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ESTIMATES FOR MILD SOLUTIONS TO SEMILINEAR CAUCHY PROBLEMS

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Abstract. The existence (and uniqueness) results on mild solutions of the abstract semilinear Cauchy problems in Banach spaces are well known. Following the results of Tartar (2008) and Burazin (2008) in the case of decoupled hyperbolic systems, we give an alternative proof, which enables us to derive an estimate on the mild solution and its time of existence. The nonlinear term in the equation is allowed to be time-dependent. We discuss the optimality of the derived estimate by testing it on three examples: the linear heat equation, the semilinear heat equation that models dynamic deflection of an elastic membrane, and the semilinear Schrödinger equation with time-dependent nonlinearity, that appear in the modelling of numerous physical phenomena.

1. Introduction

Let X be a Banach space with a norm $\|\cdot\|_X$, and let $T>0$. We consider the semilinear abstract Cauchy problem

$$\begin{gathered} \mathrm{u}'(t) - A\mathrm{u}(t) = \mathrm{f}(t,\mathrm{u}(t)) \quad \text{in } (0,T)\,, \\ \mathrm{u}(0) = \mathrm{g}\,, \end{gathered} \tag{1.1}$$

where $A : D(A) \subseteq X \to X$ is an infinitesimal generator of a C_0-semigroup on X, $\mathrm{g} \in X$, $\mathrm{f} : [0,T]\times X \to X$ and u is the unknown function. Throughout the paper we follow the terminology of [6]. Let us denote by $(T(t))_{t\geq 0}$ the C_0-semigroup generated by A, and by $M \geq 1$ and $\omega \in \mathbb{R}$ constants for which it holds

$$(\forall t \geq 0) \quad \|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}\,.$$

The open (closed) ball in X centered at point $\mathrm{v} \in X$ with radius $r>0$ we will denote by $\mathrm{B}_X(\mathrm{v},r)$ ($\mathrm{B}_X[\mathrm{v},r]$). For the open (closed) ball in $\mathbb{R}$, we will omit writing $\mathbb{R}$ and just use $\mathrm{B}(\mathrm{v},r)$ ($\mathrm{B}[\mathrm{v},r]$).

For the right-hand side we assume that f is:

(i) Borel measurable (in both variables),
(ii) locally Lipschitz in u: $(\exists \Psi \in L^\infty_{\mathrm{loc}}(\mathbb{R}))(\forall r>0)(\forall \mathrm{z},\mathrm{w} \in \mathrm{B}_X[0,r])$,

$$\|\mathrm{f}(t,\mathrm{z}) - \mathrm{f}(t,\mathrm{w})\|_X \leq \Psi(r)\|\mathrm{z}-\mathrm{w}\|_X \quad (\text{a.e. } t \in [0,T])\,.$$

It is easy to see that (ii) implies that f is locally bounded in u. More precisely, as in [2, Theorem 3], one can easily prove the following theorem.

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Theorem 1.1. *The function*

$$\Phi(t,u) := \sup_{\|\mathrm{u}\|_X \le u} \|\mathrm{f}(t,\mathrm{u})\|_X\,, \quad t\in[0,T]\,,\ u\in\mathbb{R}_0^+$$

is (the smallest) local bound for f*:*

$$(\forall r>0)(\forall \mathrm{w}\in \mathrm{B}_X[0,r]) \quad \|\mathrm{f}(t,\mathrm{w})\|_X \le \Phi(t,r) \quad (\text{a.e. } t\in[0,T])\,,$$

and has the following properties:

- $\Phi\in L^\infty_{\rm loc}([0,T]\times\mathbb{R}_0^+)$*;*
- $\Phi\ge 0$ *and* $\Phi(t,\cdot)$ *is non-decreasing for* $t\in[0,T]$*;*
- Φ *is locally Lipschitz in* u*, with the same* Ψ *as in (ii):*

$$(\forall u,v\in\mathbb{R}_0^+) \quad u\ge v \implies |\Phi(t,u)-\Phi(t,v)|\le \Psi(u)|u-v| \quad (\text{a.e. } t\in[0,T]).$$

Remark 1.2. The properties of the function Φ guarantee that the Cauchy problem

$$\begin{gathered} v'(t) = e^{-\omega t}\Phi(t, Me^{\omega t}v(t)) \\ v(0) = \|\mathrm{g}\|_X \end{gathered} \tag{1.2}$$

has the unique maximal solution $v\in W^{1,\infty}_{\rm loc}([0,S))$, for some $S>0$ (v is Lipschitz continuous on every $[a,b]\subseteq[0,S)$). This is a consequence of the function $h(t,u) := e^{-\omega t}\Phi(t,Me^{\omega t}u)$ being locally bounded on $[0,T]\times\mathbb{R}_0^+$ and locally Lipschitz in the second variable (see appendix in [2] for more details regarding solutions of such problems).

We use the following terminology from [6]: a function $\mathrm{u}\in C([0,S);X)$ is called a *mild solution* of (1.1) on $[0,S)$ (some authors use the term *weak solution*) if

$$\mathrm{u}(t) = T(t)\mathrm{g} + \int_0^t T(t-s)\mathrm{f}(s,\mathrm{u}(s))\,ds\,, \quad t\in[0,S)\,. \tag{1.3}$$

This article is organized as follows: in the second section we derive an estimate on the mild solution of (1.1) and its time of existence, while in the last section we apply the obtained estimate on three examples: the linear heat equation, the semilinear heat equation that models the dynamic deflection of an elastic membrane in MEMS device, and the semilinear Schrödinger equation with a time-dependent nonlinearity, that models numerous physical phenomena in optics, quantum physics, etc.

2. Existence and uniqueness theorem

The existence (and uniqueness) results of a mild solution of (1.1) are well known [6], and their proofs usually rely on some fixed point theorem. In the following theorem, which is the main result of this paper, we provide an alternative proof, which enables us to derive an estimate on the solution and its time of existence. It tracks the idea of [8, Ch. 14], in the case of decoupled hyperbolic systems, which was further refined in [2]. The proof follows steps of [2, Theorem 1], and uses a similar iterative procedure as it was used in the classical proof.

Theorem 2.1. *Let the assumptions from the introductory section hold, and assume that* $v\in W^{1,\infty}_{\rm loc}([0,S))$ *is the maximal solution of* (1.2) *for some* $S\in(0,T]$*. Then there exists the unique mild solution on* $[0,S)$*,* $\mathrm{u}\in C([0,S);X)$*, of the problem* (1.1)*. Additionally,* u *satisfies the estimate*

$$\|\mathrm{u}(t)\|_X \le Me^{\omega t}v(t)\,, \quad t\in[0,S)\,. \tag{2.1}$$

Proof. Let us prove the uniqueness first. Assume that $\mathrm{u}_1, \mathrm{u}_2 \in C([0,S);X)$ are two mild solutions of (1.1). From (1.3) we have

$$\begin{aligned}\|\mathrm{u}_1(t)-\mathrm{u}_2(t)\|_X &\le \int_0^t \|T(t-s)\|_{\mathcal{L}(X)}\|\mathrm{f}(s,\mathrm{u}_1(s))-\mathrm{f}(s,\mathrm{u}_2(s))\|_X\,ds\\ &\le Me^{\omega t}\int_0^t \|\mathrm{f}(s,\mathrm{u}_1(s))-\mathrm{f}(s,\mathrm{u}_2(s))\|_X\,ds\,.\end{aligned}$$

Next, we use the fact that f is locally Lipschitz (ii) and by Gronwall's inequality we get the result. One needs to be aware that these steps have to be done on $[0,S-\epsilon)$ for an arbitrary $\epsilon>0$, because our solutions can have a blow-up in S, hence possibly be unbounded.

For the existence we shall first inductively define u_n, for $n\in\mathbb{N}$, by using Picard's iterations. Let us choose $\mathrm{u}_0 \in C([0,S];X)$ such that

$$\|\mathrm{u}_0(t)\|_X \le Me^{\omega t}v(t)\,,\quad t\in[0,S)\,,$$

and then define

$$\mathrm{u}_n(t) := T(t)\mathrm{g} + \int_0^t T(t-s)\mathrm{f}(s,\mathrm{u}_{n-1}(s))\,ds\,,\quad t\in[0,S]\,. \tag{2.2}$$

The existence of such a function u_0 is trivial since any constant function is admissible. To verify that u_n is well-defined, it is enough to see that the function under the integral sign is measurable. Since u_0 is continuous and f measurable we have $s\mapsto \mathrm{f}(s,\mathrm{u}_0(s))$ is measurable, and so it is $s\mapsto T(t-s)\mathrm{f}(s,\mathrm{u}_0(s))$. In fact, $\mathrm{u}_1\in C([0,S];X)$, hence by the inductive argument we have that u_n is well-defined and $\mathrm{u}_n \in C([0,S];X)$ for $n\in\mathbb{N}$. Moreover, for $t\in[0,S)$,

$$\begin{aligned}\|\mathrm{u}_1(t)\|_X &\le \|T(t)\mathrm{g}\|_X + \int_0^t \|T(t-s)\mathrm{f}(s,\mathrm{u}_0(s))\|_X\,ds\\ &\le Me^{\omega t}\|\mathrm{g}\|_X + Me^{\omega t}\int_0^t e^{-\omega s}\|\mathrm{f}(s,\mathrm{u}_0(s))\|_X\,ds\\ &\le Me^{\omega t}\Big(\|\mathrm{g}\|_X + \int_0^t e^{-\omega s}\Phi(s,Me^{\omega s}v(s))\,ds\Big)\\ &\le Me^{\omega t}v(t)\,,\end{aligned}$$

where in the third inequality we used the local boundedness of f and $\mathrm{u}_0(s)\in \mathrm{B}_X[0,Me^{\omega s}v(s)]$ for $s\in[0,S)$. Inductively, the same estimate can be proved for each u_n.

We now distinguish the following two cases depending on whether v has a blow-up in S.

I. If $S=T$ (see appendix in [2]), then v is bounded, so there exists a constant P such that $|\Psi(Me^{\omega\cdot}v(\cdot))|\le P$ a.e. on $[0,S]$. If we subtract the formulae for u_{n+1} and u_n and use the locally Lipschitz property of f, we get (for $t\in[0,S]$)

$$\begin{aligned}\|\mathrm{u}_{n+1}(t)-\mathrm{u}_n(t)\|_X &\le Me^{\omega t}\int_0^t \|\mathrm{f}(s,\mathrm{u}_n(s))-\mathrm{f}(s,\mathrm{u}_{n-1}(s))\|_X\,ds\\ &\le Me^{\omega t}\int_0^t \Psi(Me^{\omega s}v(s))\|\mathrm{u}_n(s)-\mathrm{u}_{n-1}(s)\|_X\,ds\\ &\le Me^{\omega S}P\int_0^t \|\mathrm{u}_n(s)-\mathrm{u}_{n-1}(s)\|_X\,ds\,.\end{aligned}$$

Now one can easily prove by induction that

$$\|\mathrm{u}_{n+1}(t) - \mathrm{u}_n(t)\|_X \le \frac{RM^nP^nt^n}{n!}e^{n\omega S} \le \frac{RM^nP^nS^n}{n!}e^{n\omega S}\,,$$

where $R := \|\mathrm{u}_1 - \mathrm{u}_0\|_{C([0,S];X)}$. As

$$\sum_{j=0}^{n}\|\mathrm{u}_{j+1} - \mathrm{u}_j\|_{C([0,S];X)} \le \sum_{j=0}^{n}\frac{RM^jP^jS^j}{j!}e^{j\omega S} \le R\sum_{j=0}^{n}\frac{(MPSe^{\omega S})^j}{j!} \le Re^{MPSe^{\omega S}}\,,$$

we conclude that the series with partial sums $\sum_{j=0}^{n}(\mathrm{u}_{j+1}-\mathrm{u}_j) = \mathrm{u}_{n+1}-\mathrm{u}_0$ converge absolutely in the Banach space $C([0,S];X)$, which implies that u_n converges and we denote its limit by u. It is obvious that u satisfies (2.1), so it remains to verify that u is a mild solution of (1.1); i.e., that u satisfies (1.3). By the locally Lipschitz property of f it follows that

$$\begin{aligned}
&\Big\| \int_0^t T(t-s)\mathrm{f}(s,\mathrm{u}_n(s))\,ds - \int_0^t T(t-s)\mathrm{f}(s,\mathrm{u}(s))\,ds\Big\|_X \\
&\le MPe^{\omega S}\int_0^t \|\mathrm{u}_n(s) - \mathrm{u}(s)\|_X\,ds \\
&\le MPSe^{\omega S}\|\mathrm{u}_n - \mathrm{u}\|_{C([0,S];X)}\,,
\end{aligned}$$

and after passing to the limit in (2.2) we get the existence in this case.

II. If v has a blow-up in S, we can repeat the previous argument, thus getting a mild solution u^{S_1} on $[0,S_1]$ for every $S_1 \in (0,S)$. The uniqueness of the mild solution provides that the function $\mathrm{u} : [0,S) \to X$ given by

$$\mathrm{u}(t) := \mathrm{u}^{S_1}(t)\,, \quad t \in [0,S_1)$$

is well-defined. It is then easy to see that u is the mild solution of (1.1) on $[0,S)$, and that estimate (2.1) holds. □

Remark 2.2. For the right hand side of (1.1) we can, instead of a function defined on the whole $[0,T] \times X$, consider a function $\mathrm{f} : [0,T] \times \mathrm{B}_X(0,b) \to X$, for some $b > 0$. In this case the function Φ from Theorem 1.1 is defined on $[0,T] \times [0,b)$ and the solution v of (1.2) cannot blow-up, but it can quench when v approaches b. The statement of Theorem 2.1 is valid in this case as well, with some technical differences in its proof.

Remark 2.3. Note that Theorem 2.1 also provides an estimate on the time of existence of the mild solution of (1.1). Namely, the mild solution of (1.1) *exists at least as long as the solution to* (1.2).

Remark 2.4. Instead of Φ one can also consider some larger local bound for f in (1.2). However, an analogue of [2, Theorem 2] can be proven, which states that the best possible estimate of type (2.1) will be given by the smallest possible local bound for f, i.e. the function Φ.

Remark 2.5. The estimate in Theorem 2.1 is not optimal in general, as it can be seen from examples in [2]. Of course, due to the generality of Theorem 2.1, this is expected. However, in some examples, such as in our first example of linear heat equation in the third section, our estimate appears to be sharp. Actually, this particular example illustrates how optimality can be lost due to the imprecision in the bound

$$(\forall r>0)(\forall \mathrm{w}\in \mathrm{B}_X[0,r]) \quad \|\mathrm{f}(t,\mathrm{w})\|_X \le \Phi(t,r) \quad (\text{a.e. } t\in[0,T])\,.$$

This non-optimality can also arise due to the imprecision of the estimate $\|T(t)\|_{\mathcal{L}(X)} \le Me^{\omega t}$. In some examples this can happen even at $t=0$ (not if $(T(t))_{t\ge 0}$ is a semigroup of contractions, of course). In such situations this estimate may be improved first by introducing $\tilde{A}=A-\omega I$ (which is an infinitesimal generator of a semigroup $(\tilde{T}(t))_{t\ge 0}$, $\tilde{T}(t)=e^{-\omega t}T(t)$), and then replacing $\|\cdot\|_X$ by an equivalent norm $\|\cdot\|_{X,\tilde{A}} := \sup_{t\ge 0}\|\tilde{T}(t)\cdot\|_X$ on X in which $(\tilde{T}(t))_{t\ge 0}$ is a semigroup of contractions.

Remark 2.6. Theorem 2.1 can be stated also for non-autonomous (evolution) abstract systems

$$\begin{gathered} \mathrm{u}'(t) - A(t)\mathrm{u}(t) = \mathrm{f}(t,\mathrm{u}(t)) \quad \text{in } (0,T)\,, \\ \mathrm{u}(0) = \mathrm{g}\,, \end{gathered} \tag{2.3}$$

where $(A(t))_{t\in[0,T]}$ is a family of infinitesimal generators of C_0-semigroups $(S_t(s))_{s\ge 0}$ on X satisfying $\|S_t(s)\|_{\mathcal{L}(X)} \le e^{\omega s}$ ($\omega\in\mathbb{R}$ is independent of t) [6, p. 131], $D(A(t)) = D$ is independent of t, and $A(t)\mathrm{v}$ is continuously differentiable in X for every $\mathrm{v}\in V$. Indeed, under the above assumptions there exists an evolution system $(U(t,s))_{t\ge s\ge 0}$ [6, p. 145], and the mild solution on $[0,S)$ is given by

$$\mathrm{u}(t) = U(t,0)\mathrm{g} + \int_0^t U(t,s)\mathrm{f}(s,\mathrm{u}(s))ds\,, \quad t\in[0,S)\,.$$

Since the evolution system $(U(t,s))_{t\ge s\ge 0}$ satisfies

$$\|U(t,s)\|_{\mathcal{L}(X)} \le e^{\omega(t-s)}\,,$$

(see [6, p. 135]), all the steps of the proof of Theorem 2.1 remain the same.

One can consider even weaker assumptions on $(A(t))_{t\in[0,T]}$ [6, Ch. 5], but checking those assumptions can be very technical in general [6, p. 225].

3. Applications

In this section we will illustrate the result of Theorem 2.1. There are many problems that can be placed in the setting of Theorem 2.1, e.g. [6, Ch. 8], [4], and every (semilinear) problem that can be written as a non-stationary Friedrichs system [3]. We selected one classical example of linear equation and two nonlinear examples which arise from particular problems in physics.

Example (Linear heat equation). This example appears to be quite illuminating regarding optimality of our estimate. More precisely, it illustrates how optimality of estimate (2.1) can depend on precision of local bound Φ for f: let us consider a linear one-dimensional boundary value problem

$$\begin{gathered} \partial_t u(t,x) - \partial_{xx}u(t,x) = \lambda u(t,x) \quad \text{in } (0,T)\times(0,\pi), \\ u(\cdot,0) = u(\cdot,\pi) = 0, \\ u(0,\cdot) = u_0\,, \end{gathered} \tag{3.1}$$

where $\lambda\in\mathbb{R}$ and $u_0\in L^2((0,\pi))$.

Using separation of variables we can obtain formula for the unique global solution

$$u(t,x)=\sum_{k=1}^{\infty}A_k e^{(\lambda-k^2\pi^2)t}\sin(k\pi x)\,,$$

where

$$A_k=2\int_0^1 u_0(x)\sin(k\pi x)dx\,.$$

By the Parseval identity from the above formulae one can derive the estimate

$$\|u(t,\cdot)\|_{L^2((0,\pi))}\leq\|u_0\|_{L^2((0,\pi))}e^{(\lambda-\pi^2)t}\,,\quad t\in[0,T]\,,$$

which is sharp because in the case when u_0 equals the first eigenfunction $\sin(\pi x)$ we have an equality.

Let us now consider the problem above in the abstract sense in the space $X=L^2((0,\pi))$ and check how our estimate (2.1) reads for this example. We introduce notation for the unknown $\mathrm{u}(t):=u(t,\cdot)$ and the initial data $\mathrm{u}_0:=u_0(\cdot)$, while $A:=\partial_{xx}$ defined on

$$D(A)=\{v\in H_0^1((0,\pi)):\partial_{xx}v\in X\}\leq X\,.$$

According to [4, Prop. 2.6.1], A is an infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t\geq0}$. Moreover, by [4, Prop. 3.5.5], we have even sharper estimate

$$\|T(t)\|_{\mathcal{L}(X)}\leq e^{-\lambda_1 t}\,,$$

where $\lambda_1=\pi^2$ is the first eigenvalue of the operator $-\partial_{xx}$ in $H_0^1((0,\pi))$, and thus we can take $M=1$, $\omega=-\pi^2$.

Since the right hand side is linear in u and independent of t, it is easy to see that the local bound from Theorem 1.1 is $\Phi(u)=|\lambda|u$, so the solution of (1.2) is given by

$$v(t)=\|\mathrm{u}_0\|_X e^{|\lambda|t}\,.$$

The estimate (2.1) from Theorem 2.1 then reads

$$\|\mathrm{u}(t)\|_X\leq\|\mathrm{u}_0\|_X e^{(|\lambda|-\pi^2)t}\,,\quad t\in[0,T]\,.$$

Compering the obtained estimate with the exact solution, we can see that for $\lambda\geq0$ we have a sharp result, but for $\lambda<0$ there is a big deviation, because the local bound Φ does not distinguish the sign of the right hand side.

However, with a slight change in our abstract setting we can overcome an appearing imperfection and get the sharp estimate. Indeed, let us define $B:=A+\lambda I$, where I is the identity operator on X, which is also an infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq0}$ satisfying $S(t)=e^{\lambda t}T(t)$ (see [6, p. 12]). In terms of the operator B our abstract equation becomes

$$\mathrm{u}'(t)-B\mathrm{u}(t)=0\,.$$

Since the right hand side is zero, hence Φ is zero, the solution of (1.2) is constant function $v(t)=\|\mathrm{u}_0\|_X$. Therefore in this case the corresponding estimate (2.1) is given by

$$\|\mathrm{u}(t)\|_X\leq\|\mathrm{u}_0\|_X e^{(\lambda-\pi^2)t}\,,\quad t\in[0,T]\,,$$

where we have used

$$\|S(t)\|_{\mathcal{L}(X)}\leq e^{\lambda t}\|T(t)\|_{\mathcal{L}(X)}\leq e^{(\lambda-\pi^2)}\,.$$

The obtained estimate is indeed optimal, as it coincides with the one derived from the explicit formula for the solution.

Remark 3.1. The method from the previous example can be generalized for a wider class of abstract Cauchy problems (1.1): let a C_0-semigroup $(T(t))_{t\geq 0}$ generated by A for $\omega \in \mathbb{R}$ satisfies

$$(\forall t \geq 0) \quad \|T(t)\|_{\mathcal{L}(X)} \leq e^{\omega t} ,$$

i.e. $M = 1$, and the right hand side be of the form

$$\mathrm{f}(t, \mathrm{u}(t)) = \lambda(t)\mathrm{u}(t) + \tilde{\mathrm{f}}(t, \mathrm{u}(t)) ,$$

for $\lambda \in C^1((0,T);\mathbb{R})$ bounded from above, i.e. $\lambda_\infty := \sup_{t\in[0,T]} \lambda(t) < \infty$.

Instead of directly applying Theorem 2.1, let us rewrite our system by introducing a family of operators $B(t) := A + \lambda(t)I$. Since I is a bounded operator, for every $t \in [0, T]$ we have $D(B(t)) = D(A)$ and a C_0-semigroup $(S_t(s))_{s\geq 0}$ associated to $B(t)$ satisfies

$$\|S_t(s)\|_{\mathcal{L}(X)} = e^{\lambda(t)s}\|T(s)\|_{\mathcal{L}(X)} \leq e^{\lambda_\infty s}e^{\omega s} = e^{\tilde{\omega}s} ,$$

where $\tilde{\omega} := \lambda_\infty + \omega$ does not depend on t. Since λ is continuously differentiable, we have that the equivalent system to the starting (1.1)

$$\begin{gathered} \mathrm{u}'(t) - B(t)\mathrm{u}(t) = \tilde{\mathrm{f}}(t, \mathrm{u}(t)) \quad \text{in } (0, T) , \\ \mathrm{u}(0) = \mathrm{g} , \end{gathered}$$

satisfies the assumptions of Remark 2.6, hence we have the unique solution that satisfies the corresponding estimate (2.1).

With this approach we indeed have a better estimate then with the direct application of Theorem 2.1, since the obtained estimate distinguishes the sign of λ (as it can be seen in the previous example).

Example (Nonlinear heat equation). Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded with a Lipschitz boundary, and let $T, b, p > 0$ be positive constants. We consider a semilinear initial-boundary value problem:

$$\begin{gathered} \partial_t u(t, \mathbf{x}) - \Delta u(t, \mathbf{x}) = \frac{\gamma(\mathbf{x})}{(b - u(t, \mathbf{x}))^p} \quad \text{in } (0, T) \times \Omega, \\ u\big|_{\partial\Omega} = 0, \\ u(0, \cdot) = u_0 , \end{gathered} \tag{3.2}$$

where $\gamma, u_0 \in C_0(\Omega)$ (continuous functions on $\operatorname{cl}\Omega$ that are zero on the boundary of domain), while $u : [0, T) \times \Omega \to \mathbb{R}$ is the unknown function. Such an initial and boundary value problem models the dynamic deflection of an elastic membrane in a simple electrostatic Micro-Electromechanical System (MEMS) device (see [5] and references therein).

We will consider the problem above in the abstract sense in the space $X = C_0(\Omega)$ with a sup norm; the unknown being $\mathrm{u}(t) := u(t, \cdot)$, the initial data $\mathrm{u}_0 := u_0(\cdot)$, and $\boldsymbol{\gamma} := \gamma(\cdot)$. We take $A := \Delta$ defined on

$$D(A) = \{v \in H^1_0(\Omega) \cap X : \Delta v \in X\} \leq X ,$$

where Δv is taken in the sense of distributions. One can notice that this Laplace operator and one used in the previous example (for $d = 1$) are not the same, as they are acting on different spaces. Here we have chose $C_0(\Omega)$ so that the right hand side would be Lipschitz, as it will be seen in the sequel. According to [4, Prop. 2.6.7], A is an infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t\geq 0}$ (thus $M = 1$, $\omega = 0$). Since the mapping $w \mapsto \frac{1}{(b-w)^p}$ is locally Lipschitz on the open ball $\mathrm{B}(0,b)$, the right hand side in our equation is also locally Lipschitz on $\mathrm{B}_X(0,b)$. The local bound from Theorem 1.1 is given by $\Phi(r) := \frac{\|\boldsymbol{\gamma}\|_X}{(b-r)^p}$ for $r \in [0,b)$. Although the assumptions of Theorem 2.1 for the right hand side are satisfied only on $\mathrm{B}_X(0,b)$, based on Remark 2.2, the statement of Theorem 2.1 is valid if $\mathrm{u}_0 \in \mathrm{B}_X(0,b)$.

The solution of (1.2),

$$v(t) = b - \Big((b - \|\mathrm{u}_0\|_X)^{p+1} - (p+1)\|\boldsymbol{\gamma}\|_X t\Big)^{1/(p+1)},$$

exists until time $T_1 = \frac{(b-\|\mathrm{u}_0\|_X)^{p+1}}{(p+1)\|\boldsymbol{\gamma}\|_X}$ when it quenches. According to Theorem 2.1, the mild solution u of (3.2) exists on $[0,T_1)$ and we have

$$\|\mathrm{u}(t)\|_{C_0(\Omega)} < v(t)\,,$$

for $t \in [0,T_1)$.

As we have emphasized, the result of Theorem 2.1 is not optimal in general, but in this particular example we can say more regarding its optimality: in [1, Theorem 2.1], it was shown that under some additional (rather technical) assumptions the classical solution of (3.2) quenches in some finite time T_q which satisfies the estimate

$$0 \leq T_q - T_1 \leq \frac{1}{\|\boldsymbol{\gamma}\|_X}\Big(1 + \frac{C}{p+1}\Big)(b - \|\mathrm{u}_0\|_X)^{\frac{4p+1}{3}} + o\Big((b - \|\mathrm{u}_0\|_X)^{\frac{4p+1}{3}}\Big)\,,$$

with some constant $C > 0$ not depending on $\|\mathrm{u}_0\|_X$. Thus the quenching time T_q approaches T_1 as $\|\mathrm{u}_0\|_X \to b$, so in that sense we can say that Theorem 2.1 gives *good approximation* for the maximal time of existence for the solution of (3.2).

Example (Nonlinear Schrödinger equation). Let us now consider an initial-value problem on $\mathbb{R}^d$, for $d \leq 3$:

$$\begin{gathered}\partial_t u(t,\mathbf{x}) - i\Delta u(t,\mathbf{x}) = -\gamma(t)u(t,\mathbf{x}) - g(t)|u(t,\mathbf{x})|^2 u(t,\mathbf{x}) \quad \text{in } (0,T)\times\mathbb{R}^d,\\ u(0,\cdot) = u_0\,,\end{gathered} \tag{3.3}$$

where $\gamma, g \in C([0,T];\mathbb{C})$, $u_0 \in L^2(\mathbb{R}^d;\mathbb{C})$ and $u : [0,T) \to \mathbb{C}$ is the unknown function. Nonlinear Schrödinger equations appear in the modeling of numerous physical phenomena, such as propagation of laser beams in nonlinear media, plasma dynamics, mean field dynamics of Bose-Einstein condensates, condensed matter, etc. (see [7] and references therein). As before, we shall study (3.3) as an abstract problem, with the unknown $\mathrm{u}(t) := u(t,\cdot)$, and the initial data $\mathrm{u}_0 := u_0(\cdot)$.

The operator $A := i\Delta$ is an infinitesimal generator of a C_0-semigroup of unitary operators $(T(t))_{t\geq 0}$ on $L^2(\mathbb{R}^d;\mathbb{C})$ (thus $M = 1$, $\omega = 0$) with the domain $D(A) = H^2(\mathbb{R}^d;\mathbb{C})$ [6, p. 224]. In order to apply Theorem 2.1, it is necessary for the right hand side to be locally Lipschitz in u, which cannot be obtained in $L^2(\mathbb{R}^d;\mathbb{C})$. Therefore, we shall restrict our problem to $X := D(A) = H^2(\mathbb{R}^d;\mathbb{C})$, equipped with the graph norm of operator A. The equivalence of the graph norm and the (standard) norm on $H^2(\mathbb{R}^d;\mathbb{C})$ gives us that X coincides with $H^2(\mathbb{R}^d;\mathbb{C})$, as a Banach space.

As in [6, p. 190], we can conclude that $(T(t)\big|_X)_{t\geq 0}$ is also an unitary C_0-semigroup on X generated by the part of A in X, $A\big|_X : D(A\big|_X) \subseteq X \to X$, defined by $A\big|_X \mathrm{u} := A\mathrm{u}$ on the domain

$$D(A\big|_X) := \{\mathrm{u} \in D(A) \cap X : A\mathrm{u} \in X\} \leq X\,.$$

Using the Sobolev emmbeding theorem, in [6, Lemma 1.2.] it is shown that $\mathrm{w} \mapsto |\mathrm{w}|^2\mathrm{w}$ is locally Lipschitz in X. The proof relies on the continuous embedding of $H^2(\mathbb{R}^d;\mathbb{C})$ into $L^\infty(\mathbb{R}^d;\mathbb{C})$, which is valid only for $d \leq 3$. As γ and g are bounded, we finally obtain that the right hand side of (3.3) is locally Lipschitz in u.

With a stronger assumption on the inital data, $\mathrm{u}_0 \in X$, we can apply Theorem 2.1. As the local bound of the right hand side is $\Phi(t,r) = |\gamma(t)|r + |g(t)|r^3$, we get a Bernoulli first-order ODE for (1.2), with the solution (for the non-trivial case $\|\mathrm{u}_0\|_X \neq 0$)

$$v(t) = \Big(e^{-2\int_0^t |\gamma(\tau)|\, d\tau}\Big(-2\int_0^t |g(s)|e^{2\int_0^s |\gamma(\tau)|\, d\tau}\, ds + \|\mathrm{u}_0\|_X^{-2}\Big)\Big)^{-1/2},$$

that has a blow-up in finite time T_1 (when the term inside inner parentheses takes the value 0). Finally, we conclude that for $\mathrm{u}_0 \in H^2(\mathbb{R}^d;\mathbb{C})$ we have the existence of the unique mild solution $\mathrm{u} \in C([0,T_1);H^2(\mathbb{R}^d;\mathbb{C}))$, which is in fact the *classical solution* [6, p. 108], and which satisfies (2.1) on $[0,T_1)$.

For some specific γ and g we are far from an *optimal* result on the maximal time of existence in this example. More precisely, if we take $d = 2$, $\gamma = 0$ and $g(t) = -ik$, for some $k > 0$, it can be shown that our solution is global, i.e. it exists for all times (see [6, p. 233]). For more detailed exposition regarding globallity of solution for nonlinear Schrödinger equation we refer to [4, pp. 112–123].

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