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A NONLOCAL BOUNDARY PROBLEM FOR THE LAPLACE OPERATOR IN A HALF DISK

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Abstract. In the present work we investigate the nonlocal boundary problem for the Laplace equation in a half disk. The difference of this problem is the impossibility of direct applying of the Fourier method (separation of variables). Because the corresponding spectral problem for the ordinary differential equation has the system of eigenfunctions not forming a basis. Based on these eigenfunctions there is constructed a special system of functions that already forms the basis. This is used for solving of the nonlocal boundary equation. The existence and the uniqueness of the classical solution of the problem are proved.

1. Formulation of the problem

Our goal is to find a function $u(r,\theta) \in C^0(\bar{D}) \cap C^2(D)$ satisfying equation

$$\Delta u = 0 \tag{1.1}$$

in D, with the boundary conditions

$$u(1,\theta) = f(\theta), \quad 0 \le \theta \le \pi, \tag{1.2}$$

$$u(r,0) = 0, \quad r \in [0,1], \tag{1.3}$$

$$\frac{\partial u}{\partial \theta}(r,0) = \frac{\partial u}{\partial \theta}(r,\pi) + \alpha u(r,\pi), \quad r \in (0,1) \tag{1.4}$$

where $D = \{(r,\theta) : 0 < r < 1, 0 < \theta < \pi\}$; $\alpha > 0$; $f(\theta) \in C^2[0,\pi]$, $f(0) = 0$, $f'(0) = f'(\pi) + \alpha f(\pi)$.

Problem (1.1)–(1.4) with $\alpha = 0$ was considered in [3, 4] for the Laplace equation, and in [5, 6] for the Helmholtz equation. The existence and the uniqueness of the solution of the problem are proved by applying the method of separation of variables and proving the basis of the special function systems of the Samarskii-Ionkin type in L_p. In contrast to these papers in case of $\alpha \neq 0$ it is impossible to use directly the Fourier method of the separation of the variables. Because the corresponding spectral problem for the ordinary differential equation has the system of eigenfunctions not forming a basis.

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2. Uniqueness of the solution

Theorem 2.1. *The solution of problem* (1.1)–(1.4) *is unique.*

Proof. Suppose that there exist two functions $u_1(r,\theta)$ and $u_2(r,\theta)$ satisfying the conditions of the problem (1.1) - (1.4). We show that the function $u(r,\theta) = u_1(r,\theta) - u_2(r,\theta)$ is equal to 0.

Consider the function

$$U(r,\theta) = u(r,\theta) + u(r,\pi-\theta)$$

in $D_1 = \{(r,\theta) : 0 < r < 1,\ 0 < \theta < \pi/2\}$. It is easy to see that

$$\begin{gathered} \Delta U = 0; \\ \frac{\partial U}{\partial \theta}(r,\pi/2) = 0; \\ \frac{\partial U}{\partial \theta}(r,0) = \alpha U(r,0) \quad \text{for } 0 < r < 1; \\ U(1,\theta) = 0 \quad \text{for } 0 \le \theta \le \pi/2. \end{gathered}$$

Since $\alpha > 0$, it follows that $U = 0$ in $\bar{D_1}$ by the maximum principle and the Zaremba-Giraud principle [1, p. 26] for the Laplace equation. This means that $u(r,\theta) = -u(r,\pi-\theta)$, in particular $u(r,0) = u(r,\pi) = 0$ at $r \in [0,1]$. The equality $u(r,\theta) = 0$ in $\bar{D}$ follows from the uniqueness of the solution of the Dirichlet problem for the Laplace equation. The proof of the theorem is complete. □

3. Forming the basis

If solutions to (1.1) satisfying the conditions (1.3), (1.4) are sought in the form

$$u(r,\theta) = R(r)\varphi(\theta),$$

then $R(r) = r^{\sqrt{\lambda}}$, $\operatorname{Re}\sqrt{\lambda} \ge 0$, and for the function $\varphi(\theta)$ we have the spectral problem

$$\begin{gathered} -\varphi''(\theta) = \lambda\varphi(\theta), \quad 0 < \theta < \pi; \\ \varphi(0) = 0, \quad \varphi'(0) = \varphi'(\pi) + \alpha\varphi(\pi). \end{gathered} \tag{3.1}$$

This problem has two groups of eigenvalues. All the eigenvalues are simple and the corresponding system of eigenfunctions does not form the basis in $L_2(0,\pi)$ [2]. However, in [7] a special system of functions is built based of these eigenfunctions which forms the basis. This fact was applied for the solution of the nonlocal initial-boundary problem for the heat equation. In [8] one family of problems simulating the determination of the temperature and density of heat sources from given values of the initial and final temperature is similarly considered.

Let us present the necessary facts from [7]. Problem (3.1) has two groups of eigenvalues $\lambda_k^{(1)} = (2k)^2$, $k = 1,2,\dots$, $\lambda_k^{(2)} = (2\beta_k)^2$, $k = 0,1,2,\dots$. Herein β_k are roots of the equation $tg\beta = \alpha/2\beta$, $\beta > 0$, they satisfy the inequalities $k < \beta_k < k + 1/2$, $k = 0,1,2,\dots$, and two-side estimates are carried out for $\delta_k = \beta_k - k$ where k is large enough,

$$\frac{\alpha}{2k}\big(1 - \frac{1}{2k}\big) < \delta_k < \frac{\alpha}{2k}\big(1 + \frac{1}{2k}\big). \tag{3.2}$$

The eigenfunctions of the problem (3.1) have the form

$$\varphi_k^{(1)}(\theta) = \sin(2k\theta), \quad k = 1,2,\dots; \ \varphi_k^{(2)}(x) = \sin(2\beta_k\theta),\ k = 0,1,2,\dots.$$

This system is almost normed but does not form even an ordinary basis in $L_2(0,\pi)$. The additional system constructed from the previous one

$$\varphi_0(\theta) = (2\beta_0)^{-1}\varphi_0^{(2)}(\theta),$$
$$\varphi_{2k}(\theta) = \varphi_k^{(1)}(\theta),$$
$$\varphi_{2k-1}(\theta) = (\varphi_k^{(2)}(\theta) - \varphi_k^{(1)}(\theta))(2\delta_k)^{-1}, \quad k = 1,2,\dots$$

is a Riesz basis in $L_2(0,\pi)$. Biorthogonal to it, is the system

$$\psi_0(\theta) = 2\beta_0\psi_0^{(2)}(\theta),$$
$$\psi_{2k}(\theta) = \psi_k^{(2)}(\theta) + \psi_k^{(1)}(\theta),$$
$$\psi_{2k-1}(\theta) = 2\delta_k\psi_k^{(2)}(\theta),\ k = 1,2,\dots.$$

This system is constructed from the eigenfunctions

$$\psi_k^{(1)}(\theta) = C_k^{(1)}\cos(2k\theta + \gamma_k), \quad k = 1,2,\dots,$$
$$\psi_k^{(2)}(\theta) = C_k^{(2)}\cos(\beta_k(1-2\theta)), \quad k = 0,1,2,\dots.$$

of the problem conjugated to (3.1). The constants $C_k^{(j)}$ are taken from the biorthogonal relations $\big(\varphi_k^{(j)}, \psi_k^{(j)}\big) = 1$, $j = 1,2$.

If the function $f(\theta)$ is in $C^2[0,\pi]$ and satisfies the boundary conditions of problem (3.1), then its Fourier series by the system $\varphi_k(\theta)$ converges uniformly. We can calculate that

$$\begin{gathered}\varphi_0''(\theta) = -\lambda_0^{(2)}(\theta),\ \varphi_{2k}''(\theta) = -\lambda_k^{(1)}\varphi_{2k}(\theta),\\ \varphi_{2k-1}''(\theta) = -\lambda_k^{(2)}\varphi_{2k-1}(\theta) - \frac{\lambda_k^{(2)} - \lambda_k^{(1)}}{2\delta_k}\varphi_{2k}(\theta).\end{gathered} \tag{3.3}$$

4. Construction of the formal solution to the problem

Considering section 3, we can write any solution of (1.1)–(1.4) in the form of a biorthogonal series

$$u(r,\theta) = \sum_{k=0}^{\infty} R_k(r)\varphi_k(\theta), \tag{4.1}$$

where $R_k(r) = (u(r,\cdot)$ and $\psi_k(\cdot)) \equiv \int_0^\pi u(r,\theta)\psi_k(\theta)d\theta$. Functions (4.1) satisfy the boundary conditions (1.3) and (1.4).

Substituting (4.1) in (1.1) and the boundary conditions (1.2), taking into account (3.3), for finding unknown functions $R_k(r)$ we obtain the following problems

$$\begin{gathered}r^2R_0''(r) + rR_0'(r) - \lambda_0^{(2)}R_0(r) = 0,\\[2ex] r^2R_{2k-1}''(r) + rR_{2k-1}'(r) - \lambda_k^{(2)}R_{2k-1}(r) = 0,\\[2ex] r^2R_{2k}''(r) + rR_{2k}'(r) - \lambda_k^{(1)}R_{2k}(r) = \frac{\lambda_k^{(2)} - \lambda_k^{(1)}}{2\delta_k}R_{2k-1}(r),\end{gathered} \tag{4.2}$$

with the boundary conditions $R_k(1) = f_k$, where f_k are the Fourier coefficients of the expansion of the function $f(\theta)$ into the biorthogonal series by $\varphi_k(\theta)$.

The regular solution of (4.2) exists, is unique and can be written in the explicit form

$$\begin{gathered}
R_0(r) = f_0 r^{\sqrt{\lambda_0^{(2)}}},\ R_{2k-1}(r) = f_{2k-1} r^{\sqrt{\lambda_k^{(2)}}}, \\
R_{2k}(r) = f_{2k} r^{\sqrt{\lambda_k^{(1)}}} + f_{2k-1} \frac{r^{\sqrt{\lambda_k^{(2)}}} - r^{\sqrt{\lambda_k^{(1)}}}}{2\delta_k}.
\end{gathered} \tag{4.3}$$

Substituting (4.3) in (4.1), we obtain a formal solution

$$\begin{aligned}
u(r,\theta) = f_0 \frac{r^{2\beta_0}}{2\beta_0} \sin(2\beta_0\theta) &+ \sum_{k=1}^{\infty} f_{2k-1} \frac{r^{2k}}{2\delta_k} [r^{2\delta_k} \sin(2(k+\delta_k)\theta) - \sin(2k\theta)] \\
&+ \sum_{k=1}^{\infty} f_{2k} r^{2k} \sin(2k\theta).
\end{aligned} \tag{4.4}$$

5. Main Theorem

Our main result reads as follows.

Theorem 5.1. *If $f(\theta) \in C^2[0,\pi]$, $f(0) = 0$, $f'(0) = f'(\pi) + \alpha f(\pi)$, then there exists a unique classical solution $u(r,\theta) \in C^0(\bar{D}) \cap C^2(D)$ of problem* (1.1)–(1.4).

Proof. The uniqueness of the classical solution of the problem follows from Theorem 2.1. The formal solution of the problem is shown in the form of (4.4). To make sure that these functions are really the desired solutions we need to verify the applicability of the superposition principle. For it we need to show the convergence of the series, the possibility of termwise differentiation, and to prove the continuity of these functions on the boundary of the half-disk.

The possibility of differentiating the series (4.4) any number of times at $r < 1$ is an obvious consequence of the convergence of power series and two-sided estimates (3.2) for δ_k. Let us justify the uniform convergence of the series (4.1) at $r \le 1$. For this we use the sign of the uniform convergence of Weierstrass. By direct calculation it is easy to see that the series (4.4) is majorized by the series $C_1(|f_0| + |f_1| + |f_2| + \dots)$. This series converges [7] due to the requirements of the theorem imposed on $f(\theta)$. Since all the terms of the series (4.4) are continuous functions, then the function $u(r,\theta)$ is continuous in the boundary domain $\bar{D}$. The proof is complete. □

6. Conjugated problem: existence and uniqueness of the solution

Let us now formulate a problem conjugated to (1.1)-(1.4). We look for a function $v(r,\theta) \in C^0(\bar{D}) \cap C^2(D)$ satisfying the equation

$$\Delta v = 0 \tag{6.1}$$

in D with the boundary conditions

$$v(1,\theta) = g(\theta), \quad 0 \le \theta \le \pi, \tag{6.2}$$

$$v(r,0) = v(r,\pi), \quad r \in [0,1], \tag{6.3}$$

$$\frac{\partial v}{\partial \theta}(r,\pi) + \alpha v(r,\pi) = 0, \quad r \in (0,1), \tag{6.4}$$

where $g(\theta) \in C^2[0,\pi]$, $g(0) = g(\pi)$, $g'(\pi) + \alpha g(\pi) = 0$.

We can easily verify the conjugacy of the problems (1.1)–(1.4) and (6.1)–(6.4) by direct calculation. The uniqueness of the solution of problem (6.1)–(6.4) follows from the maximum principle and the Zaremba-Giraud principle [1, p. 26] for the Laplace equation. The existence of the solution and its representation in the form of a biorthogonal series can be proved similar to Theorem 5.1. Let us show this result without the proof.

Theorem 6.1. *If $g(\theta) \in C^2[0,\pi]$,$g(0) = g(\pi)$, $g'(\pi) + \alpha g(\pi) = 0$, then there exists a unique classical solution $v(r,\theta) \in C^0(\bar{D}) \cap C^2(D)$ of problem* (6.1)-(6.4).

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