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ENTROPY SOLUTIONS FOR NONLINEAR DEGENERATE ELLIPTIC-PARABOLIC-HYPERBOLIC PROBLEMS

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ABSTRACT. We consider the nonlinear degenerate elliptic-parabolic-hyperbolic equation

$$\partial_t g(u) - \Delta b(u) - \operatorname{div} \Phi(u) = f(g(u)) \quad \text{in } (0,T) \times \Omega,$$

where g and b are nondecreasing continuous functions, Φ is vectorial and continuous, and f is Lipschitz continuous. We prove the existence, comparison and uniqueness of entropy solutions for the associated initial-boundary-value problem where Ω is a bounded domain in $\mathbb{R}^N$. For the associated initial-value problem where $\Omega = \mathbb{R}^N$, $N \geq 3$, the existence of entropy solutions is proved. Moreover, for the case when $\Phi \circ g^{-1}$ is locally Hölder continuous of order $1 - 1/N$, and $|b(u)| \leq \omega(|g(u)|)$, where ω is nondecreasing continuous with $\omega(0) = 0$, we can prove the L^1-contraction principle, and hence the uniqueness.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, and assume that Ω has a Lipschitz boundary Γ for $N \geq 2$. Consider the initial-boundary-value problem

$$\begin{gathered} \partial_t g(u) - \Delta b(u) + \operatorname{div} \Phi(u) = f(g(u)) \quad \text{in } (0,T) \times \Omega, \\ g(u) = g_0 \quad \text{on } \{0\} \times \Omega, \\ b(u) = 0 \quad \text{on } (0,T) \times \Gamma, \end{gathered} \tag{1.1}$$

and the initial-value problem

$$\begin{gathered} \partial_t g(u) - \Delta b(u) + \operatorname{div} \Phi(u) = f(g(u)) \quad \text{in } (0,T) \times \mathbb{R}^N, \\ g(u) = g_0 \quad \text{on } \{0\} \times \mathbb{R}^N, \end{gathered} \tag{1.2}$$

where $g, b : \mathbb{R} \to \mathbb{R}$ are nondecreasing continuous with $g(0) = b(0) = 0$, $\Phi = (\phi_1, \dots, \phi_N) : \mathbb{R} \to \mathbb{R}^N$ is continuous, and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with $f(0) = 0$.

Uniqueness is not necessarily true for weak solutions, for the problem is nonlinear degenerate. To single out the correct physical solution satisfying some special conditions, many researchers have worked on this problem from 1950s. For example, Oleinik [13], Vol'pert and Hudjaev [15], Kruzkov([9, 10]) and Carrillo [4]

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investigated the problem and established the existence and uniqueness for a class of generalized (entropy) solutions.

In the hyperbolic case $b' \equiv 0$, people are interested in the Cauchy problem associated with (1.1). The works began with a simple case that $g = I$, and $f \equiv 0$ or f is independent of u. Oleinik [13] established uniqueness for generalized solutions in the class of piecewise continuous functions satisfying condition "E" for $N = 1$. In the multi-dimensional case, Kruzkov ([9, 10]) introduced a class of generalized (entropy) solutions, and proved existence and uniqueness. In addition, Kruzkov and other authors ([2, 7, 8]) proved uniqueness of entropy solutions in the case that Φ satisfies some Osgood's type conditions or local Hölder continuity condition of order $\alpha = 1 - \frac{1}{N}$, and gave an counter-example in [7] to show that the local Hölder continuity condition is sharp in a definite case. Assuming that f is continuous in u, SU [14] studied on the problem in one-dimensional space and proved comparison principle of entropy solutions.

In the mixed case that $b' \geq 0$ and $g' \geq 0$, entropy solution was introduced inspired by Vol'pert and Hudjaev [15], and Kruzkov [10], which were researched individually and published in 1969 and 1970 respectively. For the initial-boundary-value problem (1.1) in the case that f is a function dependent only on x, a famous result was given by Carrillo [4] in 1999, in which the author introduced the Kruzkov entropy solution, and proved existence, comparison and hence uniqueness under the conditions that Φ is in a class of continuous vectorial functions dependent on g and b. For the Cauchy problem (1.2) in one-dimensional space, it was studied in many articles. Vol'pert and Hudjaev [15] introduced a well-defined generalized solution for $g = I$ in 1969. For general case $g \neq I$, it is more difficult to solve the problem. In 2007, Liu and Wang [11] considered the problem in the case that $b \in C^1(\mathbb{R})$ is strictly increasing, and by using Holmgren's approach, they established the uniqueness of entropy solutions under some conditions on the growth of Φ. For the problem in multi-dimensional space, many researchers investigated the problem in the case that $g = I$ in the past few years. Karlsen and Risebro [6] established uniqueness under the conditions that b, ϕ, and f are locally Lipschitz continuous. Based on the results of Carrillo [4], Maliki and Touré [12] proved existence of entropy solution, and uniqueness was also established under some assumptions on the continuity of b and Φ, which was motivated by those in [2]. Under the condition that Φ is locally Hölder continuous of order $\alpha = 1 - \frac{1}{N}$, Andreianov and Maliki [1] established the uniqueness of entropy solutions. Golovaty and Nguyen [5] worked on the problem under the conditions that $N = 1$ and $b = \lambda I$, where λ is a nonnegative constant, and they obtained existence, L^1-contraction principle and hence uniqueness of entropy solutions.

This article is motivated by the results on [1, 2, 4, 12]. Applying the results of Carrillo [4], we obtain existence of entropy solutions for (1.1) by using the contraction mapping principle. Then we get comparison and uniqueness by arguing similarly in [4]. Using the results for (1.1), where $\Omega = B_n \subset \mathbb{R}^N$, we prove existence of entropy solution for (1.2) by passing the limit $n \to +\infty$. And inspired by the work of Andreianov and Maliki [1], we establish comparison principle for the case that $\Phi \circ g^{-1}$ is locally Hölder continuous of order $1 - 1/N$, and $|b(u)| \leq \omega(|g(u)|)$, where ω is nondecreasing continuous with $\omega(0) = 0$. And consequently, we obtain uniqueness.

2. Entropy solutions of the initial-boundary-value problem

In this section, let Ω be a bounded domain in $\mathbb{R}^N$, and assume that Ω has a Lipschitz boundary Γ for $N \geq 2$. Set $T > 0$ and denote $Q = (0,T) \times \Omega$. We consider the nonlinear degenerate problem

$$\begin{gathered}\partial_t g(u) - \Delta b(u) + \operatorname{div} \Phi(u) = f(g(u)) \quad \text{in } (0,T) \times \Omega,\\ g(u) = g_0 \quad \text{on } \{0\} \times \Omega,\\ b(u) = 0 \quad \text{on } (0,T) \times \Gamma,\end{gathered} \tag{2.1}$$

where $g, b : \mathbb{R} \to \mathbb{R}$ are nondecreasing continuous, $\Phi = (\phi_1, \dots, \phi_N) : \mathbb{R} \to \mathbb{R}^N$ is continuous, $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, and $g_0(x)$ is given.

2.1. Preliminaries.

To introduce the definition of entropy solution, we give some notation. Let γ be a maximal monotone operator, which may be multi-valued. The main section of γ, denoted by γ_0, is defined as follows:

$$\gamma_0(s) = \begin{cases} \min\{|y|; y \in \gamma(s)\}, & \text{if } \gamma(s) \neq \emptyset,\\ +\infty, & \text{if } [s, +\infty) \cap D(\gamma) = \emptyset,\\ -\infty, & \text{if } (-\infty, s] \cap D(\gamma) = \emptyset. \end{cases}$$

For any continuous and non-decreasing or non-increasing function ψ, we define

$$B_\psi(s) = \begin{cases} \int_0^s \psi(b \circ (g^{-1})_0(r)) dr & \text{if } s \in \overline{(\psi \circ b) \circ g^{-1}},\\ +\infty & \text{otherwise}, \end{cases}$$

which is proper lower semi-continuous and convex, or upper semi-continuous and concave, and we have $(\psi \circ b) \circ g^{-1} \subset \partial B_\psi$.

In this paper, H is a set-valued operator:

$$H(s) = \begin{cases} 1 & \text{if } s > 0,\\ [0,1] & \text{if } s = 0,\\ -1 & \text{if } s < 0. \end{cases}$$

The functions H_ϵ, H_0, $H_{\max}$ are defined as follows:

$$H_\epsilon(s) = \min(s^+/\epsilon, 1), \quad H_0(s) = \begin{cases} 1 & \text{if } s > 0,\\ 0 & \text{if } s \leq 0, \end{cases} \quad H_{\max}(s) = \begin{cases} 1 & \text{if } s \geq 0,\\ 0 & \text{if } s < 0. \end{cases}$$

Definition 2.1. Let $g_0 \in L^1(\Omega)$.

(1) A measurable function u is called a weak subsolution (supersolution) of (2.1), if

$$\begin{gathered} g(u) \in L^1(Q), \quad \partial_t g(u) \in L^2(0,T; H^{-1}(\Omega)),\\ b(u) \in L^2(0,T; H_0^1(\Omega)), \quad \Phi(u) \in (L^2(Q))^N,\\ \partial_t g(u) - \Delta b(u) + \operatorname{div}(\Phi(u)) \leq (\ \geq\ )\ f(g(u)) \quad \text{in } \mathcal{D}'(Q),\\ g(u(0,x)) \leq (\ \geq\ )\ g_0(x) \quad \text{a.e. in } \Omega. \end{gathered}$$

(2) A measurable function u is called a weak solution of (2.1) if it is both a weak subsolution and a weak supersolution.

Definition 2.2. Let $g_0 \in L^1(\Omega)$.

(1) A weak subsolution u is called an entropy subsolution of (2.1), if

$$\begin{aligned} &\int_Q H_0(u-s)\{(\nabla b(u) - \Phi(u) + \Phi(s)) \cdot \nabla\xi - (g(u) - g(s))\partial_t \xi \\ &- f(g(u))\xi\}\,dx\,dt - \int_\Omega (g_0 - g(s))^+ \xi(0) dx \le 0 \end{aligned}$$

for any $(s,\xi) \in \mathbb{R} \times (L^2(0,T;H_0^1(\Omega)) \cap W^{1,1}(0,T;L^\infty(\Omega)))$ with $\xi \ge 0$ and $\xi(T) = 0$, and for any $(s,\xi) \in \mathbb{R} \times (L^2(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^\infty(\Omega)))$ with $s \ge 0$, $\xi \ge 0$ and $\xi(T) = 0$.

(2) A weak supersolution u is called an entropy supersolution of (2.1) if

$$\begin{aligned} &\int_Q H_0(-s-u)\{(\nabla b(u) - \Phi(u) + \Phi(-s)) \cdot \nabla\xi - (g(u) - g(-s))\partial_t \xi \\ &- f(g(u))\xi\}\,dx\,dt + \int_\Omega (g_0 - g(-s))^- \xi(0) dx \ge 0 \end{aligned}$$

for any $(s,\xi) \in \mathbb{R} \times (L^2(0,T;H_0^1(\Omega)) \cap W^{1,1}(0,T;L^\infty(\Omega)))$ with $\xi \ge 0$ and $\xi(T) = 0$, and for any $(s,\xi) \in \mathbb{R} \times (L^2(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^\infty(\Omega)))$ with $s \ge 0$, $\xi \ge 0$ and $\xi(T) = 0$.

(3) u is called an entropy solution of (2.1) if it is both an entropy subsolution and an entropy supersolution.

For proving the existence and uniqueness of entropy solutions to (2.1), we need some of the following assumptions.

- (H1) g, $b : \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing with $g(0) = b(0) = 0$;
- (H2) $\Phi = (\phi_1, \dots, \phi_N) \in \mathcal{C}(\mathbb{R}; \mathbb{R}^N)$, $\phi_j(0) = 0$, $\forall\, 1 \le j \le N$;
- (H3) $D((g+b)^{-1}) = \mathbb{R}$;
- (H4) there exist $\Phi^{(1)}, \Phi^{(2)} \in \mathcal{C}(\mathbb{R}; \mathbb{R}^N)$ with $\phi_j^{(1)}(0) = 0$, $1 \le j \le N$, such that
$$\Phi(s) = \Phi^{(1)}(g(s)) + \Phi^{(2)}(g(s))b(s), \quad \forall s \in \mathbb{R};$$
- (H5) $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant L, and $f(0) = 0$.
- (H6) Assume that there is an integer i_0 with $1 \le i_0 \le N$, for which at least one of the following holds:
  - (1) $g(s) = g(r)$ implies $\phi_{i_0}(s) = \phi_{i_0}(r)$ for all $s, r \in \mathbb{R}$;
  - (2) ϕ_{i_0} is monotone, that is, nondecreasing or nonincreasing;
  - (3) There exists a constant $C > 0$ such that
$$|\phi_{i_0}(s) - \phi_{i_0}(r)| \le C|g(s) + b(s) - g(r) - b(r)| \quad \forall s, r \in \mathbb{R}.$$

Assumption (H4) was introduced by Carrillo [4], who considered the nonlinear degenerate problem

$$\begin{gathered} \partial_t g(u) - \Delta b(u) + \operatorname{div} \Phi(u) = f(x) \quad \text{in } (0,T) \times \Omega, \\ g(u) = g_0 \quad \text{on } \{0\} \times \Omega, \\ b(u) = 0 \quad \text{on } (0,T) \times \Gamma, \end{gathered} \tag{2.2}$$

and gave existence, comparison and uniqueness of entropy solutions as follows.

Lemma 2.3. *Let* (H1), (H3) *and* (H4) *hold. Let* $g_0 \in L^\infty(\Omega)$ *and* $g_0 \in R(g)$, $f \in L^\infty(Q)$. *Then there exists an entropy solution* u *of problem* (2.2) *such that*

$v = g(u) \in \mathcal{C}([0,T]; L^1(\Omega))$ and

$$\|g(u)\|_{L^\infty(Q)} \le \|g_0\|_{L^\infty(\Omega)} + T\|f\|_{L^\infty(Q)}.$$

Lemma 2.4. *Let* (H1) *and* (H2) *hold. Let $g_{i0} \in L^1(\Omega)$, $g_{i0} \in R(g)$, and $f_i \in L^2((0,T); H^{-1}(\Omega) \cap L^1(\Omega))$. Let u_i be an entropy solution of* (2.2) *for $i = 1,2$. Then there exists some $\kappa \in H(u_1 - u_2)$ such that*

$$\begin{aligned}
&\int_Q \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi + H_0(u_1 - u_2)(\Phi(u_2) - \Phi(u_1)) \cdot \nabla\xi \\
&- (g(u_1) - g(u_2))^+ \partial_t \xi\} \, dx \, dt - \int_\Omega (g_{10} - g_{20})^+ \xi(0) dx \\
&\le \int_Q (f_1 - f_2)\kappa\xi \, dx \, dt
\end{aligned}$$

for any nonnegative $\xi \in \mathcal{D}([0,T) \times \overline{\Omega})$.

Lemma 2.5. *Let* (H1) *and* (H2) *hold. Let $g_{i0} \in L^1(\Omega)$, $g_{i0} \in R(g)(g_{i0} = g(u_{i0}))$, let $f_i \in L^2((0,T); H^{-1}(\Omega) \cap L^1(\Omega))$, and let u_i be an entropy solution of* (2.2) *for $i = 1,2$. Then there exists some $\kappa \in H(u_1 - u_2)$ such that*

$$\int_\Omega (g(u_1(t)) - g(u_2(t)))^+ dx \le \int_\Omega (g_{10} - g_{20})^+ dx + \int_0^t (f_1 - f_2)\kappa \, dx \, dt$$

for each $t \in [0,T]$. Therefore,

$$\|g(u_1(t)) - g(u_2(t))\|_{L^1(\Omega)} \le \|g_{10} - g_{20}\|_{L^1(\Omega)} + \int_0^t \|f_1 - f_2\|_{L^1(\Omega)} dt.$$

In particular, if $g_{10} \le g_{20}$ almost everywhere in Ω, and $f_1 \le f_2$ in Q, then

$$g(u_1) \le g(u_2) \quad \text{a.e. in } Q.$$

Moreover, if $f_1 = f_2$ and $g_{10} = g_{20}$, then $g(u_1) = g(u_2)$.

2.2. **Existence of entropy solutions.**

Theorem 2.6. *Let* (H1), (H3), (H4) *and* (H5) *hold. Let $g_0 \in L^\infty(\Omega)$ and $g_0 \in R(g)$. Then there exists an entropy solution u of problem* (2.1).

Proof. Fix $h > 0$, which will be determined latter. Define an operator

$$\mathcal{T} : \mathcal{C}([0,h]; L^1(\Omega)) \to \mathcal{C}([0,h]; L^1(\Omega))$$

as follows: $w \in \mathcal{T}(v)$ if and only if there exits $u \in L^\infty((0,h)\times\Omega)$ such that $w = g(u)$ and u is an entropy solution of

$$\begin{gathered}
\frac{\partial g(u)}{\partial t} - \Delta b(u) + \operatorname{div}\Phi(u) = f(v) \quad \text{in } (0,h)\times\Omega, \\
g(u) = g_0 \quad \text{on } \{0\}\times\Omega, \\
b(u) = 0 \quad \text{on } (0,h)\times\Gamma.
\end{gathered} \tag{2.3}$$

According to Carrillo [4], for any $g_0 \in L^\infty(\Omega)$ and $v \in L^\infty((0,h)\times\Omega)$, there exists an entropy solution of (2.3). Moreover,

$$\begin{gathered}
\|w\|_{L^\infty((0,h)\times\Omega)} \le \|g_0\|_{L^\infty(\Omega)} + Lh\|v\|_{L^\infty((0,h)\times\Omega)}, \\
\|w\|_{\mathcal{C}([0,h];L^1(\Omega))} \le \|g_0\|_{L^1(\Omega)} + L\|v\|_{\mathcal{C}([0,h];L^1(\Omega))}.
\end{gathered}$$

If $\mathcal{T}$ is contractive, it has a unique fixed point $w \in \mathcal{C}([0,h];L^1(\Omega))$, that is, there exists a measurable function u such that $w = g(u)$, and u is an entropy solution of (2.1) on $[0,h]$. In fact, we can prove that $\mathcal{T}$ is a contraction if we choose h small enough. For any $v_i \in \mathcal{C}([0,h];L^1(\Omega))$, $w_i = g(u_i) \in T(v_i)$, $i = 1,2$, applying the comparison for the solutions of (2.3), we have

$$\begin{aligned}\|w_1(t) - w_2(t)\|_{L^1(\Omega)} &\le \int_0^h \|f(v_1(t)) - f(v_2(t))\|_{L^1(\Omega)} dt \\ &\le Lh\|v_1(t) - v_2(t)\|_{\mathcal{C}([0,h];L^1(\Omega))}.\end{aligned}$$

Therefore, for any $0 < \alpha < 1$, we have

$$\|w_1(t) - w_2(t)\|_{\mathcal{C}([0,h];L^1(\Omega))} \le \alpha\|v_1(t) - v_2(t)\|_{\mathcal{C}([0,h];L^1(\Omega))},$$

only if we choose $0 < h < \alpha/L$.

Take $\alpha = 1/2$, and choose an integer M large enough such that $h = T/M < \alpha/L$. Divide the interval $[0,T]$ into $[0,h]$, $[h,2h]$, $\dots$, $[(M-1)h, Mh]$, and repeat the procedure above, we eventually get an entropy solution on $[0,T]$. $\square$

2.3. **Comparison and uniqueness of entropy solutions.** We prove some properties and refer the reader to [4] for related results.

Theorem 2.7. *Let* (H1), (H2) *and* (H5) *hold. Let $g_{i0} \in L^1(\Omega)$, $g_{i0} \in R(g)$, and u_i be an entropy solution of* (2.1)*, $i = 1,2$.*

(1) There exists some $\kappa \in H(u_1 - u_2)$, such that

$$\begin{aligned}&\int_Q \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi + H_0(u_1 - u_2)(\Phi(u_2) - \Phi(u_1)) \cdot \nabla\xi \\ &- (g(u_1) - g(u_2))^+ \partial_t\xi\}\, dx\, dt - \int_\Omega (g_{10} - g_{20})^+ \xi(0) dx \\ &\le \int_Q (f(g(u_1)) - f(g(u_2)))\kappa\xi\, dx\, dt\end{aligned} \tag{2.4}$$

for any nonnegative $\xi \in \mathcal{D}([0,T) \times \overline{\Omega})$. Therefore,

$$\begin{aligned}&\int_\Omega (g(u_1(t)) - g(u_2(t)))^+ dx \\ &\le \int_\Omega (g_{10} - g_{20})^+ dx + \int_0^t \int_\Omega (f(g(u_1)) - f(g(u_2)))\kappa\, dx\, dt;\end{aligned} \tag{2.5}$$

that is,

$$\begin{aligned}&\|(g(u_1(t)) - g(u_2(t)))^+\|_{L^1(\Omega)} \\ &\le \|(g_{10} - g_{20})^+\|_{L^1(\Omega)} + \int_0^t (f(g(u_1)) - f(g(u_2)))\kappa\, dx\, dt.\end{aligned}$$

In particular, if $g_{10} \le g_{20}$ a.e. in Ω and $f_1 \le f_2$, then

$$g(u_1) \le g(u_2) \quad \text{a.e. in } Q.$$

(2) If $g_{10} = g_{20}$ a.e. in Ω, and (H4) *holds, then $b(u_1) = b(u_2)$ a.e. in Q. Thus, there exists a unique pair $(g(u), b(u))$ such that u is an entropy solution of* (2.1).

Proof. (1) The proof of (2.4) and (2.5) can be found in [4]. If $g_{10} \le g_{20}$, applying (2.5), we have

$$
\begin{aligned}
&\|(g(u_1(t)) - g(u_2(t)))^+\|_{L^1(\Omega)} \\
&\le \|(g_{10} - g_{20})^+\|_{L^1(\Omega)} + \int_0^t \int_\Omega (f(g(u_1)) - f(g(u_2)))\kappa \, dx \, dt \\
&\le L \int_0^t \|(g(u_1(\tau)) - g(u_2(\tau)))^+\|_{L^1(\Omega)} d\tau.
\end{aligned}
$$

Consequently, we deduce that

$$
\sup_{0 \le t \le h} \|(g(u_1(t)) - g(u_2(t)))^+\|_{L^1(\Omega)} \le 0,
$$

which implies $g(u_1) \le g(u_2)$ a.e. in Q.

(2) If $g_{10} = g_{20}$, then $g(u_1) = g(u_2)$. From (2.4) and (H4), we have

$$
\begin{aligned}
&\int_Q \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi + H_0(u_1 - u_2)(\Phi(u_2) - \Phi(u_1)) \cdot \nabla\xi \\
&\le \int_Q (f(g(u_1)) - f(g(u_2)))\kappa\xi \, dx \, dt \le L \int_Q |g(u_1) - g(u_2)|\kappa\xi \, dx \, dt \\
&= L \int_Q (g(u_1) - g(u_2))^+ \xi \, dx \, dt = 0
\end{aligned}
\tag{2.6}
$$

for any nonnegative $\xi \in \mathcal{D}([0,T) \times \overline{\Omega})$. Applying (2.6), we can deduce that (see [4, Corollary 14])

$$
\int_Q (b(u_1) - b(u_2))^+ \{-\Delta\xi - \Phi^{(2)}(g(u_1)) \cdot \nabla\xi\} \, dx \, dt = 0.
$$

Then by choosing $\xi = e^{\lambda x_i}$ for some i with $1 \le i \le N$, and some

$$
\lambda > \|\Phi^{(2)}(g(u_1))\|_{L^\infty(Q)},
$$

we deduce that $(b(u_1) - b(u_2))^+ = 0$. $\square$

Applying the results in [4], we can also deduce the comparison of $b(u)$.

Lemma 2.8. *Let* (H1), (H2),(H5) *and* (H6) *hold. Let $g_{i0} \in L^1(\Omega)$, $g_{i0} \in R(g)$ such that $g_{10} \le g_{20}$ a. e. in Ω. Let u_i be an entropy solution of* (2.1) *for $i = 1, 2$. Then*

$$
b(u_1) \le b(u_2) \text{ a. e. in } Q.
$$

Moreover, if $g_{10} = g_{20}$, then $b(u_1) = b(u_2)$.

Arguing as Carrillo [4], we can get a generalized comparison theorem, which will be used in section 3.

Lemma 2.9. *Let* (H1), (H2) *and* (H5) *hold. Let $g_{i0} \in L^1(\Omega)$ with $g_{i0} \in R(g)$. Let u_1 be an entropy subsolution of* (2.1) *for g_{10}, and u_2 be an entropy supersolution of* (2.1) *for g_{20}. Then*

(1) *There exists some $\kappa \in H(u_1 - u_2)$, such that*

$$
\begin{aligned}
&\int_Q \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi + H_0(u_1 - u_2)(\Phi(u_2) - \Phi(u_1)) \cdot \nabla\xi \\
&- (g(u_1) - g(u_2))^+ \partial_t\xi\} \, dx \, dt - \int_\Omega (g_{10} - g_{20})^+ \xi(0) dx
\end{aligned}
$$

$$\leq \int_Q (f(g(u_1)) - f(g(u_2)))\kappa\xi \, dx \, dt$$

for any nonnegative $\xi \in \mathcal{D}([0,T) \times \overline{\Omega})$.

(2) *Moreover, if* $g_{10} \leq g_{20}$ *a. e. in* Ω, *then*

$$g(u_1) \leq g(u_2), \quad \text{a.e. in } Q. \tag{2.7}$$

(3) *In addition, let* (H6) *hold. If* $g_{10} \leq g_{20}$ *a. e. in* Ω, *then*

$$b(u_1) \leq b(u_2), \quad \text{a.e. in } Q. \tag{2.8}$$

3. Entropy solution of the initial-value problem

Let $N \geq 3$, and $Q = (0,T) \times \mathbb{R}^N$ with $T > 0$. In this section, we consider the initial-value problem

$$\begin{gathered} \partial_t g(u) - \Delta b(u) + \operatorname{div} \Phi(u) = f(g(u)) \quad \text{in } (0,T) \times \mathbb{R}^N, \\ g(u) = g_0 \quad \text{on } \{0\} \times \mathbb{R}^N, \end{gathered} \tag{3.1}$$

where $N \geq 3$, g, b are nondecreasing continuous, $\Phi = (\phi_1, \dots, \phi_N)$ is continuous, and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous.

3.1. **Existence of entropy solutions.**

Definition 3.1. Let $g_0 \in L^\infty(\mathbb{R}^N)$.

(1) A measurable function u is called a weak subsolution (supersolution) of (3.1), if

$$\begin{gathered} g(u) \in L^\infty(Q), \quad \partial_t g(u) \in L^2(0,T; H^{-1}_{\rm loc}(\mathbb{R}^N)), \\ b(u) \in L^2(0,T; L^2_{\rm loc}(\mathbb{R}^N)) \cap L^\infty(Q), \quad \nabla b(u) \in (L^2_{\rm loc}(Q))^N, \\ \Phi(u) \in (L^2_{\rm loc}(\mathbb{R}^N))^N \cap (L^\infty(Q))^N, \\ \partial_t g(u) - \Delta b(u) + \operatorname{div}(\Phi(u)) \leq (\ \geq\ )\ f(g(u)) \quad \text{in } \mathcal{D}'(Q), \\ g(u(0,x)) \leq\ (\ \geq\ )\ g_0(x) \quad \text{a.e. in } \mathbb{R}^N. \end{gathered}$$

(2) A measurable function u is called a weak solution of (3.1) if it is both a weak subsolution and a weak supersolution.

Definition 3.2. Let $g_0 \in L^\infty(\mathbb{R}^N)$.

(1) A weak subsolution u is called an entropy subsolution of (3.1), if

$$\begin{aligned} &\int_Q H_0(u-s)\{(\nabla b(u) - \Phi(u) + \Phi(s)) \cdot \nabla\xi - (g(u) - g(s))\partial_t \xi \\ &- f(g(u))\xi\} \, dx \, dt - \int_{\mathbb{R}^N} (g_0 - g(s))^+ \xi(0) dx \leq 0 \end{aligned}$$

for any $(s,\xi) \in \mathbb{R} \times \mathcal{D}([0,T) \times \mathbb{R}^N)$, with $\xi \geq 0$.

(2) A weak supersolution u is called an entropy supersolution of (3.1), if

$$\begin{aligned} &\int_Q H_0(-s-u)\{(\nabla b(u) - \Phi(u) + \Phi(-s)) \cdot \nabla\xi - (g(u) - g(-s))\partial_t \xi \\ &- f(g(u))\xi\} \, dx \, dt + \int_{\mathbb{R}^N} (g_0 - g(-s))^- \xi(0) dx \geq 0 \end{aligned}$$

for any $(s,\xi) \in \mathbb{R} \times \mathcal{D}([0,T) \times \mathbb{R}^N)$, with $\xi \geq 0$.

(3) u is called an entropy solution of (3.1) if it is both an entropy subsolution and an entropy supersolution.

For any $R > 0$, let $B_R = \{x \in \mathbb{R}^N; |x| < R\}$, $\Gamma_R = \partial B_R = \{x \in \mathbb{R}^N; |x| = R\}$, $Q_R = (0,T) \times B_R$. For any positive integer n, let g_{0n} be a truncation of g_0 on B_n, that is, $g_{0n}(x) = g_0(x)$ for any $x \in B_n$, and extended to $\mathbb{R}$ by zero. We consider the approximate problem for (3.1):

$$
\begin{gathered}
\partial_t g(u) - \Delta b(u) + \operatorname{div} \Phi(u) = f(g(u)) \quad \text{in } (0,T) \times B_n, \\
g(u) = g_{0n} \quad \text{on } \{0\} \times B_n, \\
b(u) = 0 \quad \text{on } (0,T) \times \Gamma_n.
\end{gathered} \tag{3.2}
$$

Let (H1), (H3), (H4) and (H5) hold, then there exists a pair $(g(u_n), b(u_n))$ such that u_n is an entropy solution of (3.2). We extend u_n to $\mathbb{R}^N$ by zero, and still denote it by u_n.

Lemma 3.3. *Let* (H1), (H2), (H5), (H6) *hold. Assume that*

(H7) *There exists a nondecreasing function* $l : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$
|b(r)| \le l(|g(r)|), \quad \forall r \in \mathbb{R}.
$$

Assume $g_0 \ge 0$ *and* $g_0 \in L^1(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$. *Then, we have:*

(1) $g(u_n) \ge 0$ *and* $b(u_n) \ge 0$ *a.e. in* $\mathbb{R}^N$*;*

(2) $g(u_n(t))$ *and* $b(u_n(t))$ *are nondecreasing in* n*, that is, for any* $n < m$*,*

$$
g(u_n(t)) \le g(u_m(t)), \quad b(u_n(t)) \le b(u_m(t)) \quad \text{a.e. in } \mathbb{R}^N;
$$

(3) *There exists a constant* C *such that*

$$
\|g(u_n)\|_{L^\infty(Q)} \le C, \tag{3.3}
$$

$$
\|g(u_n)\|_{L^\infty(0,T;L^1(\mathbb{R}^N))} \le C, \tag{3.4}
$$

$$
\|\nabla b(u_n)\|_{L^2(\mathbb{R}^N)} \le C, \tag{3.5}
$$

$$
\|b(u_n)\|_{L^\infty(Q)} \le C\,. \tag{3.6}
$$

Proof. (1) If $g_0 \equiv 0$, then $u \equiv 0$ is an entropy solution of (3.2). From Lemma 2.9, we have $g(u_n(t)) \ge 0$, $b(u_n) \ge 0$ a.e. in Q_n, and so is in $\mathbb{R}^N$.

(2) For any $n < m$, it is clear that u_n is an entropy subsolution of (3.2). From (2.7) and (2.8), we have

$$
g(u_n(t)) \le g(u_m(t)), \quad b(u_n(t)) \le b(u_m(t)) \quad \text{a.e. in } \mathbb{R}^N.
$$

(3) Since $g_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, for any $0 < h < 1/(2L)$,

$$
\|g(u_n)\|_{L^\infty((0,h)\times B_R)} \le \|g_{0n}\|_{L^\infty(B_R)} + Lh\|g(u_n)\|_{L^\infty((0,h)\times B_R)},
$$

$$
\|g(u_n)\|_{L^\infty(0,h;L^1(B_R))} \le \|g_{0n}\|_{L^1(B_R)} + L\int_0^h \|g(u_n(\tau))\|_{L^1(B_R)} d\tau,
$$

and hence, we have

$$
\begin{gathered}
\|g(u_n)\|_{L^\infty((0,h)\times B_R)} \le C\|g_{0n}\|_{L^\infty(B_R)} \le C\|g_0\|_{L^\infty(\mathbb{R}^N)}, \\
\|g(u_n)\|_{L^\infty(0,h;L^1(B_R))} \le C\|g_{0n}\|_{L^1(B_R)} \le C\|g_0\|_{L^1(\mathbb{R}^N)}.
\end{gathered}
$$

Then we obtain (3.3) and (3.4) by arguing inductively and letting $R \to \infty$. In virtue of (H7), (3.3) implies (3.6).

Viewing that u_n is an entropy solution (and hence a weak solution) of (3.2), we have

$$
\begin{aligned}
&\int_0^t \int_{B_n} \partial_t(g(u_n))b(u_n)dxd\tau + \int_0^t \int_{B_n} |\nabla b(u_n)|^2 dxd\tau \\
&+ \int_0^t \int_{B_n} \Phi(u_n)\cdot \nabla b(u_n)dxd\tau \\
&= \int_0^t \int_{B_n} f(g(u_n))b(u_n)dxd\tau \\
&\le L\int_0^t \|g(u_n)\|_{L^q(B_n)}\|b(u_n)\|_{L^{2^*}(B_n)}d\tau,
\end{aligned}
$$

where $q = 2n/(n+2)$.

Applying [4, Lemma 4],

$$
\int_0^t \int_{B_n} \partial_t(g(u_n))b(u_n)dxd\tau = \int_{B_n} B_I(g(u_n(t)))dx - \int_{B_n} B_I(g_{0n})dx,
$$

where $B_I(g(u_n)) \in L^\infty(0,T;L^1(B_n))$, since $g(u_n)$ is uniformly bounded in the spaces $L^\infty(0,T;L^1(B_n))$ and $L^\infty((0,T)\times B_n)$. For the third term, from [3],

$$
\int_0^t \int_{B_n} \Phi(u_n)\cdot \nabla b(u_n)\,dx\,dt = 0.
$$

Considering that

$$
\sup_{0\le t\le T} \|g(u_n(t))\|_{L^q(B_n)} \le C \sup_{0\le t\le T} \|g(u_n(t))\|_{L^1(B_n)} \le C\|g_0\|_{L^1(\mathbb{R}^N)},
$$

we have

$$
\int_0^T \int_{B_n} |\nabla b(u_n)|^2\,dx\,dt \le C_1 + C_2\Big(\int_0^T \int_{B_n} |\nabla b(u_n)|^2\,dx\,dt\Big)^{1/2};
$$

that is,

$$
\int_0^T \int_{\mathbb{R}^N} |\nabla b(u_n)|^2\,dx\,dt \le C_1 + C_2\Big(\int_0^T \int_{\mathbb{R}^N} |\nabla b(u_n)|^2\,dx\,dt\Big)^{1/2};
$$

then we obtain (3.5). □

Theorem 3.4. *Let* (H1)-(H7) *hold. Let* $g_0 \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $0 \le g_0 \in R(g)$. *Then there exists an entropy solution of* (3.1).

Proof. Let u_n be an entropy solution of (3.2). From Lemma 3.3, it follows that $g(u_n)$ and $b(u_n)$ are nonnegative, nondecreasing and uniformly bounded in $L^\infty(Q)$. Hence, there exist $v \in L^\infty(Q)$ and $w \in L^2(0,T;H^1_{\rm loc}(\mathbb{R}^N))$, such that

$$
\begin{gathered}
g(u_n) \to v, \quad \text{in } L^p_{\rm loc}(Q),\\
b(u_n) \to w, \quad \text{in } L^2(0,T;L^2_{\rm loc}(\mathbb{R}^N)),\\
\nabla b(u_n) \to \nabla w, \quad \text{weakly in } (L^2(Q))^N,
\end{gathered}
$$

where $1 \le p < +\infty$. Since $b\circ g^{-1}$ is maximal monotone, $w \in b\circ g^{-1}(v)$, that is, there exists $\widetilde{u} \in g^{-1}(v)$ such that $w = b(\widetilde{u})$. Set

$$
u = ((b+g)^{-1})_0(v+w),
$$

which is measurable, and such that $v = g(u)$, $w = b(u)$. Since $\Phi(u) = \Phi^{(1)}(g(u)) + \Phi^{(2)}(g(u))b(u)$, we deduce that

$$\Phi(u_n) \to \Phi(u) \quad \text{in } (L^2(0,T; L^2_{\rm loc}(\mathbb{R}^N)))^N.$$

For any $\xi \in \mathcal{D}([0,T) \times \mathbb{R}^N)$, choose R large enough such that $\operatorname{supp}\xi(t) \subset B_R$, then it follows that

$$\begin{aligned}&\int_{Q_R} \{-g(u_n)\partial_t\xi + (\nabla b(u_n) - \Phi(u_n)) \cdot \nabla\xi\}\,dx\,dt\\ &= \int_{Q_R} f(g(u_n))\xi\,dx\,dt + \int_{B_R} g_{0n}\xi(0)dx,\end{aligned}$$

for any integer $n > R$. By letting $n \to +\infty$, we have

$$\begin{aligned}&\int_{Q_R} \{-g(u)\partial_t\xi + (\nabla b(u) - \Phi(u)) \cdot \nabla\xi\}\,dx\,dt\\ &= \int_{Q_R} f(g(u))\xi\,dx\,dt + \int_{B_R} g_0\xi(0)dx.\end{aligned}$$

From the choice of R, it holds that

$$\begin{aligned}&\int_{Q} \{-g(u)\partial_t\xi + (\nabla b(u) - \Phi(u)) \cdot \nabla\xi\}\,dx\,dt\\ &= \int_{Q} f(g(u))\xi\,dx\,dt + \int_{\mathbb{R}^N} g_0\xi(0)dx,\end{aligned}$$

and hence we deduce that

$$\partial_t(g(u_n)) \to g(u)_t \quad \text{weakly in } L^2(0,T; H^{-1}_{\rm loc}(\mathbb{R}^N)).$$

Therefore, u is a weak solution of (3.1).

For any nonnegative $\xi \in \mathcal{D}([0,T) \times \mathbb{R}^N)$ and $s \in \mathbb{R}$, take R so large that $\operatorname{supp}\xi(t) \subset B_R$. Since u_n is an entropy subsolution of (3.2), we have

$$\begin{aligned}&\int_{Q_R} H_0(u_n - s)\{-(g(u_n) - g(s))\partial_t\xi + (\nabla b(u_n) - \Phi(u_n) + \Phi(s)) \cdot \nabla\xi\\ &- f(g(u_n))\xi\}\,dx\,dt - \int_{B_R} (g_{0n} - g(s))^+\xi(0)dx \le 0\end{aligned}$$

for any integer $n > R$.

Since $g(u_n) + b(u_n) \to g(u) + b(u)$ in $L^2_{\rm loc}(Q)$, and

$$H_0(u_n - s) \in H(g(u) + b(u) - g(s) - b(s)),$$

we have

$$H_0(u_n - s) \to \chi_{u,s} \quad \text{weak * in } L^\infty(Q),$$

and $\chi_{u,s} \in H(g(u) + b(u) - g(s) - b(s))$.

Because $\nabla b(u_n) \to \nabla b(u)$ weakly in $L^2(Q)^N$, it follows that

$$\begin{aligned}\int_{Q_R} H_0(u_n - s)\nabla b(u_n) \cdot \nabla\xi\,dx\,dt &= \int_{Q_R} \nabla(b(u_n) - b(s))^+ \cdot \nabla\xi\,dx\,dt\\ &\to \int_{Q_R} \nabla(b(u) - b(s))^+ \cdot \nabla\xi\,dx\,dt\\ &= \int_{Q_R} H_0(u - s)\nabla b(u) \cdot \nabla\xi\,dx\,dt.\end{aligned}$$

By letting $n \to +\infty$, we have

$$\begin{aligned} &\int_{Q_R} \{-\chi_{u,s} f(g(u))\xi + H_0(u-s)(-(g(u)-g(s))\partial_t \xi \\ &+ (\nabla b(u) - \phi(u) + \phi(s)) \cdot \nabla \xi)\}\,dx\,dt - \int_{B_R} (g_0 - g(s))^+ \xi(0) dx \le 0. \end{aligned} \tag{3.7}$$

Similarly to the proof of [4, Theorem 12], applying (3.7), we can deduce that

$$\begin{aligned} &\int_Q H_0(u-s)\{(\nabla b(u) - \Phi(u) + \Phi(s)) \cdot \nabla \xi - (g(u)-g(s))\partial_t \xi \\ &- f(g(u))\xi\}\,dx\,dt - \int_{\mathbb{R}^N} (g_0 - g(s))^+ \xi(0) dx \le 0 \end{aligned}$$

for any nonnegative $\xi \in \mathcal{D}([0,T) \times \mathbb{R}^N)$ and any $s \in \mathbb{R}$.

Since u_n is also an entropy supersolution of (3.2), arguing as above, we have

$$\begin{aligned} &\int_Q H_0(-s-u)\{(\nabla b(u) - \Phi(u) + \Phi(-s)) \cdot \nabla \xi - (g(u)-g(-s))\partial_t \xi \\ &- f(g(u))\xi\}\,dx\,dt + \int_{\mathbb{R}^N} (g_0 - g(-s))^- \xi(0) dx \ge 0 \end{aligned}$$

for any nonnegative $\xi \in \mathcal{D}([0,T) \times \mathbb{R}^N)$ and any $s \in \mathbb{R}$. □

3.2. **Comparison and uniqueness of entropy solutions.** We will give the comparison and uniqueness of entropy solutions of (3.1) based on the works in [1].

To prove the comparison, instead of (H7), we assume that

(H7') For $M > 0$, there exists a nondecreasing continuous function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(0) = 0$, such that

$$|b(x)| \le \omega(|g(x)|), \quad \forall x \in [-M, M].$$

Remark 3.5. (1) From (H7'), $g(x) = 0$ implies $b(x) = 0$ for any $x \in \mathbb{R}$.

(2) Since the entropy solution u of (3.1) satisfies $g(u) \in L^\infty(Q)$, we can assume that M is fixed, ω is strictly concave, continuous, and extended to $\mathbb{R}^+$. Moreover, for any $x \in \mathbb{R}$ such that $b(x) \ge 0$, and any $r \in \mathbb{R}^+$, we have

$$b(x)r \le g(x) + \Omega^*(r),$$

where Ω^* is strictly increasing and convex (see [1]). From the convexity of Ω^*, for any $R > 0$, we have

$$\Omega^*(r) \le (\Omega^*)'(R) r, \quad \forall r \in [0, R]$$

where $(\Omega^*)'$ is increasing and $(\Omega^*)'(r) \to 0$ as $t \downarrow 0$.

Theorem 3.6. *Assume* (H1), (H5), (H7'). *Assume that* $\Phi(u) = \Psi(g(u))$, *where* $\Psi : \mathbb{R} \to \mathbb{R}^N$ *is locally Hölder continuous of order* $1 - \frac{1}{N}$, *and* $\Psi(0) = 0$. *Let* $N \ge 3$. *Let* $g_{i0} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, *and* u_1 *be an entropy subsolution of* (3.1) *for* g_{10}, *and* u_2 *be an entropy supersolution of* (3.1) *for* g_{20}. *Then for a.e.* $t \in (0,T)$, *there exists some* $\kappa \in H(u_1 - u_2)$ *such that*

$$\begin{aligned} &\|(g(u_1(t)) - g(u_2(t)))^+\|_{L^1(\mathbb{R}^N)} \\ &\le \|(g_{10} - g_{20})^+\|_{L^1(\mathbb{R}^N)} + \int_0^t (f(g(u_1)) - f(g(u_2)))\kappa\,dx\,dt. \end{aligned} \tag{3.8}$$

In particular, if $g_{10} \le g_{20}$, *then* $g(u_1) \le g(u_2)$.

The comparison principle implies the following uniqueness of entropy solutions.

Corollary 3.7. *Assume b, g, Φ, f and g_{i0} as above, and let u_i be an entropy solution of* (3.1) *for $i = 1, 2$. If $g_{10} = g_{20}$, then $g(u_1) = g(u_2)$.*

Remark 3.8. Our result is different from [1]. In fact, our problem can be rewritten of the form in [1] with the replacement of $f(x,t)$ with $f(u)$, if $g(x) = g(y)$ implies $b(x) = b(y)$ for any $x, y \in \mathbb{R}$, which maybe is invalid, unfortunately, although g and b satisfy (H7').

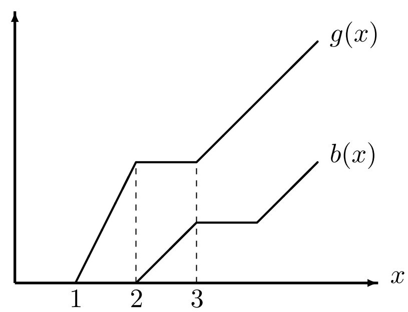

Figure 1.

For example, assume that both g and b are odd functions, and for any $x \geq 0$,

$$g(x) = \begin{cases} 0, & 0 \leq x \leq 1, \\ 2x-2, & 1 < x < 2, \\ 2, & 2 \leq x \leq 3, \\ x-1, & x > 3. \end{cases} \qquad b(x) = \begin{cases} 0, & 0 \leq x \leq 2, \\ x-2, & 2 < x < 3, \\ 1, & 3 \leq x \leq 4, \\ x-3, & x > 4. \end{cases}$$

Then (H7') is satisfied since $|b(x)| \leq |g(x)|$. However, $b(x)$ is strictly increasing on $[2,3]$ where $g = 2$ is constant. (see Figure 1)

Proof of Theorem 3.6. We need to prove only (3.8). We begin our proof by applying the Kato's Inequality; that is, there exists some $\kappa \in H(u_1 - u_2)$ such that

$$\begin{aligned}
&\int_Q \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi + H_0(u_1 - u_2)(\Phi(u_2) - \Phi(u_1)) \cdot \nabla\xi \\
&- (g(u_1) - g(u_2))^+ \partial_t \xi\} \, dx \, dt - \int_{\mathbb{R}^N} (g_{10} - g_{20})^+ \xi(0) dx \\
&\leq \int_Q (f(g(u_1)) - f(g(u_2)))\kappa\xi \, dx \, dt
\end{aligned}$$

for any nonnegative $\xi \in \mathcal{D}([0,T) \times \mathbb{R}^N)$.

Let $L > R > 0$ and $\epsilon \in (0,1)$. We take $\xi(t,x) = \mu(t)\rho_{\epsilon,L}(x)$, where $\mu \in \mathcal{D}([0,T))^+$, and

$$\rho_{\epsilon,L}(x) = R^{N-2+\epsilon}((\max|x|, R)^{2-N-\epsilon} - L^{2-N-\epsilon})^+.$$

Then we can deduce that

$$\int_0^T \int_{\mathbb{R}^N} (g(u_1) - g(u_2))^+ \rho_{\epsilon,L}(-\partial_t\mu) \, dx \, dt$$

$$
\begin{aligned}
&\le \int_0^T \int_{\mathbb{R}^N} H_0(u_1-u_2)|\Phi(u_2)-\Phi(u_1)||\nabla\rho_{\epsilon,L}|\mu\,dx\,dt \\
&\quad + \int_0^T \int_{\mathbb{R}^N} (b(u_1)-b(u_2))^+ \Delta^{ac}\rho_{\epsilon,L}\mu\,dx\,dt \\
&\quad + \int_{|x|=L} (b(u_1)-b(u_2))^+ (N-2+\epsilon)L^{1-N-\epsilon}\mu dx \\
&\quad + \int_0^T \int_{\mathbb{R}^N} (f(g(u_1))-f(g(u_2)))\kappa\rho_{\epsilon,L}\mu\,dx\,dt \\
&\quad + \int_{\mathbb{R}^N} (g_{10}-g_{20})^+\rho_{\epsilon,L}\mu(0)dx,
\end{aligned}
$$

where $\Delta^{ac}\rho_{\epsilon,L}$ is the absolutely continuous part of the measure $\Delta\rho_{\epsilon,L}$. By letting $L\to\infty$, we have

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} (g(u_1)-g(u_2))^+\rho_\epsilon(-\partial_t\mu)\,dx\,dt \\
&\le C\int_0^T \int_{|x|>R} H_0(u_1-u_2)|\Phi(u_2)-\Phi(u_1)||x|^{-1}\rho_\epsilon\,dx\,dt \\
&+ C\epsilon\int_0^T \int_{|x|>R} (b(u_1)-b(u_2))^+|x|^{-2}\rho_\epsilon\,dx\,dt \\
&+ \int_0^T \int_{\mathbb{R}^N} (f(g(u_1))-f(g(u_2)))\kappa\mu\,dx\,dt + \int_{\mathbb{R}^N}(g_{10}-g_{20})^+\mu(0)dx,
\end{aligned}
\tag{3.9}
$$

where $\rho_\epsilon(x)=(\max\frac{|x|}{R},1)^{2-N-\epsilon}$.

In particular, by taking $g_{20}=u_2=0$, we have

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} (g(u_1))^+\rho_\epsilon(-\partial_t\mu)\,dx\,dt \\
&\le C\int_0^T \int_{|x|>R} H_0(u_1)|\Psi(g(u_1))||x|^{-1}\rho_\epsilon\,dx\,dt \\
&\quad + C\epsilon\int_0^T \int_{|x|>R} (b(u_1))^+|x|^{-2}\rho_\epsilon\,dx\,dt \\
&\quad + L\int_0^T \int_{\mathbb{R}^N} (g(u_1))^+\mu\,dx\,dt + \int_{\mathbb{R}^N} g_{10}^+\mu(0)dx.
\end{aligned}
\tag{3.10}
$$

Since $\Psi:\mathbb{R}\to\mathbb{R}^N$ is locally Hölder continuous of order $(1-\frac{1}{N})$ with $\Psi(0)=0$, applying Young's inequality, we have

$$
\begin{aligned}
CH_0(u_1)|\Psi(g(u_1))||x|^{-1} &\le \frac{1}{2}H_0(u_1)|\Psi(g(u_1))|^{\frac{N}{N-1}} + C_N(C|x|^{-1})^N \\
&\le \frac{1}{2}(g(u_1))^+ + (2C|x|^{-1})^N,
\end{aligned}
\tag{3.11}
$$

where

$$
C_N = \frac{1}{N}\big(\frac{N}{2(N-1)}big)^{-(N-1)} \le 2^N.
$$

Considering (H7') and Remark 3.5 , we have

$$\begin{aligned} C\epsilon(b(u_1))^+|x|^{-2} &= C\epsilon H_0(b(u_1))b(u_1)|x|^{-2} \\ &\le H_0(u_1)(\delta g(u_1)+\delta\Omega^*(\frac{C\epsilon}{\delta}|x|^{-2})) \\ &\le \delta(g(u_1))^+ + C\epsilon(\Omega^*)'(\frac{C\epsilon}{\delta}R^{-2})|x|^{-2}, \end{aligned} \tag{3.12}$$

where $(\Omega^*)'(\frac{C\epsilon}{\delta}R^{-2}) \to 0$ as $\epsilon \to 0$, for any $R > 1$ and given δ. Combining (3.10),(3.11) and (3.12), we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} (g(u_1))^+\rho_\epsilon(-\partial_t\mu)\,dx\,dt \\ &\le (\delta+\frac{1}{2})\int_0^T\int_{|x|>R}(g(u_1))^+\mu\,dx\,dt + \int_0^T\int_{|x|>R}(2C|x|^{-1})^N\rho_\epsilon\mu\,dx\,dt \\ &\quad + C\epsilon(\Omega^*)'(\frac{C\epsilon}{\delta}R^{-2})\int_0^T\int_{|x|>R}|x|^{-2}\rho_\epsilon\mu\,dx\,dt + L\int_0^T\int_{\mathbb{R}^N}(g(u_1))^+\mu\,dx\,dt \\ &\quad + \int_{\mathbb{R}^N} g_{10}^+\mu(0)dx. \end{aligned}$$

For any $R > 1$, it holds that

$$\int_0^T\int_{|x|>R}(2C|x|^{-1})^N\rho_\epsilon\mu\,dx\,dt \le C,$$

$$C\epsilon(\Omega^*)'(\frac{C\epsilon}{\delta}R^{-2})\int_0^T\int_{|x|>R}|x|^{-2}\rho_\epsilon\mu\,dx\,dt \le C(\Omega^*)'(\frac{C\epsilon}{\delta}R^{-2})R^{N-2} \to 0,$$

as $\epsilon \to 0$. We now choose an integer M large enough such that $h = T/M < 1/(2L)$, and take $\delta \le \frac{1}{2}$, $\mu(t) = (h-t)^+$, then by letting $\epsilon \to 0$, we deduce that

$$\int_0^h\int_{|x|<R} g(u_1(t))^+dt \le 2h\|g_{10}^+\|_{L^1(\mathbb{R}^N)} + C.$$

Letting $R \to +\infty$, we have $(g(u_1))^+ \in L^1((0,h)\times\mathbb{R}^N)$.

Let $h_j(t)$ be such that $h_j(t) = h$ for $t \in [0,(j-1)h]$, and $h_j(t) = (jh-t)^+$ for $t \in [(j-1)h, T]$, $1 \le j \le M$. By arguing similarly and inductively, we conclude that

$$g(u_1),\, g(u_2) \in L^1((0,T)\times\mathbb{R}^N). \tag{3.13}$$

We now claim that $g(u_1) \le g(u_2)$ in virtue of (3.9) and (3.13). In fact, applying (3.9), we have

$$\begin{aligned} &\int_0^T\int_{\mathbb{R}^N}(g(u_1)-g(u_2))^+\rho_\epsilon(-\partial_t\mu)\,dx\,dt \\ &\le \int_0^T\int_{\mathbb{R}^N}(f(g(u_1)))-f(g(u_2)))\kappa\mu\,dx\,dt + \int_{\mathbb{R}^N}(g_{10}-g_{20})^+\mu(0)dx \\ &\quad + C\epsilon(\Omega^*)'(\frac{C\epsilon}{\delta}R^{-2})\int_0^T\int_{|x|>R}|x|^{-2}\rho_\epsilon\mu\,dx\,dt + C\delta\int_0^T\int_{|x|>R}|x|^{-N}\rho_\epsilon\mu\,dx\,dt \\ &\quad + C(\delta)\int_0^T\int_{|x|>R}(|g(u_1)|+|g(u_2)|)\mu\,dx\,dt. \end{aligned}$$

Since $g(u_1), g(u_2) \in L^1((0,T)\times\mathbb{R}^N)$, for any $\delta > 0$, we have

$$C(\delta)\int_0^h\int_{|x|>R}(|g(u_1)|+|g(u_2)|)\,dx\,dt \to 0,$$

as $R\to\infty$. Then by the facts that

$$\int_0^T\int_{|x|>R}|x|^{-N}\rho_\epsilon\mu dx \le C,$$

$$C\epsilon(\Omega^*)'(\frac{C\epsilon}{\delta}R^{-2})\int_0^T\int_{|x|>R}|x|^{-2}\rho_\epsilon\mu\,dx\,dt \to 0 \quad \text{as } \epsilon\to 0,$$

we have

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^N}(g(u_1)-g(u_2))^+(-\partial_t\mu)\,dx\,dt\\
&\le \int_{\mathbb{R}^N}(g_{10}-g_{20})^+\mu(0)dx + \int_0^T\int_{\mathbb{R}^N}(f(g(u_1))-f(g(u_2)))\kappa\mu\,dx\,dt,
\end{aligned}$$

by taking δ small, ϵ small, and R large. In the end, applying Gronwall's inequality, we can deduce (3.8). ☐

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