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APPROXIMATE SOLUTIONS OF GENERAL PERTURBED KDV-BURGERS EQUATIONS

BAOJIAN HONG, DIANCHEN LU

Abstract. In this article, we present some approximate analytical solutions to the general perturbed KdV-Burgers equation with nonlinear terms of any order by applying the homotopy analysis method (HAM). While compared with the Adomain decomposition method (ADM) and the homotopy perturbation method (HPM), the HAM contains the auxiliary convergence-control parameter $\hbar$ and the control function $H(x,t)$, which provides a useful way to adjust and control the convergence region of solution series. The numerical results reveal that HAM is accurate and effective when it is applied to the perturbed PDEs.

1. Introduction

With the development of soliton theory in nonlinear science, searching for analytical solitary wave solutions or approximate solutions of nonlinear partial differential equations (NLPDEs) plays an important and significant role in the study of dynamics of those nonlinear phenomena [10]. Many authors presented various powerful method to deal with this problem, such as inverse scattering transformation [4], Hirota bilinear method [23], homogeneous balance method [27], Bäcklund transformation [26], Darboux transformation [19], the elliptic integral method [6], the first integral method [7, 8] and so on. Because of the complexity of NLPDEs, It is difficult for us to find exact solutions in a straightforward way. One has to propose and develop some approximate methods for nonlinear theory, such as the multiple-scale method [24], the variational iteration method [9], the indirect matching method [28], the renormalization method [20], and the homotopy perturbation method [13] etc. The common essential point of these methods is to study nonlinear systems by using the approximation method.

The homotopy analysis method (HAM) was introduced in 1992 [16, 17], which yields a fast convergence for most of the selected problems. It also shows a high accuracy and a rapid convergence to solutions of the nonlinear partial evolution equations. After this, many types of nonlinear problems were solved with the aid of HAM, such as the nonlinear Vakhnenko equation [29], the Glauert-jet problem

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[5], a generalized Hirota-Satsuma coupled KdV equation [2], and a smoking habit model [11, 31] etc.

The rest of this article is organized as follows. In Section 2, we obtain some exact solutions of the general perturbed KdV-Burgers equation by using the mapping deformation method. In Section 3, we apply HAM to construct approximate solutions for the general perturbed KdV-Burgers equation. In Section 4, we discuss the accuracy of these solutions with the small perturbation term as illustrations. Also we present a short conclusion.

2. Exact solutions

Consider the general perturbed KdV-Burgers equation

$$u_t + \alpha u^p u_x + \beta u^{2p} u_x + \gamma u_{xx} + \delta u_{xxx} = f(u), \tag{2.1}$$

where $\alpha, \beta, \gamma, \delta, p$ are arbitrary constants, and $f = f(u)$ is a perturbed term, which is a sufficiently smooth function in a corresponding domain. If we let $f = 0$, we can get the well-known KdV-Burgers equation with nonlinear terms of any order [12, 6, 14, 15, 25, 30]:

$$u_t + \alpha u^p u_x + \beta u^{2p} u_x + \gamma u_{xx} + \delta u_{xxx} = 0. \tag{2.2}$$

This equation with $p \geq 1$ arises in modeling waves generated by a wavemaker in a channel and waves incoming from deep water into nearshore zones and some profound results have been described in [22]. In fact, if one takes different values for $\alpha, \beta, \gamma, \delta, p$ and f, equation (2.1) includes quite a few equations as particular cases such as KdV equation, MKdV equation, CKdV equation, Burgers equation, and KdV-Burgers equation as follows: Fitzhugh-Nagumo equation [3]:

$$u_t - u_{xx} = f = u(u-\alpha)(1-u); \tag{2.3}$$

Burgers-Huxley equation [21]:

$$u_t + \alpha u^\delta u_x - \lambda u_{xx} = f = \beta u(1-u^\delta)(\eta u^\delta - \gamma); \tag{2.4}$$

Burgers-Fisher equation [21]:

$$u_t + \alpha u^\delta u_x - u_{xx} = f = \beta u(1-u^\delta). \tag{2.5}$$

By using the general mapping deformation method [8], we know that (2.2) admits the following solutions:

$$u_1 = \{A_1(K - \sqrt{K^2}\tanh[(\frac{p\gamma}{2K(2+p)\delta} \pm \frac{p\alpha}{2K^2(2+p)}\sqrt{\frac{-K^2(1+2p)}{(1+p)\beta\delta}})\sqrt{K^2}\xi_1]\}^{1/p}; \tag{2.6}$$

$$u_2 = \{-\frac{c(1+p)}{2\alpha} - \frac{c(1+p)}{2\alpha}\tanh\frac{cp}{\gamma}(x+ct+\xi_0)\}^{1/p}, \beta = \delta = 0; \tag{2.7}$$

$$\xi_1 = x + [\frac{(1+p)\gamma^2}{(2+p)^2\delta} + \frac{(1+2p)\alpha^2}{(1+p)(2+p)^2\beta} \pm \frac{p\alpha\gamma}{K(2+p)^2\beta}\sqrt{\frac{-K^2(1+2p)}{(1+p)\beta\delta}}]t + \xi_0; \tag{2.8}$$

where

$$A_1 = -\frac{(1+2p)\alpha}{2K(2+p)\beta} \pm \frac{\gamma}{2K^2(2+p)}\sqrt{\frac{-K^2(1+p)(1+2p)}{\beta\delta}},\ K,\ \xi_0$$

and c are arbitrary constants.

Note that $i\tanh(i\xi) = -\tan\xi$, $\tanh(\xi + \frac{\pi}{2}i) = \coth(\xi)$, $i\coth(i\xi) = \cot\xi$, $i = \sqrt{-1}$. Also note that the solution $u_{1,2}$ contains all results presented in [15].

3. Homotopy analysis method (HAM)

To describe the basic idea of the HAM, let us consider the nonlinear equation, in a standard form,

$$N[u(x,t)] = 0, \tag{3.1}$$

where N is a nonlinear operator, $u(x,t)$ is an unknown function, x and t denote the spatial and temporal independent variables, respectively.

By using the basic idea of the traditional homotopy method [16], we construct the zero-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = q\hbar H(x,t)N[\phi(x,t;q)], \tag{3.2}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar$ is a nonzero auxiliary parameter, $H(x,t)$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$ is an initial estimate of $u(x,t)$ and $\phi(x,t;q)$ is an unknown function. It is important that we have much freedom to choose auxiliary things in HAM. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t). \tag{3.3}$$

Thus, as q increases from 0 to 1, the function $\phi(x,t;q)$ varies from the initial value $u_0(x,t)$ to the exact solution $u(x,t)$. Expanding $\phi(x,t;q)$ in the Taylor series with respect to q, we have

$$\phi(x,t;q) = u_0 + \sum_{m=1}^{\infty} u_m q^m; \quad u_0 = u_0(x,t), \quad u_m = u_m(x,t), \tag{3.4}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m}{\partial q^m}\phi(x,t;q)\big|_{q=0}\,. \tag{3.5}$$

If the auxiliary linear operator, the initial estimate, the auxiliary parameter and the auxiliary function are properly chosen such that they are smooth enough, the Taylor's series (3.4) with respect to q converges at $q = 1$, and we have

$$u = \phi(x,t;1) = \sum_{m=0}^{\infty} u_m. \tag{3.6}$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series has already presented by Abbaoui and Cherruault [1]. Liao proved that it must be one of the exact solutions of the original nonlinear equation [17]. As $\hbar = -1$ and $H(x,t) = 1$, equation (3.2) becomes

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] + qN[\phi(x,t;q)] = 0, \tag{3.7}$$

which is frequently used in the homotopy perturbation method (HPM). The comparison between HAM and HPM can be found in [18]. As $H(x,t) = 1$, equation (3.2) becomes

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(x,t;q)]. \tag{3.8}$$

According to definition (3.5), the governing equation can be deduced from the zero-order deformation equation (3.2). Define the vector

$$\vec{u}_m(x,t) = \{u_0, u_1, u_2, \dots, u_m\}. \tag{3.9}$$

Differentiating equation (3.2) m times with respect to the embedding parameter q, then setting $q = 0$ and dividing them by $m!$, we get the mth-order deformation equation

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_{m-1}(\vec{u}_{m-1}, x, t), \tag{3.10}$$

where

$$R_{m-1}(\vec{u}_{m-1}, x, t) = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} N[\phi(x,t;q)]\big|_{q=0}, \tag{3.11}$$

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m \ge 2. \end{cases} \tag{3.12}$$

It is notable that the m-th order deformation equation (3.10) is linear, and $u_m(x,t)$ for $m \ge 1$ can be easily solved by the boundary conditions and the symbolic computation software such as Mathematica and Matlab.

To solve (2.1) by means of HAM, we choose the initial approximation

$$u_0(x,t) = \tilde{u}_0(x,t)\big|_{t=0} = g(x), \tag{3.13}$$

where $\tilde{u}_0(x,t)$ is an arbitrary exact solution of (2.2). According to (2.1), we define the nonlinear operator

$$N[\phi] = \phi_t + \alpha\phi^p\phi_x + \beta\phi^{2p}\phi_x + \gamma\phi_{xx} + \delta\phi_{xxx} - f(\phi), \quad \phi = \phi(x,t;q). \tag{3.14}$$

By following the process above, it is straightforward to choose $H(x,t) = 1$, the base functions $g_n(x)t^n$, $n \ge 0$, and the linear operator

$$L[\phi(x,t;q)] = \frac{\partial \phi(x,t;q)}{\partial t}, \tag{3.15}$$

with the condition

$$L[c(x)] = 0. \tag{3.16}$$

From equations (3.10), (3.11) and (3.14), we have

$$\begin{aligned} R_{m-1}(\vec{u}_{m-1}, x, t) &= u_{m-1,t} + \gamma u_{m-1,xx} + \delta u_{m-1,xxx} + \alpha D_{m-1}(\phi^p\phi_x) \\ &\quad + \beta D_{m-1}(\phi^{2p}\phi_x) - F(u_0, u_1, \dots, u_{m-1}), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} D_{m-1}(\phi^n\phi_x) = &\sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{m-1}=0}^{k_{m-2}} \sum_{i=0}^{m-1} C_n^{k_1} C_{k_1}^{k_2} C_{k_2}^{k_3} \dots \\ &\times C_{k_{m-2}}^{k_{m-1}} u_0^{n-k_1} u_1^{k_1-k_2} u_2^{k_2-k_3} \dots u_{m-2}^{k_{m-2}-k_{n-1}} u_{m-1}^{k_{m-1}} u_{i\xi}, \end{aligned}$$

and $n \ge k_1 \ge k_2 \ge \dots \ge k_{m-1} \ge 0 \in N$, with $\sum_{j=1}^{m-1} k_j + i = m-1$, $i = 0, \dots, m-1$. Furthermore, we have

$$F(u_0, u_1, \dots, u_{m-1}) = \frac{1}{(n-1)!} \frac{\partial^{(m-1)}}{\partial q^{m-1}} f(\phi)\big|_{q=0}.$$

The solution of the m-th order deformation equation (3.10) with the initial condition $u_m(x,0)=0$ for $m\geq 1$ becomes

$$u_m=\chi_m u_{m-1}+L^{-1}[\hbar R_{m-1}(\vec{u}_{m-1},x,t)]. \tag{3.18}$$

Thus, from equations (3.13), (3.17) and (3.18), we can successively obtain

$$u_0=\tilde{u}_0(x,0)=g(x), \tag{3.19}$$

$$u_1=-\hbar t[\tilde{c}_0(x)+f(u_0)],\tilde{c}_0(x)=\frac{\partial}{\partial t}\tilde{u}_0(x,t)\big|_{t=0}\,, \tag{3.20}$$

$$u_2=(1+\hbar)u_1+\hbar t[\alpha u_0^p u_{1,x}+\beta u_0^{2p}u_{1,x}+\gamma u_{1,xx}+\delta u_{1,xxx}-f_u(u_0)u_1], \tag{3.21}$$

$$\dots$$

$$\begin{aligned} u_m=(1+\hbar)u_{m-1}+\hbar t[\gamma u_{1,xx}+\delta u_{1,xxx}+\alpha D_{m-1}(\phi^p\phi_x)\\ +\beta D_{m-1}(\phi^{2p}\phi_x)-F(u_0,u_1,\dots,u_{m-1})].\end{aligned} \tag{3.22}$$

Consequently, we obtain the following m-th order approximate solution, and exact solution of (2.1):

$$u_{m,\mathrm{appr}}=\sum_{k=0}^{m}u_k,\quad u_{\mathrm{exact}}=\phi(x,t;1)=\lim_{m\to\infty}\sum_{k=0}^{m}u_k. \tag{3.23}$$

4. Examples and discussion

In this section, three specific examples about equation (2.1) are presented to illustrate the effectiveness of the HAM. We plot the $\hbar$-curves of $u''_{\mathrm{appr}}(0,0)$ and $u'''_{\mathrm{appr}}(0,0)$ to discover the valid region of $\hbar$, which corresponds to the line segment nearly parallel to the horizontal axis. A comparison among the initial exact solution for the traditional unperturbed equation when $f=0$, the exact solution for the perturbed equation when $f\neq 0$ and the fourth order of approximate solution for the perturbed equation is given through numerical simulations.

Example 4.1. Consider the CKdV equation with a small perturbed term

$$u_t+6uu_x-6u^2u_x+u_{xxx}=\varepsilon u^2,\quad 0<\varepsilon\ll 1, \tag{4.1}$$

with the initial exact solution

$$\tilde{u}_0(x,t)=\frac{1}{2}-\frac{1}{2}\tanh[\frac{1}{2}(x-t)]. \tag{4.2}$$

From the preceding section, we have

$$\begin{gathered} u_0=\frac{1}{2}-\frac{1}{2}\tanh(\frac{1}{2}x),\tilde{c}_0(x)=\frac{1}{4}\sec h^2(\frac{1}{2}x),\\ u_1=-\hbar t\{\frac{1}{4}\sec h^2(\frac{1}{2}x)+\varepsilon[\frac{1}{2}-\frac{1}{2}\tanh(\frac{1}{2}x)]^2\}, \end{gathered}$$

$$
\begin{aligned}
u_2 &= -(1+\hbar)\hbar t\{\frac{1}{4}\sec h^2(\frac{1}{2}x) + \varepsilon[\frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{2}x)]^2\} \\
&\quad - \hbar^2 t^2\Big\{6[\frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{2}x)]\{\frac{1}{4}\sec h^2(\frac{1}{2}x) + \varepsilon[\frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{2}x)]^2\}_x \\
&\quad + 6\hbar^2 t^2[\frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{2}x)]^2\{\frac{1}{4}\sec h^2(\frac{1}{2}x) \\
&\quad + \varepsilon[\frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{2}x)]^2\}_x - \hbar^2 t^2\{\frac{1}{4}\sec h^2(\frac{1}{2}x) \\
&\quad + \varepsilon[\frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{2}x)]^2\}_{xxx} \\
&\quad + 2\varepsilon\hbar^2 t^2[\frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{2}x)]\{\frac{1}{4}\sec h^2(\frac{1}{2}x) + \varepsilon[\frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{2}x)]^2\} \\
&= \frac{\hbar t}{32}[\cosh(\frac{x}{2}) - \sinh(\frac{x}{2})]\sec h^5(\frac{x}{2})\{\hbar(5t - 3 - 3\varepsilon) - 3 - 3\varepsilon \\
&\quad + 2\hbar t\varepsilon(1+\varepsilon) + 2\cosh(x)[2\varepsilon - 2 - 2\hbar(1+\varepsilon) + \hbar t(2\varepsilon^2 + 7\varepsilon - 3)] \\
&\quad + [\hbar(t - \varepsilon - 1 + 2t\varepsilon^2) - \varepsilon - 1]\cosh(2x) - 2\sinh(\frac{x}{2})[1 - \varepsilon + \hbar - \varepsilon\hbar \\
&\quad + \hbar t(2 - 3\varepsilon + 2\varepsilon^2) + (1-\varepsilon)\cosh x + \hbar(1 - t - \varepsilon + 2t\varepsilon^2)\cosh x)]\Big\},
\end{aligned}
$$

$$\dots$$

$$
\begin{aligned}
u_{\rm appr} &= \frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{2}x) - \hbar\big\{\frac{1}{4}\sec h^2(\frac{1}{2}x) + \varepsilon[\frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{2}x)]^2\big\}t \\
&\quad + \frac{\hbar t}{32}[\cosh(\frac{x}{2}) - \sinh(\frac{x}{2})]\sec h^5(\frac{x}{2})\Big\{\hbar(5t - 3 - 3\varepsilon) - 3 - 3\varepsilon + 2\hbar t\varepsilon(1+\varepsilon) \\
&\quad + 2\cosh(x)[2\varepsilon - 2 - 2\hbar(1+\varepsilon) + \hbar t(2\varepsilon^2 + 7\varepsilon - 3)] \\
&\quad + [\hbar(t - \varepsilon - 1 + 2t\varepsilon^2) - \varepsilon - 1]\cosh(2x) - 2\sinh(\frac{x}{2})[1 - \varepsilon + \hbar - \varepsilon\hbar \\
&\quad + \hbar t(2 - 3\varepsilon + 2\varepsilon^2) + (1-\varepsilon)\cosh x + \hbar(1 - t - \varepsilon + 2t\varepsilon^2)\cosh x]\} + \dots.
\end{aligned}
$$

The $\hbar$-curves of $u''_{\rm appr}(0,0)$ and $u'''_{\rm appr}(0,0)$ to equation (4.1) are shown in Figure 1. A comparison between the initial exact solution and the approximate solution of the fourth order is provided in Figure 2 (a)-(b), which indicates that the solution series (3.23) is convergent when $-1.2 \le \hbar < 0$, and the approximate solution for $\hbar = -0.1$ and $\hbar = -1$ (HPM) is compared. We can see that the best value of $\hbar$ in this case is not -1.

Example 4.2. Consider the KdV-Burgers equation with a small perturbed term

$$u_t + 6uu_x + u_{xx} - u_{xxx} = \varepsilon\sin u, \quad 0 < \varepsilon \ll 1, \tag{4.3}$$

with the initial exact solution

$$\tilde{u}_0(x,t) = \frac{1}{50}\big\{1 - \coth[-\frac{1}{10}(x - \frac{6}{25}t)]\big\}^2. \tag{4.4}$$

From the preceding section, we have

$$
\begin{gathered}
u_0 = \frac{1}{50}[1 - \coth(-\frac{1}{10}x)]^2, \quad \tilde{c}_0(x) = \frac{3}{3125}\csc h^2(\frac{1}{10}x)[1 + \coth(\frac{1}{10}x)], \\
u_1 = -\hbar\varepsilon\sin\{\frac{1}{50}[1 - \coth(-\frac{1}{10}x)]^2\}t - \frac{3}{3125}\hbar t\csc h^2(\frac{1}{10}x)[1 + \coth(\frac{1}{10}x)], \\
\dots
\end{gathered}
$$

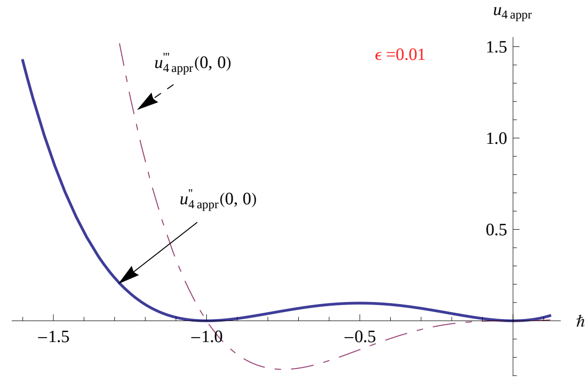
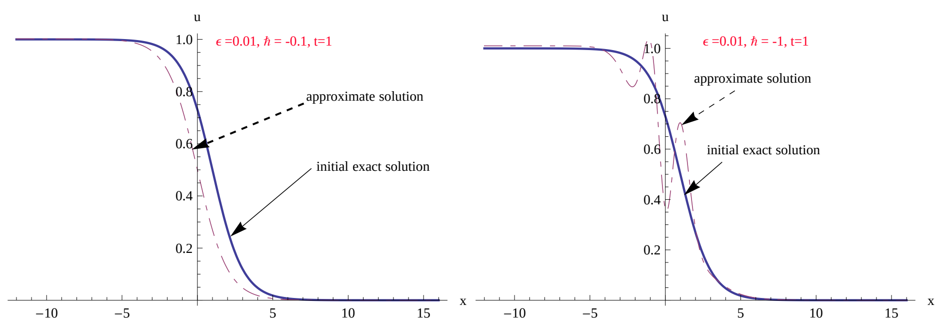


FIGURE 1. $\hbar$-curves of $u''_{\rm appr}(0,0)$ and $u'''_{\rm appr}(0,0)$ at the fourth order approximation



(a) (b)

FIGURE 2. Comparison between the curves of initial exact solution and the fourth order approximate solution with $\hbar = -0.1, -1$.

$$\begin{aligned} u_{\rm appr} &= \frac{1}{50}[1-\coth(-\frac{1}{10}x)]^2 - \hbar\varepsilon\sin\{\frac{1}{50}[1-\coth(-\frac{1}{10}x)]^2\}t \\ &\quad - \frac{3}{3125}\hbar t \csc h^2(\frac{1}{10}x)[1+\coth(\frac{1}{10}x)] + u_2 + \dots . \end{aligned}$$

The $\hbar$-curves of $u''_{\rm appr}(0,0)$ and $u'''_{\rm appr}(0,0)$ to equation (4.3) are shown in Figure 3(a). A comparison between the initial exact solution and the approximate solution of the fourth order are shown in Figure 3(b).

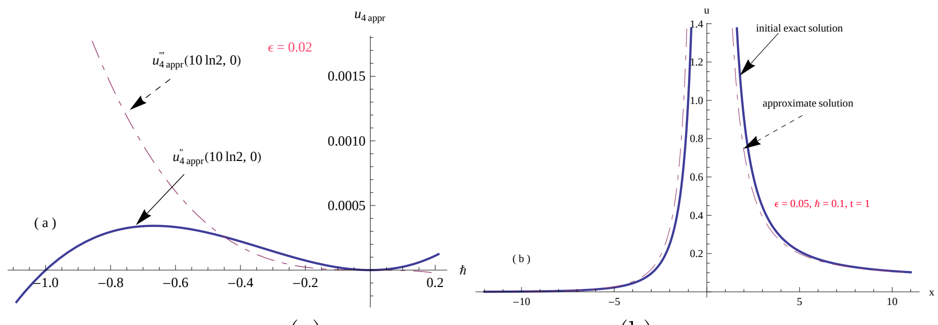
Example 4.3. Consider the Burgers-Fisher equation

$$u_t + u^2 u_x - u_{xx} = \varepsilon u(1-u^2), \quad 0 < \varepsilon \le 1, \tag{4.5}$$

with the initial exact solution and the exact solution

$$\tilde{u}_0(x,t) = \sqrt{\frac{1}{2} - \frac{1}{2}\tanh[\frac{1}{3}x - \frac{1}{9}t + \xi_0]}, \tag{4.6}$$

$$u_{\rm exact} = \sqrt{\frac{1}{2} - \frac{1}{2}\tanh[\frac{1}{3}x - \frac{1+9\varepsilon}{9}t + \xi_0]}. \tag{4.7}$$



(a) (b)

FIGURE 3. (a) The $\hbar$-curves of $u''_{\rm appr}(10\ln 2,0)$ and $u'''_{\rm appr}(10\ln 2,0)$ at the 4th order of approximation. (b) Comparison between the curves of initial exact solution and the fourth order of approximate solution.

Following the process above, we have

$$u_0=\sqrt{\frac12-\frac12\tanh(\frac13x)},\quad \tilde c_0(x)=\sec h^2(\frac13x)/18\sqrt{2-2\tanh(\frac13x)},$$

$$u_1=-\frac{\hbar t\sec h^2(\frac13x)}{18\sqrt{2-2\tanh(\frac13x)}}-\hbar t\varepsilon\sqrt{\frac12-\frac12\tanh(\frac13x)}(\frac12+\frac12\tanh(\frac13x)),$$

$$\dots$$

$$\begin{aligned} u_{\rm appr}&=\sqrt{\frac12-\frac12\tanh(\frac13x)}-\frac{\hbar t\sec h^2(\frac13x)}{18\sqrt{2-2\tanh(\frac13x)}}\\ &\quad-\hbar t\varepsilon\sqrt{\frac12-\frac12\tanh(\frac13x)}(\frac12+\frac12\tanh(\frac13x))+u_2+\dots\end{aligned}$$

The $\hbar$-curves of $u''_{\rm appr}(0,0)$ and $u'''_{\rm appr}(0,0)$ to equation (4.5) are shown in Figure 4(a). A comparison between the initial exact solution and the approximate solution of the fourth order is shown in Figure 4(b).

Conclusion. In this work, the HAM has been applied to find the approximate solutions of the general perturbed KdV-Burgers equation. Numerical simulations show that, compared to HPM, this method provides us more accuracy and reductions in the size of calculations. In addition, the results of the HPM can be obtained as a special case of the HAM when $\hbar=-1$. The parameter $\hbar$ provides us with a simpler way to adjust and control the convergence region of solution series for large values of t. It was shown that the HAM is a very powerful and efficient technique for solving various kinds of nonlinear systems in science and engineering without any assumptions and restrictions, and the auxiliary parameter $\hbar$ plays a critical role within the frame of the HAM which can be determined by the $\hbar$-curves.

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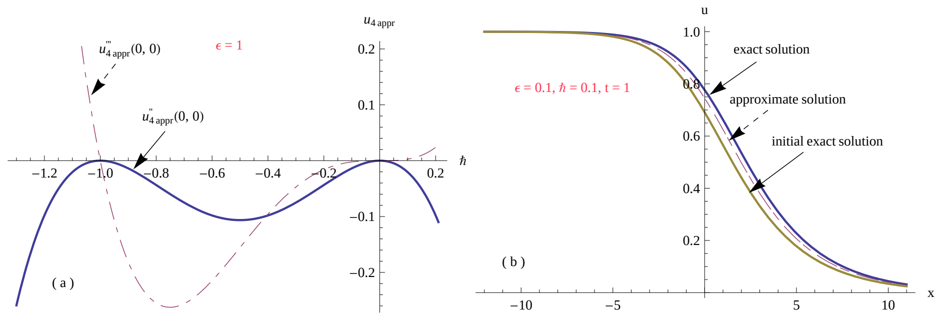

(a) (b)

FIGURE 4. (a) The $\hbar$-curves of $u''_{\rm appr}(0,0)$ and $u'''_{\rm appr}(0,0)$ at the 4th order of approximation. (b) Comparison between the curves of initial exact solution, exact solution and the fourth order of approximate solution.

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Baojian Hong

Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China.

Department of Basic Courses, Nanjing Institute of Technology, Nanjing 211167, China

E-mail address: `hbj@njit.edu.cn`

Dianchen Lu

Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China

E-mail address: `dclu@ujs.edu.cn, Tel +86 13815158840, fax +86 (511) 88791128`