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EXISTENCE OF SOLUTIONS TO FRACTIONAL-ORDER IMPULSIVE HYPERBOLIC PARTIAL DIFFERENTIAL INCLUSIONS

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Abstract. In this article we use the upper and lower solution method combined with a fixed point theorem for condensing multivalued maps, due to Martelli, to study the existence of solutions to impulsive partial hyperbolic differential inclusions at fixed instants of impulse.

1. Introduction

The theory of differential equations and inclusions of fractional order play a very important role in describing some real world problems. For example some problems in physics, mechanics, viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [16, 31]). Recently, numerous research papers and monographs have appeared devoted to fractional differential equations, for example see the monographs of Abbas et al [7], Kilbas et al [22], Lakshmikantham et al [24], and Malinowska and Torres [28], and the papers of Abbas and Benchohra [2, 5], Abbas et al [1, 6], Belarbi et al [8], Benchohra and Ntouyas [10], Kilbas et al [20], Kilbas and Marzan [21], Semenchuk [32],Vityuk and Golushkov [34], and the references therein.

The method of upper and lower solutions has been successfully applied to study the existence of solutions for fractional order ordinary and partial partial differential equations and inclusions. See the monographs by Benchohra et al [9], Heikkila and Lakshmikantham [15], Ladde et al [26], the papers of Abbas and Benchohra [3, 4], Benchohra and Ntouyas [10] and the references therein.

This article deals with the existence of solutions to impulsive fractional order initial value problems (IVP for short), for the system

$$({}^cD^r_{\theta_k}u)(x,y) \in F(x,y,u(x,y)), \quad \text{if } (x,y)\in J_k;\ k=0,\dots,m; \tag{1.1}$$

$$u(x_k^+,y) = u(x_k^-,y) + I_k(u(x_k^-,y)), \quad \text{if } y\in[0,b],\ k=1,\dots,m; \tag{1.2}$$

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$$
\begin{cases} u(x,0) = \varphi(x), & x \in [0,a], \\ u(0,y) = \psi(y), & y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{1.3}
$$

where $J_0 = [0, x_1] \times [0, b]$, $J_k := (x_k, x_{k+1}] \times [0, b]$, $k = 1, \dots, m$, $\theta_k = (x_k, 0)$, $k = 0, \dots, m$, $a, b > 0$, $\theta = (0, 0)$, ${}^cD^r_\theta$ is the fractional caputo derivative of order $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, $0 = x_0 < x_1 < \dots < x_m < x_{m+1} = a$, $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a compact valued multivalued map, $\mathcal{P}(\mathbb{R}^n)$ is the family of all subsets of $\mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \dots, m$ are given functions, $\varphi : [0, a] \to \mathbb{R}^n$, $\psi : [0, b] \to \mathbb{R}^n$ are given absolutely continuous functions. Here $u(x_k^+, y)$ and $u(x_k^-, y)$ denote the right and left limits of $u(x, y)$ at $x = x_k$, respectively.

In this article, we provide sufficient conditions for the existence of solutions for the problem (1.1)-(1.3). Our approach is based on the existence of upper and lower solutions and on a fixed point theorem for condensing multivalued maps, due to Martelli [29]. The present results extend those considered with integer order derivative [9, 11, 18, 19, 25, 30] and those with fractional derivative and without impulses [21].

2. Preliminaries

In this section, we introduce notation and preliminary facts which are used throughout this paper. By $C(J)$ we denote the Banach space of all continuous functions from J to $\mathbb{R}^n$ with the norm

$$
\|w\|_\infty = \sup_{(x,y)\in J} \|w(x, y)\|,
$$

where $\|\cdot\|$ denotes a suitable norm on $\mathbb{R}^n$. As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}^n$ and $L^1(J)$ is the space of Lebesgue-integrable functions $w : J \to \mathbb{R}^n$ with the norm

$$
\|w\|_1 = \int_0^a \int_0^b \|w(x, y)\| \, dy \, dx.
$$

Definition 2.1 ([34]). Let $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, $\theta = (0, 0)$ and $u \in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined as

$$
(I^r_\theta u)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1-1}(y - t)^{r_2-1} u(s, t) \, dt \, ds.
$$

In particular,

$$
(I^\theta_\theta u)(x, y) = u(x, y),\ (I^\sigma_\theta u)(x, y) = \int_0^x \int_0^y u(s, t) \, dt \, ds;\ \text{for almost all } (x, y) \in J,
$$

where $\sigma = (1, 1)$ For instance, $I^r_\theta u$ exists for all $r_1, r_2 \in (0, \infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I^r_\theta u) \in C(J)$, moreover

$$
(I^r_\theta u)(x, 0) = (I^r_\theta u)(0, y) = 0; \quad x \in [0, a],\ y \in [0, b].
$$

Example 2.2. Let $\lambda, \omega \in (-1, 0) \cup (0, \infty)$ and $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, then

$$
I^r_\theta x^\lambda y^\omega = \frac{\Gamma(1 + \lambda)\Gamma(1 + \omega)}{\Gamma(1 + \lambda + r_1)\Gamma(1 + \omega + r_2)} x^{\lambda+r_1} y^{\omega+r_2}, \quad \text{for almost all } (x, y) \in J.
$$

By $1 - r$ we mean $(1 - r_1, 1 - r_2) \in [0, 1) \times [0, 1)$. Denote by $D^2_{xy} := \frac{\partial^2}{\partial x \partial y}$, the mixed second order partial derivative.

Definition 2.3 ([34]). Let $r \in (0,1]\times(0,1]$ and $u \in L^1(J)$. The Caputo fractional-order derivative of order r of u is defined by the expression

$$\begin{aligned} {}^cD_\theta^r u(x,y) &= (I_\theta^{1-r} D_{xy}^2 u)(x,y) \\ &= \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} \int_0^x \int_0^y \frac{D_{st}^2 u(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}\,dt\,ds. \end{aligned}$$

The case $\sigma = (1,1)$ is included and we have

$$(D_\theta^\sigma u)(x,y) = ({}^cD_\theta^\sigma u)(x,y) = (D_{xy}^2 u)(x,y), \quad \text{for almost all } (x,y) \in J.$$

Example 2.4. Let $\lambda, \omega \in (-1,0)\cup(0,\infty)$ and $r=(r_1,r_2) \in (0,1]\times(0,1]$, then

$$D_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2}, \quad \text{for almost all } (x,y) \in J.$$

Let $a_1 \in [0,a]$, $z^+ = (a_1,0) \in J$, $J_z = [a_1,a]\times[0,b]$, $r_1, r_2 > 0$ and $r = (r_1,r_2)$. For $u \in L^1(J_z, \mathbb{R}^n)$, the expression

$$(I_{z^+}^r u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t)\,dt\,ds,$$

is called the left-sided mixed Riemann-Liouville integral of order r of u.

Definition 2.5 ([34]). For $u \in L^1(J_z, \mathbb{R}^n)$ where $D_{xy}^2 u$ is Lebesgue integrable on $[x_k, x_{k+1}]\times[0,b]$, $k = 0,\dots,m$, the Caputo fractional-order derivative of order r of u is defined by the expression $({}^cD_{z^+}^r f)(x,y) = (I_{z^+}^{1-r} D_{xy}^2 f)(x,y)$. The Riemann-Liouville fractional-order derivative of order r of u is defined by $(D_{z^+}^r f)(x,y) = (D_{xy}^2 I_{z^+}^{1-r} f)(x,y)$.

We need also some properties of set-valued Maps. Let $(X, \|\cdot\|)$ be a Banach space. Denote $\mathcal{P}(X) = \{Y \in X : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$.

Definition 2.6. A multivalued map $T : X \to \mathcal{P}(X)$ is convex (closed) valued if $T(x)$ is convex (closed) for all $x \in X$. T is bounded on bounded sets if $T(B) = \cup_{x\in B} T(x)$ is bounded in X for all $B \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in B}\sup_{y\in T(x)} \|y\| < \infty$). T is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $T(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $T(x_0)$, there exists an open neighborhood N_0 of x_0 such that $T(N_0) \subseteq N$. T is lower semi-continuous (l.s.c.) if the set $\{x \in X : T(x)\cap A \neq \emptyset\}$ is open for any open subset $A \subseteq X$. T is said to be completely continuous if $T(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(X)$. T has a fixed point if there is $x \in X$ such that $x \in T(x)$. The fixed point set of the multivalued operator T will be denoted by $FixT$. A multivalued map $G : X \to \mathcal{P}_{cl}(\mathbb{R}^n)$ is said to be measurable if for every $v \in \mathbb{R}^n$, the function $x \mapsto d(v, G(x)) = \inf\{\|v-z\| : z \in G(x)\}$ is measurable.

Lemma 2.7. [17] *Let G be a completely continuous multivalued map with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $u_n \to u$, $w_n \to w$, $w_n \in G(u_n)$ imply $w \in G(u)$).*

Definition 2.8. A multivalued map $F : J\times\mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be Carathéodory if

(i) $(x,y) \mapsto F(x,y,u)$ is measurable for each $u \in \mathbb{R}^n$;

(ii) $u \mapsto F(x, y, u)$ is upper semicontinuous for almost all $(x, y) \in J$.

F is said to be L^1-Carathéodory if (i), (ii) and the following condition holds;

(iii) for each $c > 0$, there exists $\sigma_c \in L^1(J, \mathbb{R}_+)$ such that

$$\begin{aligned}\|F(x,y,u)\|_{\mathcal{P}} &= \sup\{\|f\| : f \in F(x,y,u)\}\\ &\le \sigma_c(x,y) \quad \text{or all } \|u\| \le c \text{ and for a.e. } (x,y) \in J.\end{aligned}$$

For each $u \in C(J)$, define the set of selections of F by

$$S_{F,u} = \{w \in L^1(J) : w(x,y) \in F(x,y,u(x,y)) \text{ a.e. } (x,y) \in J\}.$$

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [23]). For more details on multi-valued maps we refer the reader to the books of Deimling [12], Gorniewicz [13], Graef et al [14], Hu and Papageorgiou [17] and Tolstonogov [33].

Lemma 2.9 ([27]). *Let X be a Banach space. Let $F : J \times X \to \mathcal{P}_{cp,cv}(X)$ be an L^1-Carathéodory multivalued map and let Λ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$, then the operator*

$$\begin{aligned}\Lambda \circ S_F : C(J,X) &\to \mathcal{P}_{cp,cv}(C(J,X)),\\ u &\mapsto (\Lambda \circ S_F)(u) := \Lambda(S_{F,u})\end{aligned}$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Lemma 2.10 ([29]). *(Martelli) Let X be a Banach space and $N : X \to \mathcal{P}_{cl,cv}(X)$ be an u. s. c. and condensing map. If the set $\Omega := \{u \in X : \lambda N(u) = N(u)$ for some $\lambda > 1\}$ is bounded, then N has a fixed point.*

3. Main Result

To define the solutions of problems (1.1)-(1.3), we shall consider the Banach space

$$\begin{aligned}PC = \big\{u : J \to \mathbb{R}^n : u \in C(J_k); k = 0,\dots,m, \text{ and there exist } u(x_k^-, y)\\ \text{and } u(x_k^+, y);\ y \in [0,b],\ k = 1,\dots,m, \text{ with } u(x_k^-, y) = u(x_k, y)\big\},\end{aligned}$$

with the norm

$$\|u\|_{PC} = \sup_{(x,y)\in J} \|u(x,y)\|.$$

Definition 3.1. A function $u \in PC \cap \cup_{k=0}^m AC(J_k)$ whose r-derivative exists on J_k is said to be a solution of (1.1)-(1.3) if there exists a function $f \in L^1(J)$ with $f(x,y) \in F(x,y,u(x,y))$ such that u satisfies $({}^cD^r_{\theta_k} u)(x,y) = f(x,y)$ on J_k, $k = 0, \dots m$ and conditions (1.2), (1.3) are satisfied.

Let $z, \bar{z} \in C(J)$ be such that

$$z(x,y) = (z_1(x,y), z_2(x,y), \dots, z_n(x,y)), \quad (x,y) \in J,$$

and

$$\bar{z}(x,y) = (\bar{z}_1(x,y), \bar{z}_2(x,y), \dots, \bar{z}_n(x,y)), \quad (x,y) \in J.$$

The notation $z \leq \bar{z}$ means that

$$z_i(x,y) \leq \bar{z}_i(x,y) \quad \text{for } i = 1,\dots,n.$$

Definition 3.2. *A function $z \in PC \cap \cup_{k=0}^{m} AC(J_k)$ is said to be a lower solution of* (1.1)-(1.3) *if there exists a function $f \in L^1(J)$ with $f(x,y) \in F(x,y,u(x,y))$ such that z satisfies*

$$\begin{gathered}
({}^cD^r_{\theta_k} z)(x,y) \leq f(x,y,z(x,y)), \quad \text{on } J_k;\\
z(x_k^+,y) \leq z(x_k^-,y) + I_k(z(x_k^-,y)), \quad \text{if } y \in [0,b],\ k = 1,\dots,m;\\
z(x,0) \leq \varphi(x),\ x \in [0,a];\\
z(0,y) \leq \psi(y), \quad y \in [0,b];\\
z(0,0) \leq \varphi(0).
\end{gathered}$$

The function z is said to be an upper solution of (1.1)-(1.3) *if the reversed inequalities hold.*

Let $h \in C(J_k)$, $k = 1,\dots,m$ and set

$$\mu(x,y) := \varphi(x) + \psi(y) - \varphi(0), \quad (x,y) \in J.$$

For the existence of solutions for problem (1.1)-(1.3), we need the following lemma.

Lemma 3.3 ([4]). *Let $r_1, r_2 \in (0,1]$ and let $h : J \to \mathbb{R}^n$ be continuous. A function u is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases}
\mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)\,dt\,ds;\\
\quad \text{if } (x,y) \in [0,x_1] \times [0,b],\\
\mu(x,y) + \sum_{i=1}^{k} (I_i(u(x_i^-,y)) - I_i(u(x_i^-,0)))\\
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)\,dt\,ds\\
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)\,dt\,ds;\\
\quad \text{if } (x,y) \in (x_k, x_{k+1}] \times [0,b],\ k = 1,\dots,m,
\end{cases}$$

if and only if u is a solution of the fractional IVP

$$\begin{gathered}
{}^cD^r u(x,y) = h(x,y), \quad (x,y) \in J_k,\\
u(x_k^+,y) = u(x_k^-,y) + I_k(u(x_k^-,y)), \quad y \in [0,b],\ k = 1,\dots,m.
\end{gathered}$$

To study problem (1.1)-(1.3), we first list the following hypotheses:

(H1) $F : J \times \mathbb{R}^n \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is L^1-Carathéodory;

(H2) There exist v and $w \in PC \cap AC(J_k)$, $k = 0,\dots,m$, lower and upper solutions for the problem (1.1)-(1.3) such that $v(x,y) \leq w(x,y)$ for each $(x,y) \in J$;

(H3) For each $y \in [0,b]$, we have

$$v(x_k^+,y) \leq \min_{u \in [v(x_k^-,y), w(x_k^-,y)]} I_k(u) \leq \max_{u \in [v(x_k^-,y), w(x_k^-,y)]} I_k(u) \leq w(x_k^+,y),$$

with $k = 1,\dots,m$.

Theorem 3.4. *Assume that hypotheses* (H1)-(H3) *hold. Then problem* (1.1)-(1.3) *has at least one solution u such that*

$$v(x,y) \leq u(x,y) \leq w(x,y), \quad \text{for all } (x,y) \in J.$$

Proof. We transform problem (1.1)-(1.3) into a fixed point problem. Consider the modified problem

$$({}^cD^r_{\theta_k}u)(x,y) \in F(x,y,g(u(x,y))), \quad \text{if } (x,y) \in J_k,\ k=0,\dots,m; \tag{3.1}$$

$$u(x_k^+,y) = u(x_k^-,y) + I_k(g(x_k^-,y,u(x_k^-,y))), \quad \text{if } y \in [0,b],\ k=1,\dots,m; \tag{3.2}$$

$$u(x,0) = \varphi(x), \quad x \in [0,a],\ u(0,y) = \psi(y)\,; y \in [0,b],\ \varphi(0) = \psi(0), \tag{3.3}$$

where $g : PC \to PC$ be the truncation operator defined by

$$(gu)(x,y) = \begin{cases} v(x,y), & u(x,y) < v(x,y), \\ u(x,y), & v(x,y) \le u(x,y) \le w(x,y), \\ w(x,y), & w(x,y) < u(x,y). \end{cases}$$

A solution to (3.1)-(3.3) is a fixed point of the operator $N : PC \to \mathcal{P}(PC)$ defined by

$$N(u) = \begin{cases} h \in PC : h(x,y) = \mu(x,y) \\ + \sum_{0<x_k<x} (I_k(g(x_k^-,y,u(x_k^-,y))) - I_k(g(x_k^-,0,u(x_k^-,0)))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)\,dt\,ds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)\,dt\,ds, \end{cases}$$

where

$$\begin{aligned} f &\in \tilde{S}^1_{F,g(u)} \\ &= \big\{ f \in S^1_{F,g(u)} : f(x,y) \ge f_1(x,y) \text{ on } A_1 \text{ and } f(x,y) \le f_2(x,y) \text{ on } A_2 \big\}, \\ A_1 &= \{(x,y) \in J : u(x,y) < v(x,y) \le w(x,y)\}, \\ A_2 &= \{(x,y) \in J : u(x,y) \le w(x,y) < u(x,y)\}, \\ S^1_{F,g(u)} &= \{ f \in L^1(J) : f(x,y) \in F(x,y,g(u(x,y))),\ \text{for } (x,y) \in J\}. \end{aligned}$$

□

Remark 3.5. (A) For each $u \in PC$, the set $\tilde{S}_{F,g(u)}$ is nonempty. In fact, $(H1)$ implies there exists $f_3 \in S_{F,g(u)}$, so we set

$$f = f_1\chi_{A_1} + f_2\chi_{A_2} + f_3\chi_{A_3},$$

where χ_{A_i} is the characteristic function of $A_i;\ i=1,2,3$ and

$$A_3 = \{(x,y) \in J : v(x,y) \le u(x,y) \le w(x,y)\}.$$

Then, by decomposability, $f \in \tilde{S}_{F,g(u)}$.

(B) By the definition of g it is clear that $F(.,.,g(u)(.,.))$ is an L^1-Carathéodory multi-valued map with compact convex values and there exists $\phi \in C(J,\mathbb{R}_+)$ such that

$$\|F(x,y,g(u(x,y)))\|_{\mathcal{P}} \le \phi(x,y);\ \text{for each } (x,y) \in J \text{and } u \in \mathbb{R}^n.$$

Set

$$\phi^* := \sup_{(x,y)\in J} \phi(x,y).$$

(C) By the definition of g and from (H3) we have

$$u(x_k^+,y) \le I_k(g(x_k,y,u(x_k,y))) \le w(x_k^+,y);\ y \in [0,b];\ k=1,\dots,m.$$

From Lemma 3.3 and the fact that $g(u) = u$ for all $v \le u \le w$, the problem of finding the solutions of the IVP (1.1)-(1.3) is reduced to finding the solutions of the operator equation $N(u) = u$. We shall show that N is a completely continuous multivalued map, u.s.c. with convex closed values. The proof will be given in several steps.

Step 1: $N(u)$ is convex for each $u \in PC$. If h_1, h_2 belong to $N(u)$, then there exist $f_1, f_2 \in \tilde{S}^1_{F,g(u)}$ such that for each $(x,y) \in J$ we have

$$\begin{aligned}(h_i u)(x,y) = \mu(x,y) &+ \sum_{0<x_k<x} (I_k(g(x_k^-, y, u(x_k^-, y))) - I_k(g(x_k^-, 0, u(x_k^-, 0)))) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} f_i(s,t)\,dt\,ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f_i(s,t)\,dt\,ds.\end{aligned}$$

Let $0 \le \xi \le 1$. Then, for each $(x,y) \in J$, we have

$$\begin{aligned}&(\xi h_1 + (1-\xi)h_2)(x,y) \\ &= \mu(x,y) + \sum_{0<x_k<x} (I_k(g(x_k^-, y, u(x_k^-, y)))) - \sum_{0<x_k<x} (I_k(g(x_k^-, 0, u(x_k^-, 0)))) \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} \\ &\quad \times [\xi f_1(s,t) + (1-\xi) f_2(s,t)]\,dt\,ds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} [\xi f_1(s,t) + (1-\xi) f_2(s,t)]\,dt\,ds.\end{aligned}$$

Since $\tilde{S}^1_{F,g(u)}$ is convex (because F has convex values), we have

$$\xi h_1 + (1-\xi)h_2 \in G(u).$$

Step 2: N sends bounded sets of PC into bounded sets. We can prove that $N(PC)$ is bounded. It is sufficient to show that there exists a positive constant ℓ such that for each $h \in N(u)$, $u \in PC$ one has $\|h\|_\infty \le \ell$. If $h \in N(u)$, then there exists $f \in \tilde{S}^1_{F,g(u)}$ such that for each $(x,y) \in J$ we have

$$\begin{aligned}(hu)(x,y) = \mu(x,y) &+ \sum_{0<x_k<x} (I_k(g(x_k^-, y, u(x_k^-, y))) - I_k(g(x_k^-, 0, u(x_k^-, 0)))) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} f(s,t)\,dt\,ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)\,dt\,ds.\end{aligned}$$

Then, for each $(x,y) \in J$ we get

$$\|(hu)(x,y)\| = \|\mu(x,y)\| + 2\sum_{k=1}^{m} \max_{y\in[0,b]} (\|v(x_k^+, y)\|, \|w(x_k^+, y)\|)$$

$$
\begin{aligned}
&+ \frac{\phi^*}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1}\,dt\,ds \\
&+ \frac{\phi^*}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\,dt\,ds.
\end{aligned}
$$

Thus,

$$
\|u\|_\infty \le \|\mu\|_\infty + 2\sum_{k=1}^m \max_{y\in[0,b]}(\|v(x_k^+,y)\|,\|w(x_k^+,y)\|) + \frac{2a^{r_1}b^{r_2}\phi^*}{\Gamma(r_1+1)\Gamma(r_2+1)} := \ell.
$$

Step 3: N sends bounded sets of PC into equicontinuous sets. Let (τ_1,y_1), $(\tau_2,y_2)\in J$, $\tau_1<\tau_2$, $y_1<y_2$ and $B_\rho=\{u\in PC:\|u\|_\infty\le\rho\}$ be a bonded set of PC. For each $u\in B_\rho$ and $h\in N(u)$, there exists $f\in\tilde{S}^1_{F,g(u)}$ such that for each $(x,y)\in J$ we have

$$
\begin{aligned}
&\|(hu)(\tau_2,y_2)-h(u)(\tau_1,y_1)\| \\
&\le \|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)\| + \sum_{k=1}^m (\|I_k(g(x_k^-,y_1,u(x_k^-,y_1))) \\
&\quad - I_k(g(x_k^-,y_2,u(x_k^-,y_2)))\|) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_0^{y_1} (x_k-s)^{r_1-1}[(y_2-t)^{r_2-1}-(y_1-t)^{r_2-1}] \\
&\quad \times \|f(s,t)\|\,dt\,ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_{y_1}^{y_2} (x_k-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t)\|\,dt\,ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\tau_1} \int_0^{y_1} [(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}-(\tau_1-s)^{r_1-1}(y_1-t)^{r_2-1}] \\
&\quad \times \|f(s,t)\|\,dt\,ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\tau_1}^{\tau_2} \int_{y_1}^{y_2} (\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t)\|\,dt\,ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\tau_1} \int_{y_1}^{y_2} (\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t)\|\,dt\,ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\tau_1}^{\tau_2} \int_0^{y_1} (\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t)\|\,dt\,ds \\
&\le \|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)\| \\
&\quad + \sum_{k=1}^m (\|I_k(g(x_k^-,y_1,u(x_k^-,y_1))) - I_k(g(x_k^-,y_2,u(x_k^-,y_2)))\|) \\
&\quad + \frac{\phi^*}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_0^{y_1} (x_k-s)^{r_1-1}[(y_2-t)^{r_2-1}-(y_1-t)^{r_2-1}]\,dt\,ds \\
&\quad + \frac{\phi^*}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_{y_1}^{y_2} (x_k-s)^{r_1-1}(y_2-t)^{r_2-1}\,dt\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\phi^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}-(\tau_1-s)^{r_1-1}(y_1-t)^{r_2-1}]\,dt\,ds\\
&+\frac{\phi^*}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,dt\,ds\\
&+\frac{\phi^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,dt\,ds\\
&+\frac{\phi^*}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_0^{y_1}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,dt\,ds.
\end{aligned}
$$

As $\tau_1 \to \tau_2$ and $y_1 \to y_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that N is completely continuous and therefore a condensing multivalued map.

Step 4: N has a closed graph. Let $u_n \to u_*$, $h_n \in N(u_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(u_*)$. $h_n \in N(u_n)$ means that there exists $f_n \in \tilde{S}^1_{F,g(u_n)}$ such that, for each $(x,y) \in J$, we have

$$
\begin{aligned}
h_n(x,y)=\mu(x,y)&+\sum_{0<x_k<x}(I_k(g(x_k^-,y,u_n(x_k^-,y)))-I_k(g(x_k^-,0,u_n(x_k^-,0))))\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{0<x_k<x}\int_{x_{k-1}}^{x_k}\int_0^y(x_k-s)^{r_1-1}(y-t)^{r_2-1}f_n(s,t)\,dt\,ds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f_n(s,t)\,dt\,ds.
\end{aligned}
$$

We must show that there exists $f_* \in \tilde{S}^1_{F,g(u_*)}$ such that, for each $(x,y) \in J$,

$$
\begin{aligned}
h_*(x,y)=\mu(x,y)&+\sum_{0<x_k<x}(I_k(g(x_k^-,y,u_*(x_k^-,y)))-I_k(g(x_k^-,0,u_*(x_k^-,0))))\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{0<x_k<x}\int_{x_{k-1}}^{x_k}\int_0^y(x_k-s)^{r_1-1}(y-t)^{r_2-1}f_*(s,t)\,dt\,ds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f_*(s,t)\,dt\,ds.
\end{aligned}
$$

Now, we consider the linear continuous operator $\Lambda : L^1(J) \to C(J)$ defined by $f \mapsto \Lambda(f)(x,y)$,

$$
\begin{aligned}
(\Lambda f)(x,y)=&\sum_{0<x_k<x}(I_k(g(x_k^-,y,u(x_k^-,y)))-I_k(g(x_k^-,0,u(x_k^-,0))))\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{0<x_k<x}\int_{x_{k-1}}^{x_k}\int_0^y(x_k-s)^{r_1-1}(y-t)^{r_2-1}f(s,t)\,dt\,ds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t)\,dt\,ds.
\end{aligned}
$$

From Lemma 2.9, it follows that $\Lambda \circ \tilde{S}^1_F$ is a closed graph operator. Clearly we have

$$
\Big\|\Big[h_n(x,y)-\mu(x,y)-\sum_{0<x_k<x}(I_k(g(x_k^-,y,u_n(x_k^-,y)))-I_k(g(x_k^-,0,u_n(x_k^-,0))))\Big]
$$

$$- \Big[h_*(x,y) - \mu(x,y) - \sum_{0<x_k<x} (I_k(g(x_k^-, y, u_*(x_k^-, y))) - I_k(g(x_k^-, 0, u_*(x_k^-, 0))))\Big]\Big\|$$
$$\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{x_1<x_k<x} \int_{x_{k-1}}^{x_k}\int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1}\|f_n(s,t) - f_*(s,t)\|\,dt\,ds$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x}\int_0^y (x - s)^{r_1-1}(y-t)^{r_2-1}\|f_n(s,t) - f_*(s,t)\|\,dt\,ds \to 0,$$

as $n \to \infty$. Moreover, from the definition of Λ, we have

$$\Big[h_n(x,y) - \mu(x,y) - \sum_{0<x_k<x} (I_k(g(x_k^-, y, u_n(x_k^-, y))) - I_k(g(x_k^-, 0, u_n(x_k^-, 0))))\Big]$$
$$\in \Lambda(\tilde{S}^1_{F,g(u_n)}).$$

Since $u_n \to u_*$, it follows from Lemma 2.9 that, for some $f_* \in \Lambda(\tilde{S}^1_{F,g(u_*)})$, we have

$$h_*(x,y)_\mu(x,y) - \sum_{0<x_k<x} (I_k(g(x_k^-, y, u_*(x_k^-, y))) - I_k(g(x_k^-, 0, u_*(x_k^-, 0))))$$
$$= \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k}\int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} f_*(s,t)\,dt\,ds$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x}\int_0^y (x - s)^{r_1-1}(y-t)^{r_2-1} f_*(s,t)\,dt\,ds, \quad (x,y) \in J.$$

From Lemma 2.7, we can conclude that N is u.s.c.

Step 5: The set $\Omega = \{u \in PC : \lambda u = N(u) \text{ for some } \lambda > 1\}$ in bounded. Let $u \in \Omega$. Then, there exists $f \in \Lambda(\tilde{S}^1_{F,g(u)})$, such that

$$\lambda u(x,y) = \mu(x,y) + \sum_{0<x_k<x} (I_k(g(x_k^-, y, u(x_k^-, y))) - I_k(g(x_k^-, 0, u(x_k^-, 0))))$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k}\int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} f(s,t)\,dt\,ds$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x}\int_0^y (x - s)^{r_1-1}(y-t)^{r_2-1} f(s,t)\,dt\,ds.$$

As in Step 2, this implies that for each $(x,y) \in J$, we have

$$\|u\|_\infty \leq \|\mu\|_\infty + 2\sum_{k=1}^{m} \max_{y\in[0,b]}(\|v(x_k^+, y)\|, \|w(x_k^+, y)\|) + \frac{2a^{r_1}b^{r_2}\phi^*}{\Gamma(r_1+1)\Gamma(r_2+1)} = \ell.$$

This shows that Ω is bounded. As a consequence of Lemma 2.10, we deduce that N has a fixed point which is a solution of (3.1)-(3.3) on J.

Step 6: The solution u of (3.1)-(3.3) satisfies

$$v(x,y) \leq u(x,y) \leq w(x,y), \quad \text{for all } (x,y) \in J.$$

Let u be the above solution to (3.1)-(3.3). We prove that

$$u(x,y) \leq w(x,y) \quad \text{for all } (x,y) \in J.$$

Assume that $u - w$ attains a positive maximum on $[x_k^+, x_{k+1}^-] \times [0,b]$ at $(\overline{x}_k, \overline{y}) \in [x_k^+, x_{k+1}^-] \times [0,b]$, for some $k = 0, \dots, m$; that is,

$$(u - w)(\overline{x}_k, \overline{y}) = \max\{u(x,y) - w(x,y) : (x,y) \in [x_k^+, x_{k+1}^-] \times [0,b]\} > 0,$$

for some $k = 0, \dots, m$. We distinguish the following cases.

Case 1. If $(\overline{x}_k, \overline{y}) \in (x_k^+, x_{k+1}^-) \times [0, b]$ there exists $(x_k^*, y^*) \in (x_k^+, x_{k+1}^-) \times [0, b]$ such that

$$\begin{aligned} &[u(x, y^*) - w(x, y^*)] + [u(x_k^*, y) - w(x_k^*, y)] - [u(x_k^*, y^*) - w(x_k^*, y^*)] \\ &\leq 0, \quad \text{for all } (x, y) \in ([x_k^*, \overline{x}_k] \times \{y^*\}) \cup (\{x_k^*\} \times [y^*, b]), \end{aligned} \tag{3.4}$$

and

$$u(x, y) - w(x, y) > 0, \quad \text{for all } (x, y) \in (x_k^*, \overline{x}_k] \times (y^*, b]. \tag{3.5}$$

By the definition of g, one has

$${}^cD_\theta^r u(x, y) \in F(x, y, w(x, y)), \quad \text{for all } (x, y) \in [x_k^*, \overline{x}_k] \times [y^*, b]. \tag{3.6}$$

An integration of (3.6), on $[x_k^*, x] \times [y^*, y]$ for each $(x, y) \in [x_k^*, \overline{x}_k] \times [y^*, b]$, yields

$$\begin{aligned} &u(x, y) + u(x_k^*, y^*) - u(x, y^*) - u(x_k^*, y) \\ &= \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k^*}^{x} \int_{y^*}^{y} (x - s)^{r_1 - 1} (y - t)^{r_2 - 1} f(s, t) \, dt \, ds, \end{aligned} \tag{3.7}$$

where $f(x, y) \in F(x, y, w(x, y))$. From (3.7) and using the fact that w is an upper solution to (1.1)-(1.3) we get

$$u(x, y) + u(x_k^*, y^*) - u(x, y^*) - u(x_k^*, y) \leq w(x, y) + w(x_k^*, y^*) - w(x, y^*) - w(x_k^*, y),$$

which gives

$$\begin{aligned} &u(x, y) - w(x, y) \\ &\leq [u(x, y^*) - w(x, y^*)] + [u(x_k^*, y) - w(x_k^*, y)] - [u(x_k^*, y^*) - w(x_k^*, y^*)]. \end{aligned} \tag{3.8}$$

Thus from (3.4), (3.5) and (3.8) we obtain the contradiction

$$\begin{aligned} 0 < [u(x, y) - w(x, y)] &\leq [u(x, y^*) - w(x, y^*)] + [u(x_k^*, y) - w(x_k^*, y)] \\ &- [u(x_k^*, y^*) - w(x_k^*, y^*)] \leq 0, \quad \text{for all } (x, y) \in [x_k^*, \overline{x}_k] \times [y^*, b]. \end{aligned}$$

Case 2. If $\overline{x}_k = x_k^+, \ k = 1, \dots, m$, then

$$w(x_k^+, \overline{y}) < I_k(g(x_k^-, u(x_k^-, \overline{y}))) \leq w(x_k^+, \overline{y}),$$

which is a contradiction. Thus

$$u(x, y) \leq w(x, y), \quad \text{for all } (x, y) \in J.$$

Analogously, we can prove that

$$u(x, y) \geq v(x, y), \quad \text{for all } (x, y) \in J.$$

This shows that problem (3.1)-(3.3) has a solution u satisfying $v \leq u \leq w$ which is solution of (1.1)-(1.3).

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