Let us do it as follows:

Lemma 3.2 (Derivative Formulas)**.** *Let $0<\alpha\le\frac{1}{K}$, $\beta>0$ and $\eta\in\mathbb{R}$, where α and β are given and fixed, but η is a varying parameter. Define $f(v,\eta)=F_{\alpha,\beta,K}(\eta v)$,*

$v \in \mathbb{R}$. Let $g(v)$ be the unique real solution of the cubic equation

$$\frac{4\beta}{(K+1)^2}g(v)^3 + (\frac{1}{K} - \alpha)g(v) + (\frac{1}{K} + 1)v = 0, \tag{3.8}$$

for a given $v \in \mathbb{R}$. Then

(i) $\frac{\partial}{\partial v}f(v,\eta) = \eta\frac{12\beta g(\eta v)^2-(\alpha+1)(K+1)^2}{12K\beta g(\eta v)^2+(1-K\alpha)(K+1)^2}$,

(ii) $\frac{\partial}{\partial \eta}f(v,\eta) = v\frac{12\beta g(\eta v)^2-(\alpha+1)(K+1)^2}{12K\beta g(\eta v)^2+(1-K\alpha)(K+1)^2}$,

(iii) $\frac{\partial^2}{\partial\eta\partial v}f(v,\eta) = \frac{12\beta g(\eta v)^2-(\alpha+1)(K+1)^2}{12K\beta g(\eta v)^2+(1-K\alpha)(K+1)^2} - \frac{24\beta(K+1)^6\eta g(\eta v)v}{[12K\beta g(\eta v)^2+(1-K\alpha)(K+1)^2]^3}$,

(iv) $\frac{\partial^2}{\partial v^2}f(v,\eta) = -\frac{24\beta(K+1)^6\eta^2 g(\eta v)}{[12K\beta g(\eta v)^2+(1-K\alpha)(K+1)^2]^3}$,

(v) $\frac{\partial^3}{\partial v^3}f(v,\eta) = -24\beta(K+1)^9\eta^3\frac{60K\beta g(\eta v)^2-(1-K\alpha)(K+1)^2}{[12K\beta g(\eta v)^2+(1-K\alpha)(K+1)^2]^5}$.

Lemma 3.3 (Intersections with the Lines $u - v = 0$ and $u + v = 0$)**.** *Let α: $0 < \alpha \le \frac{1}{K}$, $\beta > 0$, $\eta \in \mathbb{R}$ be given. Then*

(i) *If $\eta > K$ or $\eta < -\frac{1-K\alpha}{1+\alpha}$, then $u = f(v)$ intersects the line $u = v$ at the points*

$$(u,v) = (-\frac{K+1}{2(\eta-K)}\sqrt{\frac{1-K\alpha+(\alpha+1)\eta}{\beta(\eta-K)}}, -\frac{K+1}{2(\eta-K)}\sqrt{\frac{1-K\alpha+(\alpha+1)\eta}{\beta(\eta-K)}}),$$

(0,0),

$$(\frac{K+1}{2(\eta-K)}\sqrt{\frac{1-K\alpha+(\alpha+1)\eta}{\beta(\eta-K)}}, \frac{K+1}{2(\eta-K)}\sqrt{\frac{1-K\alpha+(\alpha+1)\eta}{\beta(\eta-K)}});$$

(ii) *If $\eta < -K$ or $\eta > \frac{1-K\alpha}{1+\alpha}$, then $u = f(v)$ intersects the line $u = -v$ at the points*

$$(u,v) = (-\frac{K+1}{2(\eta+K)}\sqrt{\frac{(\alpha+1)\eta+\alpha K-1}{\beta(\eta+K)}}, \frac{K+1}{2(\eta+K)}\sqrt{\frac{(\alpha+1)\eta+\alpha K-1}{\beta(\eta+K)}}),$$

$(0,0)$,

$$(\frac{K+1}{2(\eta+K)}\sqrt{\frac{(\alpha+1)\eta+\alpha K-1}{\beta(\eta+K)}}, -\frac{K+1}{2(\eta+K)}\sqrt{\frac{(\alpha+1)\eta+\alpha K-1}{\beta(\eta+K)}}).$$

Lemma 3.4 (v-aixs Intercepts)**.** *Let α: $0 < \alpha \le \frac{1}{K}$, $\beta > 0$, $\eta > 0$, $\eta \ne 1$ be given. Then $u = f(v)$ has v-axis intercepts*

$$v = -\frac{K+1}{2\eta}\sqrt{\frac{1+\alpha}{\beta}}, \quad 0, \quad \frac{K+1}{2\eta}\sqrt{\frac{1+\alpha}{\beta}}.$$

Lemma 3.5 (Local Maximum, Minimum and Piecewise Monotonicity)**.** *Let α: $0 < \alpha \le \frac{1}{K}$, $\beta > 0$, $\eta \in \mathbb{R}$ be given. Then If $\eta > 0$, then f has local extremal values*

$$M = f(-v_c) = \frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}},$$
$$m = f(v_c) = -\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}},$$

where $v_c = \frac{3+(1-2\alpha)K}{6\eta}\sqrt{\frac{1+\alpha}{3\beta}}$, and M, m are, respectively, the local maximum and minimum of f. The function f is strictly increasing on $(-\infty, -\tilde{v}_c)$ and $(\tilde{v}_c, \infty)$, but strictly decreasing on $(-\tilde{v}_c, \tilde{v}_c)$.

On the other hand, if $\eta < 0$, then f has local extremal values

$$\begin{gathered}
m = f(-v_c) = -\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}}, \\
M = f(v_c) = \frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}}.
\end{gathered}$$

The function f is strictly decreasing on $(-\infty, -v_c)$ and (v_c, ∞), but strictly increasing on $(-v_c, v_c)$.

Lemma 3.6 (Bounded Invariant Intervals). *Let $0 < \alpha \le 1/K$, $\beta > 0$, $\eta \in \mathbb{R}$.*

(i) *If $\eta > K$, and*

$$\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}} < \frac{K+1}{2(\eta - K)}\sqrt{\frac{1-K\alpha+(\alpha+1)\eta}{\beta(\eta-K)}},$$

then the iterates of every point in the set

$$\begin{aligned}
U \equiv (-\infty, -&\frac{K+1}{2(\eta - K)}\sqrt{\frac{1-K\alpha+(\alpha+1)\eta}{\beta(\eta-K)}}) \\
&\cup (\frac{K+1}{2(\eta - K)}\sqrt{\frac{1-K\alpha+(\alpha+1)\eta}{\beta(\eta-K)}}, \infty)
\end{aligned}$$

escape to $\pm\infty$, while those of any point in $\mathbb{R}\backslash\bar{U}$ are attracted to the bounded invariant interval $\mathcal{I} \equiv [-M, M]$ of f;

(ii) *If $\eta < -K$, and*

$$\frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}} < -\frac{K+1}{2(\eta + K)}\sqrt{\frac{(\alpha+1)\eta+\alpha K-1}{\beta(\eta+K)}},$$

then the iterates of every point in the set

$$\begin{aligned}
U = (-\infty, &\frac{K+1}{2(\eta + K)}\sqrt{\frac{(\alpha+1)\eta+\alpha K-1}{\beta(\eta+K)}}) \\
&\cup (-\frac{K+1}{2(\eta + K)}\sqrt{\frac{(\alpha+1)\eta+\alpha K-1}{\beta(\eta+K)}}, \infty)
\end{aligned}$$

escape to $\pm\infty$, while those of any point in $\mathbb{R}\backslash\bar{U}$ are attracted to the bounded invariant interval $\mathcal{I} \equiv [-M, M]$ of f.

Theorem 3.7 (Period-Doubling Bifurcation Theorem). *Let $K > 0$, α: $0 < \alpha \le \frac{1}{K} \le 2\alpha + 1$, $\beta > 0$ be fixed, and let η: $\eta > 0$ be a varying parameter. Then*

(i) *For $0 < \eta < \frac{1-K\alpha}{1+\alpha}$, 0 is the unique fixed point of f, and it is stable;*

(ii) *With the same α, β and K as in (i), but $\eta > \frac{1-K\alpha}{1+\alpha}$, then 0 becomes unstable, and there appear stable period-2 orbit*

$$\{\frac{K+1}{2(\eta + K)}\sqrt{\frac{(\alpha+1)\eta+\alpha K-1}{\beta(\eta+K)}}, -\frac{K+1}{2(\eta + K)}\sqrt{\frac{(\alpha+1)\eta+\alpha K-1}{\beta(\eta+K)}}\}$$

of f;

(iii) *The curve of the period-2 points:*

$$v = \pm \frac{K+1}{2(\eta+K)} \sqrt{\frac{(\alpha+1)\eta + \alpha K - 1}{\beta(\eta+K)}}$$

is smooth in the η-v plane, and tangent to the line $\{\frac{1-K\alpha}{1+\alpha}\} \times \mathbb{R}$ at point $(\frac{1-K\alpha}{1+\alpha}, 0)$;

(iv) *The period-2 orbit becomes unstable when η increases through*

$$\frac{K+1+\sqrt{(K+1)^2-(\alpha+1)(1-K\alpha)K}}{\alpha+1}.$$

Proof. (i) It follows from Lemma 3.2 (i) that

$$\begin{aligned} f'(0) &= \eta \frac{12\beta g(0)^2 - (\alpha+1)(K+1)^2}{12K\beta g(0)^2 + (1-K\alpha)(K+1)^2} \\ &= \eta \frac{-(\alpha+1)(K+1)^2}{(1-K\alpha)(K+1)^2} \\ &= -\eta \frac{\alpha+1}{1-K\alpha}. \end{aligned}$$

So $-1 < f'(0) < 0$, for $0 < \eta < \frac{1-K\alpha}{1+\alpha}$. So the origin point is a stable fixed point of f. As is shown in Lemma 3.3, there are no other fixed points of f when $0 < \eta < \frac{1-K\alpha}{1+\alpha} \le K$, so 0 is the unique fixed point of f.

(ii) Let

$$v_0 = \frac{K+1}{2(\eta+K)} \sqrt{\frac{(\alpha+1)\eta + \alpha K - 1}{\beta(\eta+K)}};$$

i.e., the positive intersection of f with the line $u = -v$. Then

$$\begin{gathered} F(\eta v_0) = -v_0, \\ \frac{\eta v_0 + g(\eta v_0)}{K} = -v_0, \\ g(\eta v_0) = -(K+\eta)v_0. \end{gathered}$$

So

$$g(\eta v_0)^2 = (\eta+K)^2 v_0^2 = \frac{(K+1)^2[(\alpha+1)\eta+\alpha K-1]}{4\beta(\eta+K)}.$$

Furthermore,

$$\begin{aligned} \frac{\partial f}{\partial v}|_{v=v_0} &= \eta \frac{12\beta \frac{(K+1)^2[(\alpha+1)\eta+\alpha K-1]}{4\beta(\eta+K)} - (\alpha+1)(K+1)^2}{12K\beta \frac{(K+1)^2[(\alpha+1)\eta+\alpha K-1]}{4\beta(\eta+K)} + (1-K\alpha)(K+1)^2} \\ &= \eta \frac{3[(\alpha+1)\eta+\alpha K-1] - (\alpha+1)(\eta+K)}{3K[(\alpha+1)\eta+\alpha K-1] + (1-K\alpha)(\eta+K)} \\ &= \eta \frac{2(\alpha+1)\eta + 2\alpha K - K - 3}{(2\alpha K + 3K + 1)\eta + 2(\alpha K - 1)K}. \end{aligned} \tag{3.9}$$

On the other hand,

$$2(\alpha+1)\eta + 2\alpha K - K - 3 > 2(1-\alpha K) + 2\alpha K - K - 3 > -K - 1,$$

and thus

$$|2(\alpha+1)\eta + 2\alpha K - K - 3| < K + 1 \tag{3.10}$$

for η greater than $\frac{1-\alpha K}{\alpha+1}$ and close to $\frac{1-\alpha K}{\alpha+1}$ enough,

$$\begin{aligned}
&(2\alpha K+3K+1)\eta+2(\alpha K-1)K \\
&=2K(\alpha+1)\eta+(K+1)\eta+2(\alpha K-1)K \\
&>2K(1-\alpha K)+(K+1)\eta+2(\alpha K-1)K \\
&>(K+1)\eta,
\end{aligned} \tag{3.11}$$

for η greater than $\frac{1-\alpha K}{\alpha+1}$.

Combining (3.9), (3.10) and (3.11), we have

$$|f'(v_0)|<1$$

for η greater than $\frac{1-\alpha K}{\alpha+1}$ and close to $\frac{1-\alpha K}{\alpha+1}$ enough. By similar arguments we have

$$|f'(-v_0)|<1$$

for η greater than $\frac{1-\alpha K}{\alpha+1}$ and close to $\frac{1-\alpha K}{\alpha+1}$ enough.

Combining the two aspects above, we conclude that the new emerging period-2 orbit ares stable. This completes the proof of the period-2 bifurcations of f at the origin.

(iii) It is easy to verify that

$$v=\pm\frac{K+1}{2(\eta+K)}\sqrt{\frac{(\alpha+1)\eta+\alpha K-1}{\beta(\eta+K)}}$$

is differentiable with respect to η for η in $(\frac{1-\alpha K}{\alpha+1},\infty)$. The derivative is ∞ at $\eta=\frac{1-\alpha K}{\alpha+1}$. This shows that the curve is smooth in the η-v plane, and tangent to line $\{\frac{1-K\alpha}{1+\alpha}\}\times\mathbb{R}$ at point $(\frac{1-K\alpha}{1+\alpha},0)$.

(iv) Let

$$\frac{\partial f}{\partial v}=\eta\frac{2(\alpha+1)\eta+2\alpha K-K-3}{(2\alpha K+3K+1)\eta+2(\alpha K-1)K}>1,$$

and then it follows that

$$\eta>\frac{K+1+\sqrt{(K+1)^2-(\alpha+1)(1-\alpha K)K}}{\alpha+1},$$

or

$$\eta<\frac{K+1-\sqrt{(K+1)^2-(\alpha+1)(1-\alpha K)K}}{\alpha+1}.$$

So the period-2 orbit becomes unstable when η increases through

$$\frac{K+1+\sqrt{(K+1)^2-(\alpha+1)(1-\alpha K)K}}{\alpha+1}.$$

□

Remark 3.8. We just conclude that 0 is stable by $|f'(0)|<1$ when $0<\eta<\frac{1-K\alpha}{1+\alpha}$. However, $|f'(0)|<1$ just implies the local stability of 0. In fact, it follows from Lemma 3.3 that $u=f(v)$ does not intersect with line $u=v$ or $u=-v$ at other points except the origin, so we have $|f(v)|<|v|$ for $v\neq 0$. Therefore, 0 attracts $(-\infty,+\infty)$, illustrated by Figure 3.1:

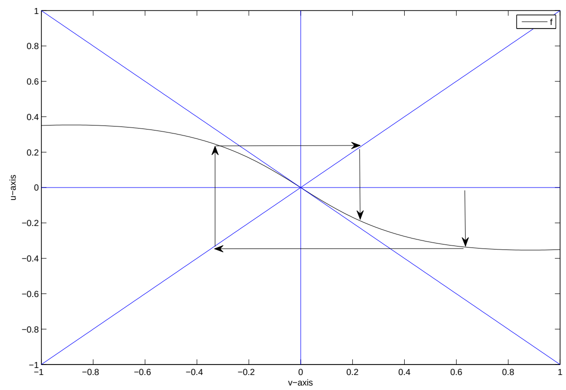

FIGURE 3.1. Global attraction diagram of 0 for f, where $\alpha = 0.5$, $\beta = 1$, $K = 0.7$, $\eta = 0.4$.

Remark 3.9. The stable period-2 orbit in Theorem 3.7 (ii) attracts $(-\infty, 0) \cup (0, +\infty)$ for η larger than and close to $\frac{1-K\alpha}{1+\alpha}$. Since $-f(-f(v)) = f(f(v))$, so the period-2 stability under f is equivalent to that under $-f$. The global attraction of its period-2 orbit can be easily illustrated by its graph, e.g., Figure 3.2.

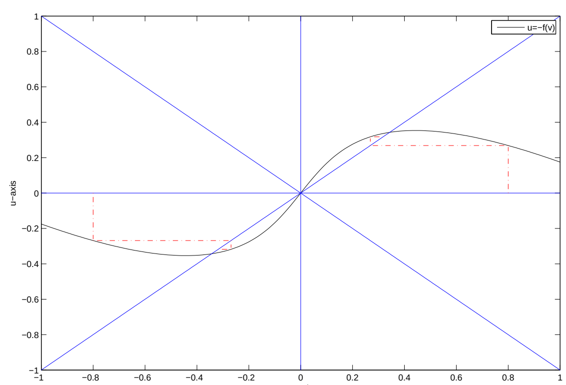

FIGURE 3.2. Global attraction diagram of the period-2 orbit for f, where $\alpha = 0.5$, $\beta = 1$, $K = 0.7$, $\eta = 0.8$.

Theorem 3.10 (Homoclinic Orbits for the Case $\eta > 0$)**.** *Let $K > 0$, α: $0 < \alpha \leq 1/K$ and $\beta > 0$ be fixed, and $\eta \geq \frac{3\sqrt{3}(K+1)}{2(1+\alpha)}$, then the repelling fixed point 0 of f has homoclinic orbits.*

Proof. For a homoclinic orbit of 0 to exist, the local maximum of f must be no less than the positive v-axis intercept of f, i.e.,

$$\frac{K+1}{2\eta}\sqrt{\frac{1+\alpha}{\beta}} < \frac{1+\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}},$$

which is equivalent to

$$\eta > \frac{3\sqrt{3}(K+1)}{2(1+\alpha)}. \tag{3.12}$$

On the other hand, it follows from Lemma 3.2 (i) that

$$\frac{\partial}{\partial v} f(v,\eta)|_{v=0} = -\eta \frac{\alpha+1}{1-K\alpha}.$$

Therefore 0 is a repelling fixed point of f for η larger than $\frac{1-K\alpha}{\alpha+1}$, which is implied by (3.12). This completes the proof. $\square$

For $\eta = \frac{3\sqrt{3}(K+1)}{2(1+\alpha)}$, v_c (or $-v_c$) lies on a degenerated homoclinic orbit. When $\eta < \frac{3\sqrt{3}(K+1)}{2(1+\alpha)}$, f has maximum less than the v-axis intercept. Hence there are no points homoclinic to 0 for these η-values. On the other hand, when $\eta > \frac{3\sqrt{3}(K+1)}{2(1+\alpha)}$, there are infinitely many distinct homoclinic orbits. Consequently, f is not structurally stable when $\eta = \frac{3\sqrt{3}(K+1)}{2(1+\alpha)}$, i.e., a small change in f can change the number of homoclinic orbits.

Example 3.11. The parameters chosen are $\alpha = 0.5$, $\beta = 1$, $\lambda = 0.85$, $k_1 = k_2 = 0.7$, $K = 0.7$, $b(x) = 1 + 3x^2$,

$$w(x,0) = \sin^2(\pi x), \quad w_t(x,0) = 0.$$

Figures 3.3–3.6 show the spatiotemporal profiles of u, v, w_x and w_t for $x \in [0,1]$ and $t \in [7.34, 8.80]$ respectively; Figures 3.7 and 3.8 illustrate the reflection maps $F(G(e^{-(1+\frac{1}{K})cL}\cdot))$ and $G(e^{(d-\frac{c}{K})L}F(e^{-(c+d)L}\cdot))$ respectively.

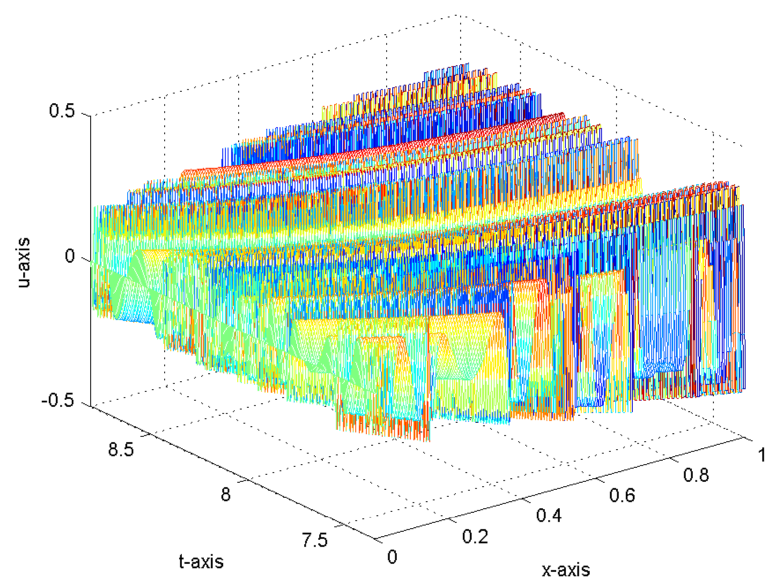

FIGURE 3.3. The spatiotemporal profile of $u(x,t)$ for $x \in [0,1]$ and $t \in [7.34, 8.80]$.

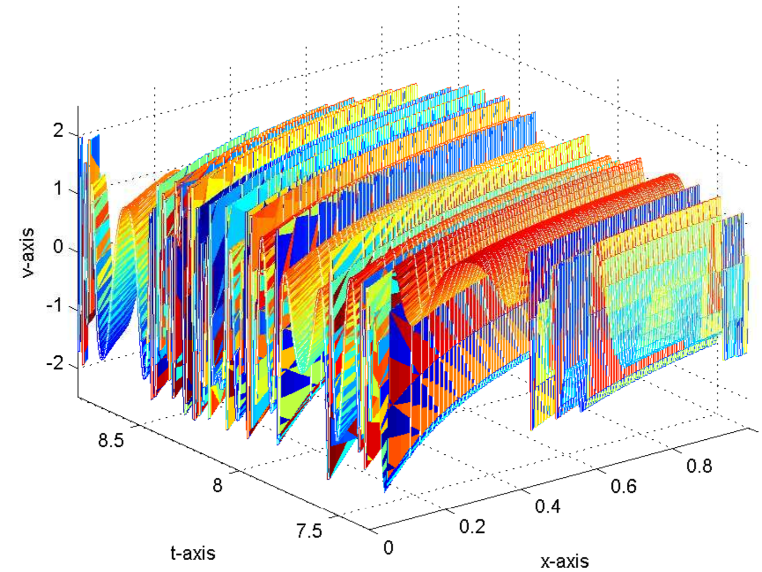

FIGURE 3.4. The spatiotemporal profile of $v(x,t)$ for $x \in [0,1]$ and $t \in [7.34, 8.80]$.

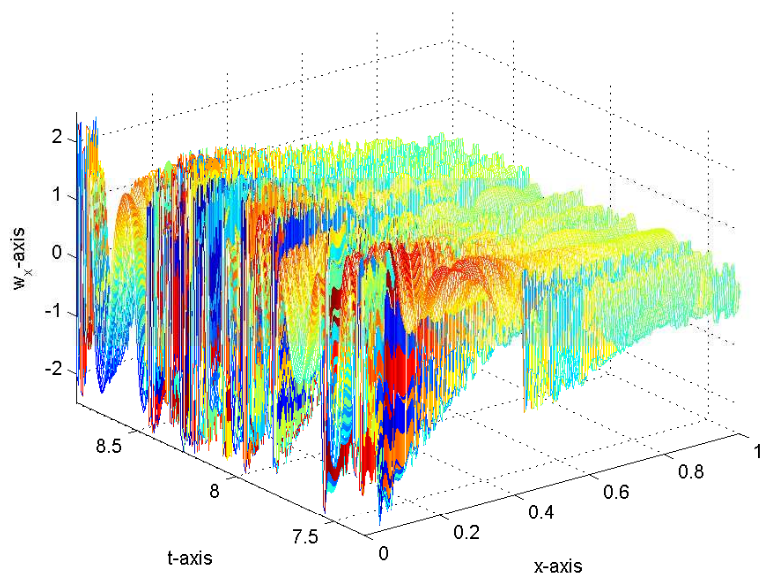

FIGURE 3.5. The spatiotemporal profile of $w_x(x,t)$ for $x \in [0,1]$ and $t \in [7.34, 8.80]$.

Figures 3.7 and 3.8 show that H_1 and H_2 are topologically transitive, so probably they are chaotic according to Devaney's definition [8].

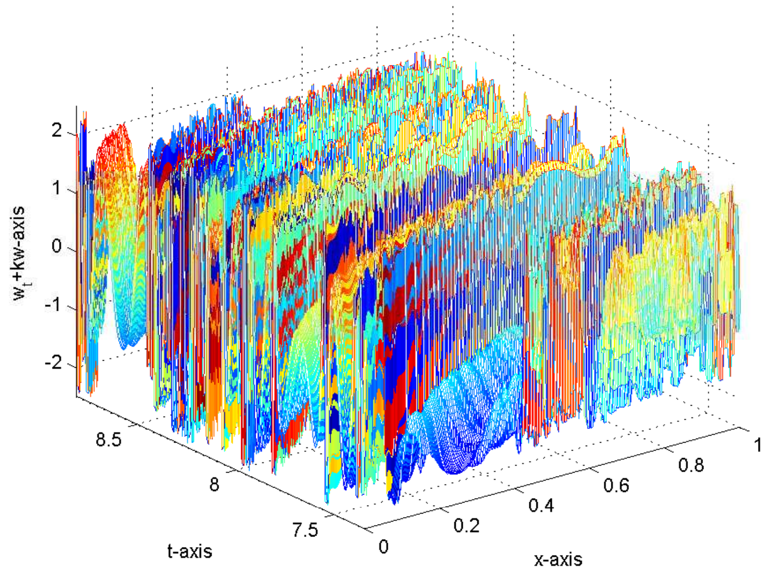

FIGURE 3.6. The spatiotemporal profile of $w_t(x,t) + kw(x,t)$ for $x \in [0,1]$ and $t \in [7.34, 8.80]$.

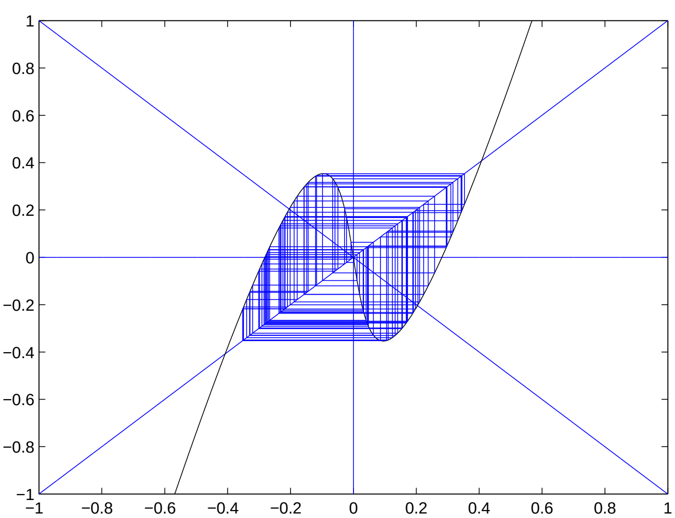

FIGURE 3.7. Orbits of $H_1 = F(G(e^{-(\frac{k_1}{K}+k_2)L}\cdot))$, $\alpha = 0.5$, $\beta = 1$, $\lambda = 0.85$, $k_1 = k_2 = 0.7$, $K = 0.7$.

4. Period doubling bifurcation and pitchfork bifurcation route to chaos

The mapping f_η (or H_1, H_2) has a unique fixed point or periodic point (0), which is stable when $\eta > 0$ is small enough. As η increases, the fixed point 0 becomes unstable, and there appears a stable periodic-2 orbit, then the period-2

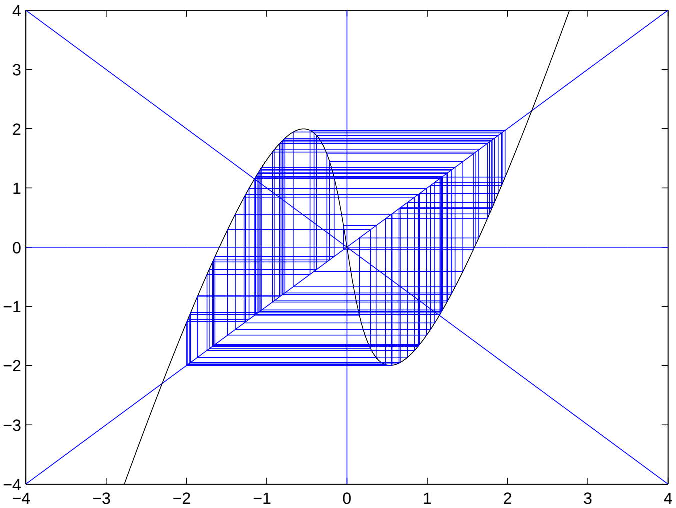

FIGURE 3.8. Orbits of $H_2 = G(e^{-\frac{k_1}{K}L}F(e^{-k_2L}\cdot))$, $\alpha = 0.5$, $\beta = 1$, $\lambda = 0.85$, $k_1 = k_2 = 0.7$, $K = 0.7$.

orbit becomes unstable, too. Finally, homoclinic orbits appear when η is large enough. We have proved these facts in Theorem 3.7 and 3.10. In this section, we try to explore more about the bifurcation routes.

Let us start by a bifurcation diagram, we take $\alpha = 0.5$, $\beta = 1$, $K = 0.7$, and let η vary from 0.4 to 3. The stable fixed point 0 bifurcates into a stable symmetric period-2 orbit at $\eta \approx 0.43$, then the symmetric period-2 orbit bifurcates into two new stable period-2 orbits at $\eta \approx 2.2$, then they bifurcate into two period-4 orbits at $\eta \approx 2.6$. The bifurcations are illustrated by Figures 4.1 and 4.2. To distinguish the pitchfork period-2 bifurcation from the period doubling bifurcation of period-4, we start our iteration at $v = 0.3$ and $v = -0.3$ respectively, and found that they are stablized by different period-2 orbits.

It is easy to see that there is a pitchfork bifurcation of period-2 following the period doubling bifurcation of period-2 described by Theorem 3.7.

Let us compare this experiment results with Theorem 3.7.

• The first bifurcation: from the fixed point to period-2 orbit. According to Theorem 3.7 (i)-(ii), the first bifurcation parameter value is

$$\eta = \frac{1 - K\alpha}{1 + \alpha}. \tag{4.1}$$

Substitute the experiment parameter values $\alpha = 0.5$ and $K = 0.7$ to (4.1), we obtain

$$\eta = \frac{1 - K\alpha}{1 + \alpha} = 0.4333,$$

which agrees with the bifurcation diagrams.

• The second bifurcation: from the symmetric period-2 orbit to the nonsymmetric period-2 orbits. According to Theorem 3.7 (iv), the second bifurcation

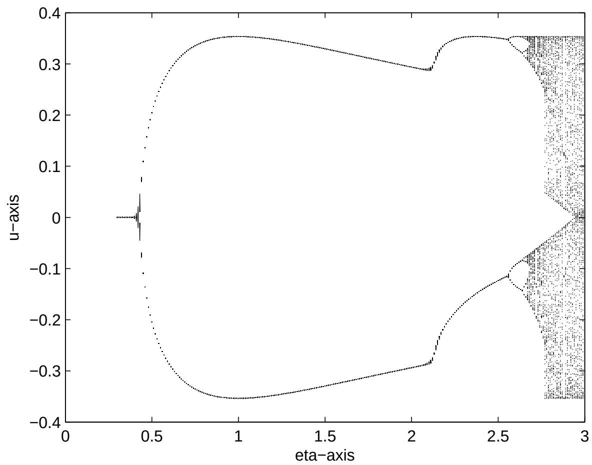

FIGURE 4.1. Bifurcation diagram of f, where $\alpha = 0.5$, $\beta = 1$, $K = 0.7$, iteration starts at $v = 0.3$.

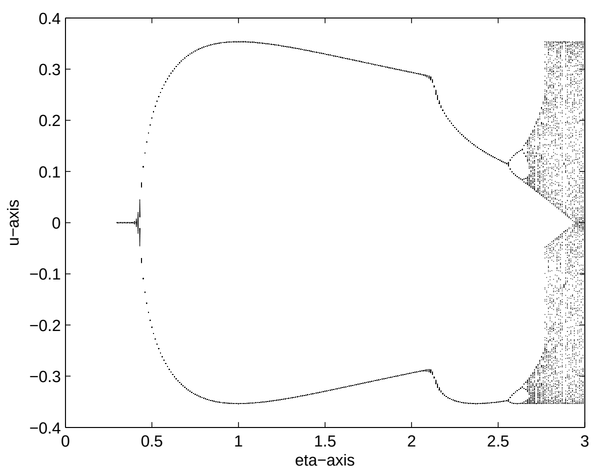

FIGURE 4.2. Bifurcation diagram of f, where $\alpha = 0.5$, $\beta = 1$, $K = 0.7$, iteration starts at $v = -0.3$.

parameter value is

$$\eta = \frac{K + 1 + \sqrt{(K+1)^2 - (\alpha+1)(1-K\alpha)K}}{\alpha+1}. \tag{4.2}$$

Substitute the experiment parameters to (4.2), we obtain $\eta = 2.1238$, which agrees with the bifurcation diagrams.

The *old* period-2 points $\pm\frac{K+1}{2(\eta+K)}\sqrt{\frac{(\alpha+1)\eta+\alpha K-1}{\beta(\eta+K)}}$ of f are fixed points of $-f$. Suppose $-f$ has a period-doubling bifurcation at

$$\eta=\frac{K+1+\sqrt{(K+1)^2-(\alpha+1)(1-K\alpha)K}}{\alpha+1},$$

then the period-2 orbits of $-f$ are just the *new* period-2 orbits of f. Let us check it as follows.

Proof. Let $h=-f$, we have found the parameter value and the fixed point for which $\frac{\partial h}{\partial v}=-1$. So it suffices to verify that

$$A\equiv\Big[\frac{\partial^2 h}{\partial\eta\partial v}+\frac{1}{2}(\frac{\partial h}{\partial\eta})\frac{\partial^2 h}{\partial v^2}\Big]\neq 0,$$
$$B\equiv\frac{1}{3}\frac{\partial^3 h}{\partial v^3}+\frac{1}{2}(\frac{\partial^2 h}{\partial v^2})^2\neq 0,$$

for

$$v=v_0\triangleq\frac{K+1}{2(\eta+K)}\sqrt{\frac{(\alpha+1)\eta+\alpha K-1}{\beta(\eta+K)}},\tag{4.3}$$

$$\eta=\eta_0\triangleq\frac{K+1+\sqrt{(K+1)^2-(\alpha+1)(1-K\alpha)K}}{\alpha+1}.\tag{4.4}$$

It follows from Theorem 3.2 that

$$\begin{aligned}A=&-\frac{12\beta g(\eta_0v_0)^2-(\alpha+1)(K+1)^2}{12K\beta g(\eta_0v_0)^2+(1-K\alpha)(K+1)^2}\\&+\frac{24\beta(K+1)^6\eta_0g(\eta_0v_0)v_0}{[12K\beta g(\eta_0v_0)^2+(1-K\alpha)(K+1)^2]^3}\\&-\frac{12\beta(K+1)^6\eta_0^2g(\eta_0v_0)v_0[12\beta g(\eta_0v_0)^2-(\alpha+1)(K+1)^2]}{[12K\beta g(\eta_0v_0)^2+(1-K\alpha)(K+1)^2]^4},\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}B=&\,8\beta(K+1)^9\eta_0^3\frac{60K\beta g(\eta_0v_0)^2-(1-K\alpha)(K+1)^2}{[12K\beta g(\eta_0v_0)^2+(1-K\alpha)(K+1)^2]^5}\\&+\frac{288\beta^2(K+1)^{12}\eta_0^4g(\eta_0v_0)^2}{[12K\beta g(\eta_0v_0)^2+(1-K\alpha)(K+1)^2]^6}.\end{aligned}$$

Combing Theorem 3.2 (i) and the fact that $\frac{\partial f}{\partial v}=1$ for $\eta=\eta_0$, $v=v_0$, we have

$$12\beta g(\eta_0v_0)^2-(\alpha+1)(K+1)^2>0.$$

Noting that $g(\eta v)$ and v have opposite sign, so all the terms in the RHS of (4.5) are negative. Therefore $A<0$.

By similar arguments we have $B>0$. This completes the proof. $\square$

The *old* period-2 orbit $\{p(\eta),-p(\eta)\}$ becomes unstable after the second bifurcation, and a pair of stable period-2 orbits appear. Denote them by $\{p_1(\eta),q_1(\eta)\}$ and $\{p_2(\eta),q_2(\eta)\}$ respectively, where p_1 and p_2 are around p, q_1 and q_2 are around $-p$. Let

$$p_1>p,\quad p_2<p.$$

Then

$$q_1>-p,\quad q_2<-p,$$

since f is increasing around p and $-p$. This pair of stable period-2 orbits can be illustrated by Figure 4.1 and Figure 4.2 (curves over $\eta \in (2.12, 2.55)$).

Let us look at the period-4 bifurcation. By period doubling bifurcation theorems, it occurs where $\frac{\partial}{\partial v}(f \circ f)|_{v=p_i} = f'(p_i)f'(q_i) = -1$, $i = 1, 2$.

On the other hand,

$$\frac{\partial}{\partial v}(f \circ f) = f'(p)f'(-p) = 1$$

at the pitchfork bifurcation point of period-2. Since $\frac{\partial}{\partial v}(f \circ f)$ varies continuously with respect to parameters and arguments, so $\frac{\partial}{\partial v}(f \circ f)$ must vanish at some period-2 point before period-4 bifurcation. Since $\frac{\partial}{\partial v}(f \circ f) = 0$ if and only if the period-2 cycle contains extremal point v_c or $-v_c$, the extremal point v_c or $-v_c$ must be contained in a period-2 orbit before period-4 bifurcation. This process can be illustrated by the following experiment results and figures:

We take $\alpha = 0.5$, $\beta = 1$, $K = 0.7$, $\eta = 1.8$, then Theorem 3.7 (ii) tells that the unique symmetric period-2 orbit is $\{0.3079, -0.3079\}$. Figure 4.3 illustrates this fact.

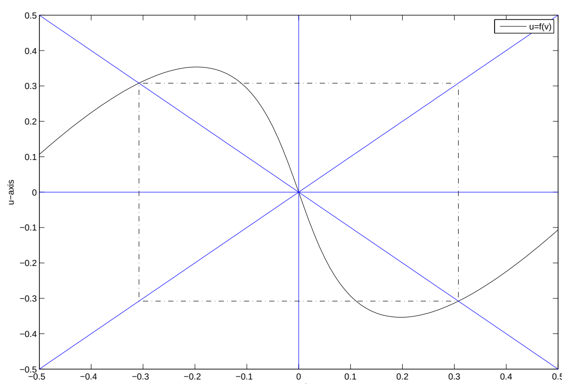

FIGURE 4.3. Stable symmetric period-2 orbit of f, where $\alpha = 0.5$, $\beta = 1$, $K = 0.7$, $\eta = 1.8$.

Then we take larger $\eta = 2.2$, the symmetric period-2 orbit bifurcates into two branches of stable period-2 orbits, illustrated by Figure 4.4.

Take $\eta = 2.3$, the two branches of period-2 orbits go apart, and pass by the extremal points, illustrated by Figure 4.5

Take $\eta = 2.6$, each period-2 orbit bifurcates into a stable period-4 orbit, illustrated by fig. 4.6.

It is well known that a discrete dynamical system is chaotic if it has a homoclinic orbit. According to Theorem 3.10, f has homoclinic orbits and chaos when $\eta \geq \frac{3\sqrt{3}(K+1)}{2(1+\alpha)}$. For $\alpha = 0.5$ and $K = 0.7$, the condition is as $\eta \geq 2.9445$, which agrees with bifurcation diagrams Figure 4.1 and Figure 4.2.

In addition to homoclinic orbits, period three is a classical criteria for chaos.

Theorem 4.1. *Given α, β and K, there exists η_3 such that $F(\eta_3 \cdot)$ has period* 3.

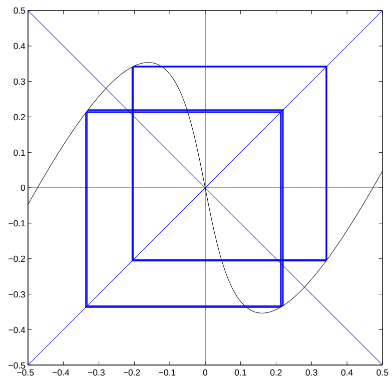

FIGURE 4.4. Two branches of stable period-2 orbits of f, where $\alpha = 0.5$, $\beta = 1$, $K = 0.7$, $\eta = 2.2$.

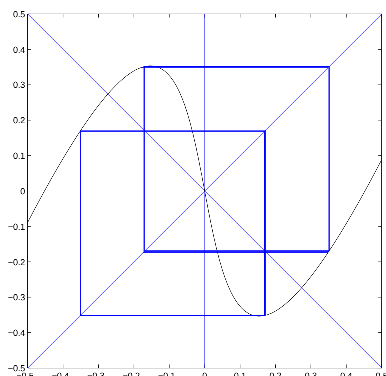

FIGURE 4.5. Two branches of period-2 orbits of f, where $\alpha = 0.5$, $\beta = 1$, $K = 0.7$, $\eta = 2.3$, period-2 orbits pass by extremal points $\pm v_c$.

Proof. It is easy to verify that $F(\bar{\eta}\cdot)$ has period three, where $\bar{\eta}$ is the critical value of η such that the local maximum equals to the positive intercept with line $u = v$. Let $d = M$, $c = -v_c$, $b \in (0, v_c)$ such that

$$F(\bar{\eta}b) = -v_c, \tag{4.6}$$

and $a \in (v_c, M]$ such that

$$F(\bar{\eta}a) = b. \tag{4.7}$$

It is easy to see that $c < b < a < d$. Then the period three follows from the Li-York's Theorem. □

Remark 4.2. By continuity, there exists $a \in (v_c, M]$ such that

$$f_\eta^2(a) < f_\eta(a) < a < f_\eta^3(a)$$

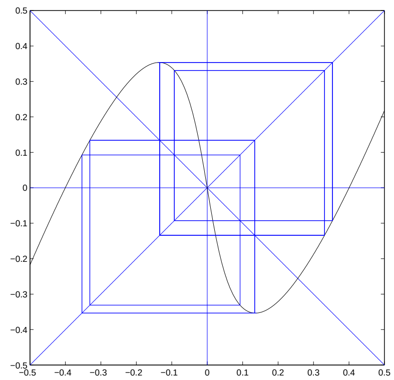

FIGURE 4.6. Two branches of stable period-4 orbits of f, where $\alpha = 0.5$, $\beta = 1$, $K = 0.7$, $\eta = 2.6$.

for η around $\bar{\eta}$. So f_η has period three for η in a neighbor of $\bar{\eta}$. This is why we often see *Period Three Windows* in bifurcation diagrams. Of course, a rigorous proof must include the stability of the period-3 orbits in the *window*, e.g., an extremal point is in one of the period-3 cycles. We omit the rigorous proof here.

We give the period-3 orbit for $\alpha = 0.5$, $\beta = 1$, $K = 0.7$ and $\eta = 4.1$ in Figure 4.7.

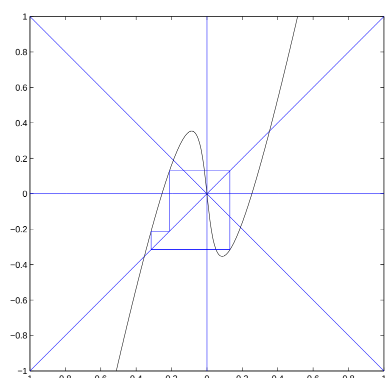

FIGURE 4.7. The period-3 orbit of f, where $\alpha = 0.5$, $\beta = 1$, $K = 0.7$, $\eta = 4.1$.

We give the bifurcation diagram of f for $\alpha = 0.5$, $\beta = 1$ and $\eta \in [3.5, 4.1]$ in Figure 4.8. Two windows of period-6 seem emerge, over $\eta = 3.77$ and 3.92 respectively.

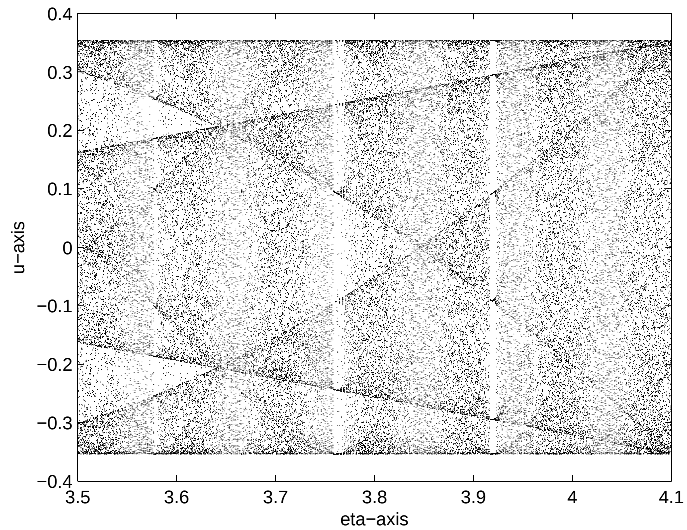

FIGURE 4.8. The bifurcation diagram of f, where $\alpha = 0.5$, $\beta = 1$, $K = 0.7$, $\eta \in [3.5, 4.1]$. Two windows of period-6 seem emerge.

5. A more general case

Let $W = e^{-ct-\int_0^\eta d(\eta)d\eta}w$, where $d(\eta)$ is a real function defined on $[0, L]$, and w satisfies (2.1). Then $w = e^{ct+\int_0^\eta d(\eta)d\eta}W$, and

$$w_t = e^{ct+\int_0^\eta d(\eta)d\eta}(W_t + cW),$$
$$w_\eta = e^{ct+\int_0^\eta d(\eta)d\eta}(W_\eta + d(\eta)W).$$

Then it follows immediately that

$$[\frac{\partial}{\partial t} - K\frac{\partial}{\partial \eta} + c - Kd(\eta)][\frac{\partial}{\partial t} + \frac{\partial}{\partial \eta} + c + d(\eta)]W = 0, \tag{5.1}$$

or

$$[\frac{\partial}{\partial t} - a(x)\frac{\partial}{\partial x} + c - Kd(\psi(x))][\frac{\partial}{\partial t} + b(x)\frac{\partial}{\partial x} + c + d(\psi(x))]W = 0. \tag{5.2}$$

Let $k_1(x) = c - Kd(\psi(x))$, $k_2(x) = c + d(\psi(x))$, then

$$[\frac{\partial}{\partial t} - a(x)\frac{\partial}{\partial x} + k_1(x)][\frac{\partial}{\partial t} + b(x)\frac{\partial}{\partial x} + k_2(x)]W = 0. \tag{5.3}$$

On the other hand, given $k_1(x)$ and $k_2(x)$, assume that $k_1(x) + Kk_2(x)$ is a constant. Then

$$d(\eta) = \frac{-k_1(\psi^{-1}(\eta)) + k_2(\psi^{-1}(\eta))}{K+1}, \quad c = \frac{k_1 + Kk_2}{K+1}.$$

Let

$$u = \frac{1}{2}[b(x)W_x + W_t + k_2(x)W], \quad v = \frac{1}{2}[a(x)W_x - W_t - k_1(x)W].$$

Lemma 5.1 (Constancy along characteristics)**.**

$$\begin{aligned} e^{ct+\int_0^\eta d(\eta)d\eta}u &= c_1', \quad \textit{along each characteristic } \eta + Kt = c_1, \\ e^{ct+\int_0^\eta d(\eta)d\eta}v &= c_2', \quad \textit{along each characteristic } \eta - t = c_2. \end{aligned} \tag{5.4}$$

We impose boundary conditions

$$\begin{gathered} W_t(0,t) + cW(0,t) = -\lambda[b(0)W_x(0,t) + d(0)W(0,t)], \\ b(1)W_x(1,t) + d(L)W(1,t) = \alpha[W_t(1,t) + cW(1,t)] - \beta[W_t(1,t) + cW(1,t)]^3, \end{gathered}$$

and obtain the following result.

Lemma 5.2 (Composite reflection relations)**.**

$$\begin{gathered} u(1,t) = F_{\alpha,\beta,K}(G_\lambda(e^{-(1+\frac{1}{K})cL}u(1,t-(1+\frac{1}{K})L))), \\ v(0,t) = G_\lambda(e^{\int_0^L d(\eta)d\eta - c\frac{L}{K}}F_{\alpha,\beta}(e^{-cL-\int_0^L d(\eta)d\eta}v(0,t-(1+\frac{1}{K})L))), \end{gathered}$$

for any $t > 0$.

Then the dynamics of u and v are determined by the iterative compositions of $F_{\alpha,\beta,K}(G_\lambda(e^{-(1+\frac{1}{K})cL}\cdot))$ and $G_\lambda(e^{\int_0^L d(\eta)d\eta - c\frac{L}{K}}F_{\alpha,\beta}(e^{-cL-\int_0^L d(\eta)d\eta}\cdot))$.

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