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OSCILLATION OF SOLUTIONS OF LINEAR IMPULSIVE PARTIAL DIFFERENCE EQUATIONS WITH CONTINUOUS VARIABLES

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Abstract. This article studies oscillation of linear partial difference equations with continuous arguments under impulse perturbations through both variables. The results improve on previously established results; furthermore a new connection is established between impulsive partial difference equations with continuous arguments and the more developed area of partial difference equations with discrete variables.

1. Introduction

The theory of impulsive equations is an important area of scientific activity. Since every nonimpulsive equation can be regarded as an impulsive equation with no impulse effect, this fact makes the theory of impulsive equations more interesting than the corresponding theory of nonimpulsive equations. Moreover, such equations appear in the modeling of several real-world phenomena in many areas such as physics, biology and engineering.

To the best of our knowledge, first paper on impulsive equations was published on the oscillation differential equations [1]. From the publication of this paper up to the present time, impulsive delay differential equations started receiving attention of many mathematicians and numerous papers have been published on various types of equations. Most of the publications are devoted to first-order impulsive delay differential equations and there is just a few works in the direction of impulsive partial difference equations with continuous arguments (IPDEWCA). In [2], the authors studied the oscillation of solutions to IPDEWCA

$$\begin{gathered} p_1 z(t+a,s+b)+p_2 z(t+a,s)+p_3 z(t,s+b) \\ -p_4 z(t,s)+p(t,s)z(t-\tau,s-\sigma)=0 \quad \text{for } (t,s)\in\big(\mathbb{R}_0^+\backslash\{t_k\}_{k\in\mathbb{N}_0}\big)\times\mathbb{R}_0^+ \\ z(t_k^+,s)=\alpha_k z(t_k^-,s) \quad \text{for } k\in\mathbb{N}_0 \text{ and } s\in\mathbb{R}_0^+, \end{gathered} \tag{1.1}$$

where the impulse points $\{t_k\}_{k\in\mathbb{N}_0}$ are assumed to be placed equidistancely through the first axes. Note that only the first variable of the unknown function is exposed to impulse effects.

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In this article, we shall draw our attention to the qualitative behavior of solutions to IPDEWCA by introducing impulse effects to both of the variables of the unknown function, which is more compatible with the nature of partial difference equations with continuous arguments. Here, we shall adopt a method, which is similar to that of [3], for building a bridge between the solutions of IPDEWCA and the solutions of partial difference equations with continuous arguments (PDEWCA), and combining this technique with the one in [4], we will be able to establish a new connection between oscillation of IPDEWCA and difference equations with discrete arguments (PDEWDA). Therefore, the detailed process taking place in the proofs for IPDEWCA will be cleared away. Finally, we would like to mention that our main result is also new even for the nonimpulsive case, improves the ones in [2] for the autonomous case, and includes the results of [5]. For fundamental results in the theory of PDEWDA and PDEWCA, the readers are referred to the books [6, 7] and the survey [8] devoted to the study of various (including qualitative) properties of the solutions.

This article is organized as follows: In Section 2, we study the oscillation of PDEWDA by removing a condition in the well-known oscillation criteria introduced in [7]; in Section 3, we construct a connection between the oscillation of PDEWCA and the oscillation of PDEWDA, which extends almost all of the oscillation results given for PDEWDA to PDEWCA. In Section 4, we relate the oscillation of IPDEWCA with the oscillation of PDEWCA, so that the results for the oscillation of PDEWDA can be also applied to reveal the oscillation of IPDEWCA. Finally in Section 5, we make our final comments and compare our results with the ones introduced in [2], and an illustrative example concerning the autonomous case is given to mention the importance of our results.

2. Oscillation of PDEWDA

In this section, we confine our attention to the difference inequality with discrete arguments

$$\begin{aligned} &p_1A(m+1,n+1)+p_2A(m+1,n)+p_3A(m,n+1)\\ &-p_4A(m,n)+p(m,n)A(m-\kappa,n-\ell)\leq 0 \quad \text{for } (m,n)\in\mathbb{Z}_0\times\mathbb{Z}_0, \end{aligned} \tag{2.1}$$

where $\mathbb{Z}_k := \{n\in\mathbb{Z}:\ n\geq k\}$ for $k\in\mathbb{Z}$, under the following conditions:

(A1) $p_1\geq 0$ and $p_2,p_3,p_4>0$;
(A2) $\kappa,\ell\in\mathbb{Z}_0$;
(A3) $p:\mathbb{N}_0\times\mathbb{N}_0\to\mathbb{R}_0^+$.

Definition 2.1. A double sequence $A:\mathbb{Z}_{-\kappa}\times\mathbb{Z}_{-\ell}\to\mathbb{R}$ satisfying (2.1) identically on $\mathbb{Z}_0\times\mathbb{Z}_0$ is called a solution of (2.1).

Definition 2.2. A solution A of (2.1) is called eventually positive if there exists $(m_0,n_0)\in\mathbb{Z}_0\times\mathbb{Z}_0$ such that $A(m,n)>0$ for all $(m,n)\in\mathbb{Z}_{m_0}\times\mathbb{Z}_{n_0}$. and is called eventually negative if negative of A is eventually positive. A solution neither eventually positive nor negative is called oscillatory. If every solution of (2.1) oscillates, then it is called oscillatory.

Theorem 2.3 ([7, Theorem 2.15])**.** *Assume that* (A1)–(A3) *hold, and that*

(i) $\limsup_{\substack{m\to\infty\\ n\to\infty}} p(m,n)>0$;

(ii) (a) *If* $\kappa \geq \ell \geq 1$, *then*
$$\limsup_{\substack{m\to\infty\\ n\to\infty}} \sup_{\lambda\in E} \Big\{\lambda \prod_{i=1}^{\ell} \big[p_4 - \lambda p(m-i,n-i)\big] \times \prod_{j=1}^{\kappa-\ell} \big[p_4 - \lambda p(m-\ell-j,n-\ell)\big]\Big\} < \Big(p_1 + 2\frac{p_2p_3}{p_4}\Big)^{\ell} p_2^{\kappa-\ell},$$
where
$$E := \big\{\lambda > 0 : \ p_4 - \lambda p(m,n) > 0 \textit{ for all large } (m,n) \in \mathbb{Z}_0 \times \mathbb{Z}_0\big\}.$$
(b) *If* $\ell \geq \kappa \geq 1$, *then*
$$\limsup_{\substack{m\to\infty\\ n\to\infty}} \sup_{\lambda\in E} \Big\{\lambda \prod_{i=1}^{\kappa} \big[p_4 - \lambda p(m-i,n-i)\big] \times \prod_{j=1}^{\ell-\kappa} \big[p_4 - \lambda p(m-\kappa,n-\kappa-j)\big]\Big\} < \Big(p_1 + 2\frac{p_2p_3}{p_4}\Big)^{k} p_3^{\ell-\kappa}.$$

Then (2.1) *has no eventually positive solutions.*

The proof of Theorem 2.3 uses the property that the set E is bounded, which is ensured by (i) in Theorem 2.3. In the following result, we shall remove the requirement for (i) in Theorem 2.3 by introducing a new proof. To this end, we need to introduce
$$\Lambda(m,n) := \big\{\lambda > 0 : \ p_4 - \lambda p(i,j) > 0 \text{ for all } (i,j) \in [m-\kappa,m)\times[n-\ell,n)\cap\mathbb{Z}_0\times\mathbb{Z}_0\big\}. \tag{2.2}$$

Theorem 2.4. *Assume that* (A1)–(A3) *hold. Assume also that*
$$\limsup_{\substack{m\to\infty\\ n\to\infty}} \sup_{\lambda\in \Lambda(m,n)} \Big\{\lambda \prod_{i=1}^{\ell} \big[p_4 - \lambda p(m-i,n-i)\big] \times \prod_{j=1}^{\kappa-\ell} \big[p_4 - \lambda p(m-\ell-j,n-\ell)\big]\Big\} < p_2^{\kappa-\ell}\Big(p_1 + 2\frac{p_2p_3}{p_4}\Big)^{\ell} \quad \textit{if } \kappa \geq \ell \geq 1, \tag{2.3}$$
or
$$\limsup_{\substack{m\to\infty\\ n\to\infty}} \sup_{\lambda\in \Lambda(m,n)} \Big\{\lambda \prod_{i=1}^{\kappa} \big[p_4 - \lambda p(m-i,n-i)\big] \times \prod_{j=1}^{\ell-\kappa} \big[p_4 - \lambda p(m-\kappa,n-\kappa-j)\big]\Big\} < p_3^{\ell-\kappa}\Big(p_1 + 2\frac{p_2p_3}{p_4}\Big)^{k} \quad \textit{if } \ell \geq \kappa \geq 1.$$
Then (2.1) *has no eventually positive solutions.*

Proof. We shall only give a proof for the case $\kappa \geq \ell$ since the remaining case follows by using similar arguments. Let A be an eventually positive solution of (2.1). We may find $(m_0,n_0) \in \mathbb{Z}_0 \times \mathbb{Z}_0$ such that $A(m,n) > 0$ and $A(m-\kappa,n-\ell) > 0$ for all $(m,n) \in \mathbb{Z}_{m_0} \times \mathbb{Z}_{n_0}$. Now, define
$$B(m,n) := \frac{A(m-\kappa,n-\ell)}{A(m,n)} > 0 \quad \text{for } (m,n) \in \mathbb{Z}_{m_0} \times \mathbb{Z}_{n_0}. \tag{2.4}$$

We shall first show that

$$\lim_{\substack{m\to\infty\\ n\to\infty}} B(m,n)=\infty. \tag{2.5}$$

Using (2.4), we may rewrite (2.1) as

$$\begin{aligned} &p_1A(m+1,n+1)+p_2A(m+1,n)+p_3A(m,n+1)\\ &\le \big[p_4-B(m,n)p(m,n)\big]A(m,n)\end{aligned} \tag{2.6}$$

for all $(m,n)\in\mathbb{Z}_{m_0}\times\mathbb{Z}_{n_0}$. It follows from (2.1) that

$$A(m+1,n)\le\frac{p_4}{p_2}A(m,n)\quad\text{and}\quad A(m,n+1)\le\frac{p_4}{p_3}A(m,n) \tag{2.7}$$

for all $(m,n)\in\mathbb{Z}_{m_0}\times\mathbb{Z}_{n_0}$. Using (2.7) in (2.6), we obtain

$$\Big(p_1+2\frac{p_2p_3}{p_4}\Big)A(m+1,n+1)\le\big[p_4-B(m,n)p(m,n)\big]A(m,n)$$

for all $(m,n)\in\mathbb{Z}_{m_0}\times\mathbb{Z}_{n_0}$, or

$$A(m+1,n+1)\le\Big(p_1+2\frac{p_2p_3}{p_4}\Big)^{-1}\big[p_4-B(m,n)p(m,n)\big]A(m,n)$$

for all $(m,n)\in\mathbb{Z}_{m_0}\times\mathbb{Z}_{n_0}$. This yields

$$\begin{aligned} &A(m,n)\\ &\le\Big(p_1+2\frac{p_2p_3}{p_4}\Big)^{-1}\big[p_4-B(m-1,n-1)p(m-1,n-1)\big]A(m-1,n-1)\\ &\le\Big(p_1+2\frac{p_2p_3}{p_4}\Big)^{-2}\big[p_4-B(m-1,n-1)p(m-1,n-1)\big]\\ &\quad\times\big[p_4-B(m-2,n-2)p(m-2,n-2)\big]A(m-2,n-2)\\ &\quad\dots\\ &\le\Big(p_1+2\frac{p_2p_3}{p_4}\Big)^{-\ell}\Big(\prod_{i=1}^{\ell}\big[p_4-B(m-i,n-i)p(m-i,n-i)\big]\Big)A(m-\ell,n-\ell)\end{aligned} \tag{2.8}$$

for all $(m,n)\in\mathbb{Z}_{m_0+\ell}\times\mathbb{Z}_{n_0+\ell}$. Using (2.6), we obtain

$$A(m+1,n)<\frac{1}{p_2}\big[p_4-B(m,n)p(m,n)\big]A(m,n)\quad\text{for all } (m,n)\in\mathbb{Z}_{m_0}\times\mathbb{Z}_{n_0},$$

which yields

$$\begin{aligned} &A(m-\ell,n-\ell)\\ &<\frac{1}{p_2}\big[p_4-B(m-\ell-1,n-\ell)p(m-\ell-1,n-\ell)\big]A(m-\ell-1,n-\ell)\\ &\quad\dots\\ &\le\frac{1}{p_2^{\kappa-\ell}}\Big(\prod_{j=1}^{\kappa-\ell}\big[p_4-B(m-\ell-j,n-\ell)p(m-\ell-j,n-\ell)\big]\Big)A(m-\kappa,n-\ell)\end{aligned} \tag{2.9}$$

for all $(m,n)\in\mathbb{Z}_{m_0+\kappa}\times\mathbb{Z}_{n_0+\kappa}$. Define

$$C(m,n):=\min\big\{B(i,j):\ (i,j)\in[m-\kappa,m]\times[n-\ell,n]\cap\mathbb{Z}^2\big\} \tag{2.10}$$

for $(m,n) \in \mathbb{Z}_{m_0} \times \mathbb{Z}_{n_0}$. Using (2.8), (2.9) and (2.10), we obtain

$$\begin{aligned} A(m,n) &\le \frac{1}{p_2^{\kappa-\ell}}\Big(p_1 + 2\frac{p_2p_3}{p_4}\Big)^{-\ell}\Big(\prod_{i=1}^{\ell}\big[p_4 - B(m-i,n-i)p(m-i,n-i)\big]\Big) \\ &\quad\times \Big(\prod_{j=1}^{\kappa-\ell}\big[p_4 - B(m-\ell-j,n-\ell)p(m-\ell-j,n-\ell)\big]\Big)A(m-\kappa,n-\ell) \\ &\le \frac{1}{p_2^{\kappa-\ell}}\Big(p_1 + 2\frac{p_2p_3}{p_4}\Big)^{-\ell}\Big(\prod_{i=1}^{\ell}\big[p_4 - C(m,n)p(m-i,n-i)\big]\Big) \\ &\quad\times \Big(\prod_{j=1}^{\kappa-\ell}\big[p_4 - C(m,n)p(m-\ell-j,n-\ell)\big]\Big)A(m-\kappa,n-\ell) \end{aligned}$$

or equivalently

$$\begin{aligned} 1 &\le \frac{1}{p_2^{\kappa-\ell}}\Big(p_1 + 2\frac{p_2p_3}{p_4}\Big)^{-\ell}\Big(\prod_{i=1}^{\ell}\big[p_4 - C(m,n)p(m-i,n-i)\big]\Big) \\ &\quad\times \Big(\prod_{j=1}^{\kappa-\ell}\big[p_4 - C(m,n)p(m-\ell-j,n-\ell)\big]\Big)B(m,n) \end{aligned} \tag{2.11}$$

for all $(m,n) \in \mathbb{Z}_{m_0+2\kappa} \times \mathbb{Z}_{n_0+2\kappa}$. It follows from (2.6) and (2.10) that

$$C(m,n) \in \Gamma(m,n) \quad \text{for all } (m,n) \in \mathbb{Z}_{m_0+2\kappa} \times \mathbb{Z}_{n_0+2\kappa}. \tag{2.12}$$

Then from (2.11) and (2.12), we have

$$\begin{aligned} C(m,n) &\le \frac{1}{p_2^{\kappa-\ell}}\Big(p_1 + 2\frac{p_2p_3}{p_4}\Big)^{-\ell}C(m,n)\Big(\prod_{i=1}^{\ell}\big[p_4 - C(m,n)p(m-i,n-i)\big]\Big) \\ &\quad\times \Big(\prod_{j=1}^{\kappa-\ell}\big[p_4 - C(m,n)p(m-\ell-j,n-\ell)\big]\Big)B(m,n) \end{aligned} \tag{2.13}$$

for all $(m,n) \in \mathbb{Z}_{m_0+2\kappa} \times \mathbb{Z}_{n_0+2\kappa}$. The condition (2.3) implies existence of a constant $\mu < 1$ and $(m_1,n_1) \in \mathbb{Z}_{m_0+2\kappa} \times \mathbb{Z}_{n_0+2\kappa}$ such that

$$\begin{aligned} &\frac{1}{p_2^{\kappa-\ell}}\Big(p_1 + 2\frac{p_2p_3}{p_4}\Big)^{-\ell}C(m,n)\prod_{i=1}^{\ell}\big[p_4 - C(m,n)p(m-i,n-i)\big] \\ &\times \prod_{j=1}^{\kappa-\ell}\big[p_4 - C(m,n)p(m-\ell-j,n-\ell)\big] < \mu \end{aligned} \tag{2.14}$$

for all $(m,n) \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{n_1}$. Using (2.14) in (2.13), we obtain

$$C(m,n) \le \mu B(m,n) \quad \text{for all } (m,n) \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{n_1}. \tag{2.15}$$

To prove that (2.5) is true assume the contrary; i.e.,

$$\rho := \liminf_{\substack{m\to\infty \\ n\to\infty}} B(m,n) < \infty. \tag{2.16}$$

Note that from (2.10), we have

$$\liminf_{\substack{m\to\infty\\ n\to\infty}} C(m,n) = \rho. \tag{2.17}$$

First assume that $\rho > 0$. Then taking lim inf on both sides of (2.15), we obtain $\rho \leq \mu\rho$, which is a contradiction since $\mu < 1$. Next assume that $\rho = 0$, then there exists $(m_2, n_2) \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{n_1}$ such that

$$\min\{B(i,j) : (i,j) \in [m_1, m_2] \times [n_1, n_2] \cap \mathbb{Z}_0 \times \mathbb{Z}_0\} = B(m_2, n_2). \tag{2.18}$$

Then (2.10), (2.15) and (2.18) yield that

$$\begin{aligned} \mu B(m_2, n_2) &\geq C(m_2, n_2) \\ &\geq \min\{B(i,j) : (i,j) \in [m_1, m_2] \times [n_1, n_2] \cap \mathbb{Z}_0 \times \mathbb{Z}_0\} = B(m_2, n_2), \end{aligned}$$

which is a contradiction since $\mu < 1$. Therefore, we have just proved that (2.5) is true. Now, we show that (2.3) implies

$$\limsup_{\substack{m\to\infty\\ n\to\infty}} \sum_{i=m-\kappa}^{m-1} \sum_{j=n-\ell}^{n-1} p(i,j) > 0. \tag{2.19}$$

If (2.19) is not true, then we have

$$\lim_{\substack{m\to\infty\\ n\to\infty}} p(m,n) = 0. \tag{2.20}$$

We may find $(m_2, n_2) \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{n_1}$ such that

$$p(m,n) < \varepsilon \quad \text{for all } (m,n) \in \mathbb{Z}_{m_2} \times \mathbb{Z}_{n_2}, \tag{2.21}$$

where

$$\varepsilon := \frac{1}{2\kappa}\Big(\frac{p_4\kappa}{\kappa+1}\Big)^{\kappa+1} \frac{1}{p_2^{\kappa-\ell}}\Big(p_1 + 2\frac{p_2p_3}{p_4}\Big)^{-\ell} > 0. \tag{2.22}$$

From (2.21), we have

$$p_4 - \frac{p_4}{\varepsilon} p(m,n) > 0 \quad \text{for all } (m,n) \in \mathbb{Z}_{m_2} \times \mathbb{Z}_{n_2}, \tag{2.23}$$

which implies $(0, p_4/\varepsilon) \subset \Lambda(m,n)$ for all $(m,n) \in \mathbb{Z}_{m_2} \times \mathbb{Z}_{n_2}$. Then, using (2.22) and (2.23), we have

$$\begin{aligned} &\sup_{\lambda\in\Lambda(m,n)} \Big\{\lambda \prod_{i=1}^{\ell} \big[p_4 - \lambda p(m-i, n-i)\big] \prod_{j=1}^{\kappa-\ell} \big[p_4 - \lambda p(m-\ell-j, n-\ell)\big]\Big\} \\ &\geq \sup_{\lambda\in(0,p_4/\varepsilon)} \Big\{\lambda\big(p_4 - \lambda\varepsilon\big)^{\kappa}\Big\} = \frac{1}{\varepsilon\kappa}\Big(\frac{p_4\kappa}{\kappa+1}\Big)^{\kappa+1} \\ &\geq 2p_2^{\kappa-\ell}\Big(p_1 + 2\frac{p_2p_3}{p_4}\Big)^{\ell} \end{aligned}$$

for all $(m,n) \in \mathbb{Z}_{m_2} \times \mathbb{Z}_{n_2}$. This contradicts (2.3), and proves that (2.19) is true. Now, we will use (2.19) to show

$$\liminf_{\substack{m\to\infty\\ n\to\infty}} B(m,n) < \infty. \tag{2.24}$$

The proof will be completed if we can show that (2.24) holds, which contradicts (2.5). From (2.19), there exist a constant $\delta > 0$ and an increasing divergent double sequence $\{(\xi_r, \zeta_r)\}_{r\in\mathbb{N}_0} \subset \mathbb{Z}_{m_2} \times \mathbb{Z}_{n_2}$ such that

$$\sum_{i=\xi_r-\kappa}^{\xi_r-1} \sum_{j=\zeta_r-\ell}^{\zeta_r-1} p(i,j) \geq \delta \quad \text{for all } r \in \mathbb{N}_0. \tag{2.25}$$

Dirichlet's Pigeonhole principle implies existence of a sequence $\{(\alpha_r, \beta_r)\}_{r\in\mathbb{N}_0} \subset \mathbb{Z}_0 \times \mathbb{Z}_0$ such that

$$\xi_r - \kappa \leq \alpha_r < \xi_r, \quad \zeta_r - \ell \leq \beta_r < \zeta_r, \quad p(\alpha_r, \beta_r) > \frac{\delta}{\kappa\ell} \quad \text{for all } r \in \mathbb{N}_0. \tag{2.26}$$

From (2.1), for all $r \in \mathbb{N}_0$, we obtain

$$\begin{aligned} 0 &\geq p_1 A(\alpha_r + 1, \beta_r + 1) + p_2 A(\alpha_r + 1, \beta_r) + p_3 A(\alpha_r, \beta_r + 1) \\ &\quad - p_4 A(\alpha_r, \beta_r) + p(\alpha_r, \beta_r) A(\alpha_r - \kappa, \beta_r - \ell) \\ &> -p_4 A(\alpha_r, \beta_r) + \frac{\delta}{\kappa\ell} A(\alpha_r - \kappa, \alpha_r - \ell), \end{aligned}$$

which yields

$$B(\alpha_r, \beta_r) = \frac{A(\alpha_r - \kappa, \beta_r - \ell)}{A(\alpha_r, \beta_r)} \leq \frac{1}{\delta} p_4 \kappa\ell \quad \text{for all } r \in \mathbb{N}_0. \tag{2.27}$$

Letting $r \to \infty$ in (2.27), we reach at (2.24), and thus the proof is complete. $\square$

Remark 2.5. Under the assumptions of Theorem 2.4, every solution of the following PDEWDA

$$\begin{aligned} &p_1 A(m+1, n+1) + p_2 A(m+1, n) + p_3 A(m, n+1) \\ &- p_4 A(m,n) + p(m,n) A(m-\kappa, n-\ell) = 0 \quad \text{for } (m,n) \in \mathbb{Z}_0 \times \mathbb{Z}_0 \end{aligned}$$

is oscillatory. This result therefore improves Theorem 2.3.

3. Oscillation of PDEWCA via PDEWDA

In this section, we reduce the oscillation of PDEWCA to the oscillation of PDEWDA, which is a relatively more developed area. Let us consider the PDEWCA

$$\begin{aligned} &p_1 z(t+a, s+b) + p_2 z(t+a, s) + p_3 z(t, s+b) \\ &- p_4 z(t,s) + p(t,s) z(t-\tau, s-\sigma) \leq 0 \quad \text{for } (t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \end{aligned} \tag{3.1}$$

under the following assumptions:

(A4) $a, b > 0$ and $\tau, \sigma > 0$;
(A5) $p_2, p_3 \geq p_4$;
(A6) $p : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function.

Definition 3.1. A continuous function $z : [-\tau, \infty) \times [-\sigma, \infty) \to \mathbb{R}$ satisfying (3.1) identically on $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ is called a solution of (3.1).

Definition 3.2. A solution z of (3.1) is called eventually positive if there exists $(t_0, s_0) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ such that $z(t,s) > 0$ for all $(t,s) \in [t_0, \infty) \times [s_0, \infty)$. and is called eventually negative if negative of z is eventually positive. A solution neither eventually positive nor negative is called oscillatory. If every solution of (3.1) oscillates, then it is called oscillatory.

We define the minimized function q of p by

$$q(t,s) := \min\big\{p(\eta,\zeta) : (\eta,\zeta) \in [t,t+a]\times[s,s+b]\big\} \quad \text{for } (t,s)\in\mathbb{R}_0^+\times\mathbb{R}_0^+. \tag{3.2}$$

For simplicity of notation, we let

$$\upsilon := \Big\lfloor \frac{\tau}{a} \Big\rfloor \quad \text{and} \quad \nu := \Big\lfloor \frac{\sigma}{b} \Big\rfloor, \tag{3.3}$$

where $\lfloor\cdot\rfloor : \mathbb{R}\to\mathbb{Z}$ denotes the least integer function.

Below, we quote one of the most important results for oscillation of the PDEWCA

$$\begin{aligned} &p_1 z(t+a,s+b) + p_2 z(t+a,s) + p_3 z(t,s+b) \\ &- p_4 z(t,s) + p(t,s) z(t-\tau,s-\sigma) = 0 \quad \text{for } (t,s)\in\mathbb{R}_0^+\times\mathbb{R}_0^+. \end{aligned} \tag{3.4}$$

Theorem 3.3 ([7, Theorem 2.37]). *Assume that* (A4)–(A6) *hold, and that*

(i) $\limsup_{\substack{t\to\infty\\ s\to\infty}} q(t,s) > 0$;

(ii) (a) *If* $\upsilon \ge \nu \ge 1$, *then*

$$\begin{aligned} &\limsup_{\substack{t\to\infty\\ s\to\infty}} \sup_{\lambda\in E} \Big\{\lambda \prod_{i=1}^{\ell} \big[p_4 - \lambda q(t-ai,s-bi)\big] \\ &\times \prod_{j=1}^{\kappa-\ell} \big[p_4 - \lambda q(t-a\ell-aj,n-b\ell)\big]\Big\} < \Big(p_1 + 2\frac{p_2p_3}{p_4}\Big)^{\ell} p_2^{\kappa-\ell}, \end{aligned}$$

where

$$E := \big\{\lambda > 0 : \ p_4 - \lambda q(t,s) > 0 \textit{ for all large } (t,s)\in\mathbb{R}_0^+\times\mathbb{R}_0^+\big\}. \tag{3.5}$$

(b) *If* $\nu \ge \upsilon \ge 1$, *then*

$$\begin{aligned} &\limsup_{\substack{t\to\infty\\ s\to\infty}} \sup_{\lambda\in E} \Big\{\lambda \prod_{i=1}^{\kappa} \big[p_4 - \lambda p(t-ai,s-bi)\big] \\ &\times \prod_{j=1}^{\ell-\kappa} \big[p_4 - \lambda p(t-a\kappa,s-b\kappa-bj)\big]\Big\} < \Big(p_1 + 2\frac{p_2p_3}{p_4}\Big)^{k} p_3^{\ell-\kappa}. \end{aligned}$$

Then (3.4) *is oscillatory.*

The removal of the condition (i) of Theorem 3.3 is just a simple corollary of the main result of this section.

Lemma 3.4 ([7, Lemma 2.36]). *Assume that* (A1), (A4)–(A6) *hold. If z is an eventually positive solution of the partial difference inequality with continuous variables*

$$\begin{aligned} &p_1 z(t+a,s+b) + p_2 z(t+a,s) + p_3 z(t,s+b) \\ &- p_4 z(t,s) + p(t,s) z(t-\tau,s-\sigma) \le 0 \quad \textit{for } (t,s)\in\mathbb{R}_0^+\times\mathbb{R}_0^+, \end{aligned} \tag{3.6}$$

then z eventually satisfies the partial difference inequality with continuous variables

$$\begin{aligned} &p_1 z(t+a,s+b) + p_2 z(t+a,s) + p_3 z(t,s+b) \\ &- p_4 z(t,s) + q(t,s) z(t-a\upsilon,s-b\nu) \le 0 \quad \textit{for } (t,s)\in\mathbb{R}_0^+\times\mathbb{R}_0^+. \end{aligned} \tag{3.7}$$

Corollary 3.5. *Assume that* (A1), (A4)–(A6) *hold. If* (3.7) *has no eventually positive solutions, then* (3.6) *also has no eventually positive solutions.*

For the next result, which builds a bridge between the oscillation of PDEWCA and of PDEWDA, we introduce

$$r_{t,s}(m,n) := q(t+am, s+bn) \quad \text{for } (m,n) \in \mathbb{Z}_0 \times \mathbb{Z}_0, \tag{3.8}$$

where $(t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$.

Theorem 3.6. *Assume that* (A1), (A4)–(A6) *hold. Moreover, assume that for some fixed* $(\eta,\zeta) \in [0,a) \times [0,b)$*, the partial difference inequality with discrete variables*

$$\begin{aligned} &p_1 A(m+1,n+1) + p_2 A(m+1,n) + p_3 A(m,n+1) \\ &- p_4 A(m,n) + r_{\eta,\zeta}(m,n) A(m-\upsilon, n-\nu) \le 0 \quad \textit{for } (m,n) \in \mathbb{N}_0 \times \mathbb{N}_0 \end{aligned} \tag{3.9}$$

has no eventually positive solutions. Then, (3.7) *has no eventually positive solutions.*

Proof. Assume the contrary that z is an eventually positive solution of (3.7). Now we define the double sequence

$$A(m,n) := z(\eta+am, \zeta+bn) \quad \text{for } (m,n) \in \mathbb{N}_0 \times \mathbb{N}_0.$$

Then A is eventually positive. Substituting $(t,s) = (\eta+am, \zeta+bn)$ for $(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0$ into (3.7), we see that the double sequence A satisfies (3.9). This is a contradiction. ☐

Corollary 3.7. *Assume that* (A1), (A4)–(A6) *hold. Assume also that there exists* $(\eta,\zeta) \in [0,a) \times [0,b)$ *such that*

$$\begin{aligned} &\limsup_{\substack{m\to\infty \\ n\to\infty}} \sup_{\lambda \in \Lambda_{\eta,\zeta}} \Big\{ \lambda \prod_{i=1}^{\nu} \big[p_4 - \lambda r_{\eta,\zeta}(m-i, n-i)\big] \\ &\times \prod_{j=1}^{\upsilon-\nu} \big[p_4 - \lambda r_{\eta,\zeta}(m-\nu-j, n-\nu)\big] \Big\} < p_2^{\upsilon-\nu} \Big(p_1 + 2\frac{p_2 p_3}{p_4}\Big)^{\nu} \end{aligned} \tag{3.10}$$

if $\upsilon \ge \nu \ge 1$*, or*

$$\begin{aligned} &\limsup_{\substack{m\to\infty \\ n\to\infty}} \sup_{\lambda \in \Lambda_{\eta,\zeta}} \Big\{ \lambda \prod_{i=1}^{\upsilon} \big[p_4 - \lambda r_{\eta,\zeta}(m-i, n-i)\big] \\ &\times \prod_{j=1}^{\nu-\upsilon} \big[p_4 - \lambda r_{\eta,\zeta}(m-\upsilon, n-\upsilon-j)\big] \Big\} < p_3^{\nu-\upsilon} \Big(p_1 + 2\frac{p_2 p_3}{p_4}\Big)^{k} \end{aligned} \tag{3.11}$$

if $\nu \ge \upsilon \ge 1$*, where*

$$\Lambda_{t,s} := \{\lambda > 0 : \ p_4 - \lambda r_{t,s}(m,n) > 0 f \textit{ or all large } (m,n) \in \mathbb{N}_0 \times \mathbb{N}_0\}$$

for $(t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$*. Then,* (3.7) *has no eventually positive solutions.*

Proof. Under the conditions of the corollary, (3.9) has no eventually positive solutions by Theorem 2.4. Therefore, it follows from Theorem 3.6 that (3.7) has no eventually positive solutions. Finally, an application of Corollary 3.5 completes the proof. ☐

Remark 3.8. Note that for all $(\eta,\zeta) \in [0,a) \times [0,b)$, we have

$$\limsup_{\substack{m\to\infty \\ n\to\infty}} \sup_{\lambda \in \Lambda_{\eta,\zeta}} \Big\{ \lambda \prod_{i=1}^{\nu} \big[p_4 - \lambda r_{\eta,\zeta}(m-i, n-i)\big] \prod_{j=1}^{\upsilon-\nu} \big[p_4 - \lambda r_{\eta,\zeta}(m-\nu-j, n-\nu)\big] \Big\}$$

$$\leq \limsup_{\substack{t\to\infty\\ s\to\infty}} \sup_{\lambda\in E} \Big\{\lambda \prod_{i=1}^{\nu} \big[p_4 - \lambda q(t-ai, s-bi)\big] \prod_{j=1}^{\upsilon-\nu} \big[p_4 - \lambda q(t-a\nu-aj, s-b\nu)\big]\Big\},$$

where

$$E := \big\{\lambda > 0 :\ p_4 - \lambda q(t,s) > 0 \text{ for all large } (t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+\big\}. \tag{3.12}$$

Hence, Corollary 3.7 improves Theorem 3.3.

4. Oscillation of IPDEWCA via PDEWDA

The primary assumptions of this section are as follows:

(A7) $\{t_k\}_{k\in\mathbb{N}_0}$ and $\{s_\ell\}_{\ell\in\mathbb{N}_0}$ are increasing divergent sequences of nonnegative reals;

(A8) $\{\alpha_k\}_{k\in\mathbb{N}_0}$ and $\{\beta_\ell\}_{\ell\in\mathbb{N}_0}$ are sequences of real numbers, which involve no zero terms;

(A9) $p : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous function.

For simplicity of notation, we define

$$D := \big((\mathbb{R}_0^+\backslash\{t_k\}_{k\in\mathbb{N}_0}) \times \mathbb{R}_0^+\big) \cup \big(\mathbb{R}_0^+ \times (\mathbb{R}_0^+\backslash\{s_\ell\}_{\ell\in\mathbb{N}_0})\big).$$

This section is concerned with the oscillation of solutions of the IPDEWCA

$$\begin{gathered} p_1 z(t+a, s+b) + p_2 z(t+a, s) + p_3 z(t, s+b) \\ - p_4 z(t,s) + p(t,s) z(t-\tau, s-\sigma) = 0 \quad \text{for } (t,s) \in D \\ z(t_k^+, s) = \alpha_k z(t_k^-, s) \quad \text{for } k \in \mathbb{N}_0 \text{ and } s \in \mathbb{R}_0^+ \\ z(t, s_\ell^+) = \beta_\ell z(t, s_\ell^-) \quad \text{for } \ell \in \mathbb{N}_0 \text{ and } t \in \mathbb{R}_0^+. \end{gathered} \tag{4.1}$$

Definition 4.1. A function $z : [-\tau,\infty) \times [-\sigma,\infty) \to \mathbb{R}$ is called a solution of (4.1) provided that each of the following conditions are satisfied:

(i) z is continuous on each of the intervals of the form $(t_{k-1}, t_k] \times (s_{\ell-1}, s_\ell]$ for each $k, \ell \in \mathbb{N}$;

(ii) the limit value $z(t_k^-, \cdot)$ for each $k \in \mathbb{N}_0$ and the limit value $z(\cdot, s_\ell^-)$ for each $\ell \in \mathbb{N}_0$ exist and are finite;

(iii) z satisfies
  (a) the first equation in (4.1) if $(t,s) \in (\mathbb{R}_0^+ \times \mathbb{R}_0^+)\backslash\{(t_k, s_\ell)\}_{k,\ell\in\mathbb{N}_0}$;
  (b) the first equation in (4.1) together with the second one if $(t,s) \in D$ with $t \in \{t_k\}_{k\in\mathbb{N}}$;
  (c) the first equation in (4.1) together with the third one if $(t,s) \in D$ with $s \in \{s_\ell\}_{\ell\in\mathbb{N}}$;
  (d) the last two equations in (4.1) if $(t,s) \in \{(t_k, s_\ell)\}_{k,\ell\in\mathbb{N}_0}$.

Oscillation and nonoscillation of solutions to (4.1) are defined similar to Definition 2.2. Below, we quote the first result on the oscillation of IPDEWCA.

Theorem 4.2 ([2, Theorem 1])**.** *Assume that* (A1), (A4)–(A6) *and* (i) *of Theorem 3.3 hold, and that*

(i) $\{\alpha_k\}_{k\in\mathbb{N}_0}$ *is a sequence of reals such that* $\alpha_k > 1$ *for all* $k \in \mathbb{N}_0$*;*

(ii) $\sum_{k=0}^{\infty}(\alpha_k - 1) < \infty$*;*

(iii) (a) *If* $\upsilon \geq \nu \geq 1$*, then*

$$\limsup_{\substack{t\to\infty\\ s\to\infty}} \sup_{\lambda\in E} \Big\{\lambda \prod_{i=1}^{\ell} \Big[p_1 + 2\frac{p_2 p_3}{p_4} \prod_{0<t_k<t-a(i-2)} \frac{1}{\alpha_k}\Big]^{-1}$$

$$\times \prod_{j=1}^{\ell} \Big[p_4 \prod_{0<t_k\le t-a(i-1)} \alpha_k - \lambda q(t-ai, s-bi)\Big]$$
$$\times \prod_{j=1}^{\ell-\kappa} \Big[p_4 \prod_{0<t_k\le t-a(\ell+j-1)} \alpha_k - \lambda q(t-a\ell-aj, s-b\ell)\Big]\Big\} < p_2^{\kappa-\ell},$$

where q, υ and ν are defined as in (3.2) *and* (3.3), *and*

$$E := \big\{\lambda > 0 :\ p_4 \prod_{0<t_k<t+a} \alpha_k - \lambda q(t,s) > 0 \text{ for all large } (t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+\big\}.$$

(b) *If $\nu \ge \upsilon \ge 1$, then*

$$\limsup_{\substack{t\to\infty\\ s\to\infty}} \sup_{\lambda\in E} \Big\{\lambda \prod_{i=1}^{\kappa} \Big[p_1 + 2\frac{p_2p_3}{p_4} \prod_{0<t_k<t-a(i-2)} \frac{1}{\alpha_k}\Big]^{-1}$$
$$\times \prod_{j=1}^{\kappa} \Big[p_4 \prod_{0<t_k\le t-a(i-1)} \alpha_k - \lambda q(t-ai, s-bi)\Big]$$
$$\times \prod_{j=1}^{\ell-\kappa} \Big[p_4 \prod_{0<t_k\le t-a(\ell+j-1)} \alpha_k - \lambda q(t-a\kappa, s-b\kappa-bj)\Big]\Big\} < p_3^{\kappa-\ell}.$$

Then (1.1) *is oscillatory.*

Now consider the following companion partial difference inequality with continuous arguments

$$\begin{aligned} &p_1\Big(\prod_{\substack{t\le t_i<t+a\\ s\le s_j<s+b}} \alpha_i\beta_j\Big)\omega(t+a,s+b) + p_2\Big(\prod_{t\le t_i<t+a} \alpha_i\Big)\omega(t+a,s)\\ &+ p_3\Big(\prod_{s\le s_j<s+b} \beta_j\Big)\omega(t,s+b) - p_4\omega(t,s)\\ &+ \Big(\prod_{\substack{t-\tau\le t_i<t\\ s-\sigma\le s_j<s}} \frac{1}{\alpha_i\beta_j}\Big)p(t,s)\omega(t-\tau,s-\sigma) = 0 \quad \text{for } (t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+. \end{aligned} \tag{4.2}$$

Lemma 4.3. *Assume that* (A1), (A4), (A7)–(A9) *hold.*

(i) *If z is a solution of* (4.1), *then the companion function ω defined by*

$$\omega(t,s) := \Big(\prod_{\substack{0\le t_i<t\\ 0\le s_j<s}} \frac{1}{\alpha_i\beta_j}\Big)z(t,s) \quad \text{for } (t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \tag{4.3}$$

is a solution of (4.2);

(ii) *If ω is a solution of* (4.2), *then the function z defined by*

$$z(t,s) := \Big(\prod_{\substack{0\le t_i<t\\ 0\le s_j<s}} \alpha_i\beta_j\Big)\omega(t,s) \quad \text{for } (t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \tag{4.4}$$

is a solution of (4.1).

Proof. We only give the proof of the part (i) since the part (ii) can be proved similarly. Assume that z is a solution of (4.1). First, we have to show that ω defined by (4.3) is continuous. Let $(t,s) \in (t_{k-1}, t_k) \times (s_{\ell-1}, s_\ell)$ for some $k, \ell \in \mathbb{N}$. Then using (4.1), we obtain

$$\begin{aligned}
\omega(t_k^+, s) &= \Big(\prod_{\substack{0 \le t_i \le t_k \\ 0 \le s_j < s}} \frac{1}{\alpha_i \beta_j}\Big) z(t_k^+, s) = \Big(\prod_{\substack{0 \le t_i \le t_k \\ 0 \le s_j < s}} \frac{1}{\alpha_i \beta_j}\Big) \alpha_k z(t_k, s) \\
&= \Big(\prod_{\substack{0 \le t_i < t_k \\ 0 \le s_j < s}} \frac{1}{\alpha_i \beta_j}\Big) z(t_k, s) = \omega(t_k, s),
\end{aligned}$$

for $k \in \mathbb{N}_0$ and

$$\begin{aligned}
\omega(t, s_\ell^+) &= \Big(\prod_{\substack{0 \le t_i < t \\ 0 \le s_j \le s_\ell}} \frac{1}{\alpha_i \beta_j}\Big) z(t, s_\ell^+) = \Big(\prod_{\substack{0 \le t_i < t \\ 0 \le s_j \le s_\ell}} \frac{1}{\alpha_i \beta_j}\Big) \beta_\ell z(t, s_\ell) \\
&= \Big(\prod_{\substack{0 \le t_i < t_k \\ 0 \le s_j < s}} \frac{1}{\alpha_i \beta_j}\Big) z(t, s_\ell) = \omega(t, s_\ell)
\end{aligned}$$

for $\ell \in \mathbb{N}_0$. Combining the conclusion above we have $\omega(t_k^+, s_\ell^+) = \omega(t_k, s_\ell)$ for all $k, \ell \in \mathbb{N}$. Therefore, we have just verified that ω is continuous. Now, we show that ω solves (4.1). For all $(t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$, we have

$$\begin{aligned}
& p_1 \Big(\prod_{\substack{t \le t_i < t+a \\ s \le s_j < s+b}} \alpha_i \beta_j\Big) \omega(t+a, s+b) + p_2 \Big(\prod_{t \le t_i < t+a} \alpha_i\Big) \omega(t+a, s) \\
& + p_3 \Big(\prod_{s \le s_j < s+b} \beta_j\Big) \omega(t, s+b) - p_4 \omega(t,s) + \Big(\prod_{\substack{t-\tau \le t_i < t \\ s-\sigma \le s_j < s}} \frac{1}{\alpha_i \beta_j}\Big) p(t,s) \omega(t-\tau, s-\sigma) \\
& = p_1 \Big(\prod_{\substack{t \le t_i < t+a \\ s \le s_j < s+b}} \alpha_i \beta_j\Big) \Big(\prod_{\substack{0 \le t_i < t+a \\ 0 \le s_j < s+b}} \frac{1}{\alpha_i \beta_j}\Big) z(t+a, s+b) \\
& \quad + p_2 \Big(\prod_{t \le t_i < t+a} \alpha_i\Big) \Big(\prod_{\substack{0 \le t_i < t+a \\ 0 \le s_j < s}} \frac{1}{\alpha_i \beta_j}\Big) z(t+a, s) \\
& \quad + p_3 \Big(\prod_{s \le s_j < s+b} \beta_j\Big) \Big(\prod_{\substack{0 \le t_i < t \\ 0 \le s_j < s+b}} \frac{1}{\alpha_i \beta_j}\Big) z(t, s+b) - p_4 \Big(\prod_{\substack{0 \le t_i < t \\ 0 \le s_j < s}} \frac{1}{\alpha_i \beta_j}\Big) z(t,s) \\
& + \Big(\prod_{\substack{t-\tau \le t_i < t \\ s-\sigma \le s_j < s}} \frac{1}{\alpha_i \beta_j}\Big) \Big(\prod_{\substack{0 \le t_i < t-\tau \\ 0 \le s_j < s-\sigma}} \frac{1}{\alpha_i \beta_j}\Big) p(t,s) z(t-\tau, s-\sigma),
\end{aligned}$$

which is equal to

$$\Big(\prod_{\substack{0 \le t_i < t \\ 0 \le s_j < s}} \frac{1}{\alpha_i \beta_j}\Big) \Big[p_1 z(t+a, s+b) + p_2 z(t+a, s) + p_3 z(t, s+b)$$

$$- p_4 z(t,s) + p(t,s) z(t-\tau, s-\sigma)\Big] = 0.$$

This completes the proof of the part (i). ☐

Next, to give a result on the oscillation we introduce the following assumption.

(A10) $\{\alpha_k\}_{k\in\mathbb{N}_0}$ and $\{\beta_\ell\}_{\ell\in\mathbb{N}_0}$ are sequences of positive reals.

Theorem 4.4. *Assume that* (A1), (A4), (A7), (A9), (A10) *hold. Then,* (4.1) *is oscillatory if and only if so is* (4.2).

Proof. Clearly, (A10) implies that the transforms in (4.3) and (4.4) are oscillation invariant. ☐

(A11) There exist positive constants q_1, q_2, q_3, q_4 such that

$$\Big(\prod_{\substack{t\le t_i<t+a\\ s\le s_j<s+b}} \alpha_i\beta_j\Big) \ge q_1, \quad \Big(\prod_{t\le t_i<t+a} \alpha_i\Big) \ge q_2, \quad \Big(\prod_{s\le s_j<s+b} \beta_j\Big) \ge q_3,$$

$$\Big(\prod_{\substack{t-\tau\le t_i<t+a\\ s-\sigma\le s_j<s+b}} \frac{1}{\alpha_i\beta_j}\Big) \ge q_4$$

for all sufficiently large $(t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$;

(A12) $p_2q_2, p_3q_3 \ge p_4 > 0$.

Lemma 4.5. *Assume that* (A1), (A4), (A6), (A7), (A10)–(A12) *hold. If* (4.1) *is nonoscillatory, then the following difference inequality with continuous arguments*

$$\begin{aligned} &p_1q_1\omega(t+a,s+b) + p_2q_2\omega(t+a,s) + p_3q_3\omega(t,s+b)\\ &- p_4\omega(t,s) + q_4 q(t,s)\omega(t-\upsilon a, s-\nu b) \le 0 \quad \text{for } (t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+, \end{aligned} \tag{4.5}$$

where q, υ and ν are defined as in (3.2) *and* (3.3)*, has an eventually positive solution.*

Proof. The proof follows from Theorem 4.4 and (A11), we learn that

$$\begin{aligned} &p_1q_1\omega(t+a,s+b) + p_2q_2\omega(t+a,s) + p_3q_3\omega(t,s+b)\\ &- p_4\omega(t,s) + q_4 q(t,s)\omega(t-\tau, s-\sigma) \le 0 \quad \text{for } (t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \end{aligned}$$

admits an eventually positive solution. An application of Lemma 3.4 shows that (4.5) has an eventually positive solution. The proof is complete. ☐

Lemma 4.6. *Assume that* (A1), (A4), (A6), (A7), (A10)–(A12) *hold. Moreover assume that for some fixed $\eta,\zeta \in [0,a)\times[0,b)$, the partial difference inequality with discrete variables*

$$\begin{aligned} &p_1q_1A(m+1,n+1) + p_2q_2A(m+1,n) + p_3q_3A(m,n+1)\\ &- p_4A(m,n) + q_4 r_{\eta,\zeta}(m,n)A(m-\upsilon, n-\nu) \le 0 \quad \text{for } (m,n) \in \mathbb{Z}_0 \times \mathbb{Z}_0, \end{aligned}$$

where r is defined as in (3.8)*, has no eventually positive solutions. Then,* (4.1) *is oscillatory.*

The proof of the above lemma follows from Theorem 3.6 and Lemma 4.5.

Corollary 4.7. *Assume that* (A1), (A4), (A6), (A7), (A10)–(A12) *hold. Moreover, assume that there exists $\eta,\zeta \in [0,a)\times[0,b)$ such that* (3.10) *holds if $\upsilon \ge \nu \ge 1$ or* (3.11) *holds if $\nu \ge \upsilon \ge 1$. Then,* (4.1) *is oscillatory.*

The proof of the above corollary follows from Corollary 3.7 and Lemma 4.6.

5. Discussion and final comments

In this section, we restrict our attention to the autonomous case for emphasizing the significance of results. In [2], authors assumed that $\alpha_k > 1$ for all $k \in \mathbb{N}_0$ and $\sum_{k\in\mathbb{N}_0}(\alpha_k - 1) < \infty$, which is equivalent to the condition $\prod_{k\in\mathbb{N}_0} \alpha_k < \infty$ (see [9, Theorem 7.4.6]). A necessary condition for this condition is $\lim_{k\to\infty} \alpha_k = 1$ (see [9, Corollary 7.4.3]), which is strong and not required in our results.

Example 5.1. Consider the autonomous IDEWCA

$$\begin{gathered}
p_1 z(t+a, s+b) + p_2 z(t+a, s) + p_3 z(t, s+b) - p_4 z(t, s) \\
+ p z(t - \upsilon a, s - \nu b) = 0 \quad \text{for } (t,s) \in \left(\mathbb{R}_0^+ \backslash \mathbb{N}_0\right) \times \left(\mathbb{R}_0^+ \backslash \mathbb{N}_0\right) \\
z(k^+, s) = \alpha z(k^-, s) \quad \text{for } k \in \mathbb{N}_0 \text{ and } s \in \mathbb{R}_0^+ \\
z(t, \ell^+) = \beta z(t, \ell^-) \quad \text{for } \ell \in \mathbb{N}_0 \text{ and } t \in \mathbb{R}_0^+,
\end{gathered} \tag{5.1}$$

where $p_1, p_2, p_3, p_4 > 0$, $p > 0$, $a, b > 0$, $\upsilon, \nu \in \mathbb{N}$ and $\alpha, \beta > 0$ with $p_2\alpha, p_3\beta \geq p_4$. Then Corollary 4.7 implies that every solution of (5.1) oscillates if

$$p > \begin{cases} \dfrac{(\alpha\beta)^{\nu-1} p_2^{\nu-\upsilon}}{\upsilon\big(p_1 + 2\alpha^{\upsilon}\beta^{\nu} p_2 p_3\big)^{\nu}} \Big(\dfrac{\upsilon p_4}{\upsilon+1}\Big)^{\upsilon+1} & \text{if } \upsilon \geq \nu \\ \dfrac{(\alpha\beta)^{\upsilon-1} p_3^{\upsilon-\nu}}{\nu\big(p_1 + 2\alpha^{\upsilon}\beta^{\nu} p_2 p_3\big)^{\upsilon}} \Big(\dfrac{\nu p_4}{\nu+1}\Big)^{\nu+1} & \text{if } \nu \geq \upsilon. \end{cases}$$

If we let $\beta = 1$, then (5.1) reduces to a particular case of the equation studied in [2]. But unfortunately all the results therein fail to apply to this equation because of the condition $\alpha > 1$ (see (ii) of Theorem 4.2).

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