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FEYNMAN-KAC THEOREM IN HILBERT SPACES

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Abstract. In this article we study the relationship between solutions to Cauchy problems for the abstract stochastic differential equation $dX(t) = AX(t)dt + BdW(t)$ and solutions to Cauchy problems (backward and forward) for the infinite dimensional deterministic partial differential equation

$$\pm\frac{\partial g}{\partial t}(t,x)+\frac{\partial g}{\partial x}(t,x)Ax+\frac{1}{2}\operatorname{Tr}[(BQ^{1/2})^*\frac{\partial^2 g}{\partial x^2}(t,x)(BQ^{1/2})]=0,$$

where g is the probability characteristic $g = \mathbb{E}^{t,x}[h(X(T))]$ in the backward case and $g = \mathbb{E}^{0,x}[h(X(t))]$ in the forward case. This relationship, that is the inifinite dimensional Feynman-Kac theorem, is proved in both directions: from stochastic to deterministic and from deterministic to stochastic. Special attention is given to the definition and interpretation of objects in the equations.

1. Introduction

Many practical problems lead to stochastic equations and it is not always possible or necessary to find their solutions, sometimes is sufficient to have only probability characteristics of the solutions.

The famous Feynman-Kac theorem allows to pass from solving stochastic differential equations to deterministic partial differential equations for probability characteristics in the finite dimensional case. It relates solutions of the Cauchy problem for stochastic differential equations with Brownian motion $\beta(t)$, $t \ge 0$:

$$dX(t) = a(t,X(t))dt + b(t,X(t))d\beta(t), \quad t \in [0,T],\ X(0) = \xi, \tag{1.1}$$

and solutions of the Cauchy problem for deterministic partial differential equations

$$g_t(t,x) + a(t,x)g_x(t,x) + \frac{1}{2}b^2(t,x)g_{xx}(t,x) = 0, \quad g(T,x) = h(x), \tag{1.2}$$

for the probability characteristic $g(t,x) = \mathbb{E}^{t,x}[h(X(T))]$ with an arbitrary Borel function h. Here $\mathbb{E}^{t,x}$ means the mathematical expectation of a solution to the equation (1.1) under the condition $X(t) = x$, $0 \le t \le T$.

The study of the relationship between problems (1.1)–(1.2) was initially caused by needs from physics. For example, the process $\{X(t), t \in [0,T]\}$ describes the

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random motion of particles in a liquid or gas and $g(t,x)$ is a probability characteristic such as temperature, determined by the Kolmogorov equation. Recent years the importance of the relationship between stochastic and deterministic problems became more acute with the development of numerical methods (see, e.g. [13]) and applications in financial mathematics. For example, $X(t)$ is a stock price at time t, then $g(t,x)$ is the value of stock options, determined by the famous Black-Scholes equation [3, 8, 16].

Along with applications to mathematical physics and financial mathematics in finite dimensional case (see, e.g. [1, 3, 8, 14, 16]) there exist recent infinite dimensional applications, for instance, stochastic equations in financial mathematics [10, 11]. As an example, let $P(t,T)$ be a price at time $t \le T$ of a coupon bond with maturity date T parametrized as $P(t,t) = 1$ for all t and let $f(t,T), t \le T$ be the forward curve, i.e. $P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$. Then the Musiela reparametrization $r(t,\cdot) := f(t,\cdot + t)$ in the special case of zero HJM-shift satisfies the following equation in Hilbert space $H = L_2(\mathbb{R}_+)$

$$dr(t) = Ar(t)dt + b(t,r(t))dW(t), \quad t \ge 0, \quad r(0) = \xi, \tag{1.3}$$

where A is the generator of the right-shifts semigroup in H, W is an $\mathbb{H}$-valued Q-Wiener process, and b is a random mapping from a Hilbert space $\mathbb{H}$ to H. The value of bond options may be calculated, at least numerically, via $g(t,x)$ defined for $X(t) = r(t)$. So the relationship between infinite dimensional stochastic and deterministic problems is important both in theory and applications. An extension of the Feynman-Kac theorem to the infinite dimensional case raises many questions related with the very formulation of the problems in infinite dimensional spaces, the definition of relevant objects and a rigorous rationale for the relationship.

The present article concerns the stochastic Cauchy problem in Hilbert spaces

$$dX(t) = AX(t)dt + BdW(t), \quad t \in [0,T], \quad X(0) = \xi. \tag{1.4}$$

We prove the infinite dimensional case of the Feynman-Kac theorem under the basic condition on A to be the generator of a C_0-semigroup in a Hilbert space H. We suppose $B \in \mathcal{L}(\mathbb{H}, H)$ in the case of an $\mathbb{H}$-valued Q-Wiener process W and $B \in \mathcal{L}_{\mathtt{HS}}(\mathbb{H}, H)$ in the case of a cylindrical Wiener process W. For the problem (1.4) we associate a problem for a deterministic partial differential equation which is an extension of the problem (1.2) in the case of Hilbert spaces:

$$\frac{\partial g}{\partial t}(t,x) + \frac{\partial g}{\partial x}(t,x)Ax + \frac{1}{2}\operatorname{Tr}\left[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(t,x)(BQ^{1/2})\right] = 0, g(T,x) = h(x), \tag{1.5}$$

and show that g satisfies the infinite dimensional deterministic problem (1.5) with the trace class operator Q in the case of Q-Wiener process and with $Q = I$ in the case of cylindrical Wiener process. We give the rigorous interpretation of objects included in stochastic and deterministic equations and prove the connection between their solutions in both directions: "from stochastic to deterministic" and "from deterministic to stochastic".

The implication "from stochastic to deterministic" consists of several steps. The first step is the proof of the Markov property for the Cauchy problem solution X, then the martingal property for the function $g(t,x)|_{x=X(t)}$ and the last step is the formal usage of infinite dimensional Ito's formula for $g(t,X(t))$. Particular attention is paid to the subtle issue of transition from zero expectation for a function of g to equality for g itself. The implication "from deterministic to stochastic" also uses the infinite dimensional Ito's formula for $g(t,X(t))$ too.

Comparing with our previous article [12] in the current paper we proved the implication "from deterministic to stochastic" and improved the proof of the implication "from stochastic to deterministic", we proved the relationship for the stochastic problem (1.4) with Q-Wiener and cylindrical Wiener processes, and at last along with $g(t,x)=\mathbb{E}^{t,x}[h(X(T))]$ we introduce a different probability characteristic $\tilde{g}(t,x)=\mathbb{E}^{0,x}[h(X(t))]$ and show that g leads to the backward deterministic Cauchy problem and $\tilde{g}$ leads to the forward deterministic Cauchy problem.

2. Definitions and auxiliary statements

We start with interpretation for objects of the stochastic problem (1.4). Let the operator A be the generator of a C_0-semigroup in Hilbert space H. This ensures uniform well-posedness of the Cauchy problem for the corresponding homogeneous equation $X'(t)=AX(t)$ and existence of strongly continuous solution operators $U(t), t\ge 0$ to the homogeneous problem, as well as existence and uniqueness of a weak solution to the stochastic problem (1.4) with an $\mathbb{H}$-valued Wiener process W (see, e.g. [2, 5]):

$$X(t)=U(t)\xi+W_A(t)=U(t)\xi+\int_0^t U(t-s)BdW(s),\quad t\in[0,T].$$

The stochastic convolution $W_A(t)$ with respect to Wiener process (as Q-Wiener as cylindrical Wiener) formally is defined under the following condition

$$\mathbb{E}\Big[\int_0^t \|U(s)B\|^2_{\mathrm{HS}}ds\Big]<\infty, \tag{2.1}$$

where $\|U(s)B\|^2_{\mathrm{HS}}:=\operatorname{Tr}[U(s)BQ(U(s)B)^*]$. In the case of a Q-Wiener process the operator Q is a trace class operator and for the validity of (2.1) it is sufficient the operators $U(s)B$, $s\in[0,T]$, hence B, to be bounded. In the case of cylindrical Wiener process the operator Q is only bounded operator with $\operatorname{Tr}Q=\infty$ (for simplicity we will suppose $Q=I$) and for the validity of (2.1) it is sufficient the operators $U(s)B$, $s\in[0,T]$, hence B, to be Hilbert-Schmidt operator.

Now we give the interpretation for the objects of the deterministic partial differential equation (1.5). Define the function $g(t,x):=\mathbb{E}^{t,x}[h(X(T))]$ that transforms $[0,T]\times H$ into $\mathbb{R}$, supposing h a measurable function from H to $\mathbb{R}$. We show that g satisfies the infinite dimensional deterministic Cauchy problem (1.5) corresponding to the stochastic one (1.4).

As we mentioned above, in the case of cylindrical Wiener process W the operator Q is equal to I and the problem (1.5) becomes as follows

$$\frac{\partial g}{\partial t}(t,x)+\frac{\partial g}{\partial x}(t,x)Ax+\frac{1}{2}\operatorname{Tr}\big[B^*\frac{\partial^2 g}{\partial x^2}(t,x)B\big]=0,\quad g(T,x)=h(x). \tag{2.2}$$

In this case condition (2.1) on operator B guarantees that operator under the trace sign, $B^*\frac{\partial^2 g}{\partial x^2}(t,x)B$, is really of trace class.

First we make sense to terms in (1.5). The derivatives $\frac{\partial g}{\partial x}$ and $\frac{\partial^2 g}{\partial x^2}$ are understood in the sense of Frechet; that means $\frac{\partial g}{\partial x}:[0,T]\times H\to H^*$ and $\frac{\partial^2 g}{\partial x^2}:[0,T]\times H\to$

$\mathcal{L}(H, H^*)$. More precisely

$$\frac{\partial g}{\partial x}(t,x)(\cdot) : H \to \mathbb{R}, \quad \frac{\partial^2 g}{\partial x^2}(t,x)(\cdot) : H \to H^*, \quad \text{for any fixed } t \in [0,T],\, x \in H,$$
$$BQ^{1/2} : \mathbb{H} \to H, \quad (BQ^{1/2})^* : H^* \to \mathbb{H}^*.$$

The term $\mathrm{Tr}[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(BQ^{1/2})]$ requires special attention. Expression Tr is usually defined as the trace of an operator acting in the same Hilbert space. The operator under the trace sign in equation (1.5) maps Hilbert space $\mathbb{H}$ to its adjoint $\mathbb{H}^*$.

Using the traditional definition of the trace and the Riesz theorem on the isomorphism $\mathbb{H}$ and $\mathbb{H}^*$; that is identifying $\mathbb{H}^*$ with $\mathbb{H}$, we can make sense to the trace sign. Note that the isomorphism allows us to consider operators $BQ^{1/2}$, $(BQ^{1/2})^*$, and $\frac{\partial^2 g}{\partial x^2}$ as mappings from $\mathbb{H}$ to H, from H to $\mathbb{H}$, and H to H, respectively. Then operator $(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(BQ^{1/2})$ transfers the Hilbert space $\mathbb{H}$ to $\mathbb{H}$ and trace of this operator can be understood in the usual sense: in the case of a Q-Wiener process with a trace class operator Q, bounded operators $\frac{\partial^2 g}{\partial^2 x}$ and B we have

$$\begin{aligned} \big| \,\mathrm{Tr}\,\big[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(BQ^{1/2})\big]\big| &\le \sum_{j=1}^{\infty} \big|\big\langle \frac{\partial^2 g}{\partial x^2}(BQ^{1/2})e_j, (BQ^{1/2})e_j \big\rangle\big| \\ &\le \sum_{j=1}^{\infty} \sigma_j^2 \|B\|^2 \|\frac{\partial^2 g}{\partial x^2}\|^2 < \infty. \end{aligned}$$

In the case of a cylindrical Wiener process the estimate for $\mathrm{Tr}[B^* \frac{\partial^2 g}{\partial x^2} B]$ takes place for the bounded operator $\frac{\partial^2 g}{\partial^2 x}$ and a Hilbert-Schmidt operator B.

3. From stochastic to deterministic

At first we prove necessary properties of the process X that is a solution of (1.4), and the function $g(t,x)$ that determines the relationship between solutions of problems (1.4) and (1.5). We obtain required properties for the case of more general processes, diffusion processes, to which the solution of the equation (1.4) is a special case.

An H-valued Ito process $\{X(t), t \ge 0\}$ is called diffusion if it can be written in the form

$$dX(t) = a(X(t))dt + b(X(t))dW(t), \tag{3.1}$$

where a and b are some measurable mappings. Consider the Cauchy problem for the equation (3.1),

$$dX(t) = a(X(t))dt + b(X(t))dW(t), \quad t \in [0,T], \quad X(0) = \xi. \tag{3.2}$$

In this article we consider only diffusion processes such that existence and uniqueness of a solution to the stochastic Cauchy problem (3.2) take place. It may be reached by different ways. For example, it is guaranteed by the estimate to coefficients a and b: $\|a(z_1) - a(z_2)\| + \|b(z_1) - b(z_2)\| \le c\|z_1 - z_2\|$, $z_1, z_2 \in H$, $c \in \mathbb{R}$ (see [6, Theorem 2.1, ch. VII]. In this article we are not interested in a concrete form of conditions, we only suppose that existence and uniqueness of the Cauchy problem solution hold.

As pointed out in previous section this unique solution can be written as a sum of the term depending on the initial data and the stochastic convolution term. To prove the infinite dimensional Feynman-Kac theorem it is important to establish the Markov property for the solution of the Cauchy problem (3.2). The following statement is an extension of the finite dimensional case result (see [14, Theorem 7.1.2]) to the case of Hilbert spaces.

Proposition 3.1. *Let $h = h(z)$, $z \in H$ be Borel-measurable and $X = X(t)$, $t \in [0,T]$, be a unique solution of* (3.2). *Then X satisfies the Markov property with respect to the filter $\mathfrak{F}_t$, $t \geq 0$ defined by Wiener process W:*

$$\mathbb{E}\left[h(X(t+s))|\mathfrak{F}_t\right] = \mathbb{E}^{0,X(t)}[h(X(s))]. \tag{3.3}$$

Proof. Let $X^{t,x}$, $t \in [0,T]$, $x \in H$ be the solution of the Cauchy problem for the equation (3.1) with the initial data $X(t) = x$. In this notation X, which is the solution of the Cauchy problem (3.2), may be written as $X^{0,\xi}$. It is easy to see that a constriction of $X(\tau)$ to the segment $\tau \in [t,T]$ is also a solution of the Cauchy problem for the equation (3.1) with the initial data $x = X(t)$. By the uniqueness of a solution to the Cauchy problem (3.2) we have $X(\tau) = X^{t,X(t)}(\tau)$, $\tau \geq t$, almost surely. So we have

$$\mathbb{E}[h(X(\tau))|\mathfrak{F}_t] = \mathbb{E}[h(X^{t,X(t)}(\tau))|\mathfrak{F}_t] = \mathbb{E}[h(X^{t,X(t)}(\tau))] = \mathbb{E}[h(X(\tau))]. \tag{3.4}$$

The first and the last equalities are consequences of the uniqueness of a solution (3.2). The validity of the second equality follows from the independence $X^{t,x}(\tau)$ from $\mathfrak{F}_t$ because t is a starting point for the solution $X^{t,x}$.

Suppose s is defined as $\tau = t + s$. Then

$$\mathbb{E}[h(X(t+s))|\mathfrak{F}_t] = \mathbb{E}[h(X(t+s))] = \mathbb{E}[h(X^{t,X(t)}(t+s))] = \mathbb{E}[h(X^{t,z}(t+s))]_{z=X(t)}. \tag{3.5}$$

Here the first equality follows from (3.4). The second equality is a consequence of solution's uniqueness.

Using the diffusion property of process X, we obtain

$$h(X^{t,z}(t+s)) = h(X^{0,z}(s))$$

and hence the equality for mathematical expectations

$$\mathbb{E}[h(X^{t,z}(t+s))]_{z=X(t)} = \mathbb{E}[h(X^{0,z}(s))]_{z=X(t)} \tag{3.6}$$

is also valid. Equalities (3.5) and (3.6) imply the desired relation (3.3). $\square$

Corollary 3.2. *Suppose a process X is a solution of* (1.4). *Then it is unique, diffusive, and by Proposition 3.1 has the Markov property.*

By the homogeneity in time of diffusion processes, equation (3.6) can be written in the ensuing form:

$$\mathbb{E}^{0,X(t)}[h(X(s))] = \mathbb{E}^{t,X(t)}[h(X(t+s))]. \tag{3.7}$$

As a consequence of Proposition 3.1 and (3.7) we obtain the following result.

Corollary 3.3. *Let $h(z), z \in H$ be Borel-measurable and $X(t)$, $t \geq 0$, be a unique solution of* (3.2). *Then the Markov property for X may be rewritten as follows:*

$$\mathbb{E}[h(X(t+s))|\mathfrak{F}_t] = \mathbb{E}^{t,X(t)}[h(X(t+s))].$$

On the basis of Markov property for X we prove the martingale property for g, useful for the proof of Feynman-Kac theorem. The following statement generalizes [15, Theorem 5.50].

Proposition 3.4. *Suppose a process X satisfies conditions of Proposition 3.1. Then the process $g(t,X(t)) := \mathbb{E}^{t,x}[h(X(T))]|_{x=X(t)}$ is martingale; i.e.,*

$$\mathbb{E}\left[g(\tau,X(\tau))|\mathfrak{F}_t\right] = g(t,X(t)), \quad 0 \le t \le \tau \le T.$$

Proof. According to Proposition 3.1, X has the Markov property. Therefore

$$\mathbb{E}\big[h(X(T))|\mathfrak{F}_\tau\big] = \mathbb{E}^{\tau,X(\tau)}[h(X(T))] = g(\tau,X(\tau)),$$

and we obtain

$$\begin{aligned}\mathbb{E}[g(\tau,X(\tau))|\mathfrak{F}_t] &= \mathbb{E}[\mathbb{E}[h(X(T))|\mathfrak{F}_\tau]|\mathfrak{F}_t] = \mathbb{E}[h(X(T))|\mathfrak{F}_t]\\ &= \mathbb{E}^{t,X(t)}[h(X(T))] = g(t,X(t)).\end{aligned}$$

The first equality implies the obtained representation for the process $g(\tau,X(\tau))$ via the conditional expectation. The second equality follows from properties of conditional expectation. The third one is the direct consequence of the Markov property for X. The last equality follows from the definition of the process $g(t,X(t))$ and completes the proof. □

Now we can prove the connection between problems (1.4) and (1.5).

Theorem 3.5. *Consider the stochastic Cauchy problem* (1.4) *where A is the generator of a C_0-semigroup in a Hilbert space H, $B \in \mathcal{L}(\mathbb{H},H)$ in the case of a Q-Wiener process W and $B \in \mathcal{L}_{\mathrm{HS}}(\mathbb{H},H)$ in the case of a cylindrical Wiener process W. Define $g = g(t,x) := \mathbb{E}^{t,x}[h(X(T))] : [0,T]\times H \to \mathbb{R}$, h is a measurable function from H to $\mathbb{R}$. Suppose $\mathbb{E}^{t,x}|h(X(T))| < \infty$ and derivatives $\frac{\partial g}{\partial t}$, $\frac{\partial g}{\partial x}$, $\frac{\partial^2 g}{\partial x^2}$ exist for all pairs (t,x). Then g is a solution of the infinite dimensional backward Cauchy problem* (1.5) *in the case of Q-Wiener process and the problem* (2.2) *in the case of cylindrical Wiener process.*

Proof. Let X be a solution of (1.4). Fix some $\tau \in [0,T]$ and consider the stochastic Cauchy problem with the initial data $x = X(\tau)$:

$$dX^{\tau,x}(t) = AX^{\tau,x}(t)dt + BdW(t), \quad t \in [\tau,T], \quad X^{\tau,x}(\tau) = x. \tag{3.8}$$

Then $X^{\tau,x}$ is a constriction of X on the segment $[\tau,T]$; i.e., they coincide on $[\tau,T]$. It means that g also coincide for Cauchy problems (1.4) and (3.8). So further we will simply write X instead of $X^{\tau,x}$.

Applying the Ito formula in Hilbert spaces [5] to g as a function from the solution of the problem (3.8) we obtain

$$\begin{aligned}dg(t,X(t)) = \frac{\partial g}{\partial x}(t,X(t))BdW(t) + \Big(&\frac{\partial g}{\partial t}(t,X(t)) + \frac{\partial g}{\partial x}(t,X(t))AX(t)\\ &+ \frac{1}{2}\operatorname{Tr}\big[(BQ^{1/2})^*\frac{\partial^2 g}{\partial x^2}(t,X(t))(BQ^{1/2})\big]\Big)dt.\end{aligned}$$

This equality is written in the form of differentials (increments). In the integral form it can be written as

$$\begin{aligned}g(t,X(t)) = g(\tau,x) + \int_\tau^t \frac{\partial g}{\partial x}(s,X(s))BdW(s) + \int_\tau^t \Big(&\frac{\partial g}{\partial s}(s,X(s))\\ + \frac{\partial g}{\partial x}(s,X(s))AX(s) + \frac{1}{2}\operatorname{Tr}&\big[(BQ^{1/2})^*\frac{\partial^2 g}{\partial x^2}(s,X(s))(BQ^{1/2})\big]\Big)ds.\end{aligned}$$

Apply the expectation to both sides of the equation. From the definition of an Ito integral (via the approximation in the mean square by step processes) and the properties of the Q-Wiener process, we obtain $\mathbb{E}[\int_\tau^t \frac{\partial g}{\partial x}(s, X(s))BdW(s)] = 0$. Further, since the process g is martingale, we have

$$\mathbb{E}[g(t, X(t))] = \mathbb{E}[g(t, X(t))|\mathfrak{F}_\tau] = g(\tau, x). \tag{3.9}$$

Hence using the theorem of Tonelli-Fubbini in Hilbert spaces and equalities (3.9) we conclude that

$$\begin{aligned} 0 &= \mathbb{E}\Big[\int_\tau^t \Big(\frac{\partial g}{\partial s}(s, X(s)) + \frac{\partial g}{\partial x}(s, X(s))AX(s) \\ &\quad + \frac{1}{2}\operatorname{Tr}\big[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(s, X(s))(BQ^{1/2})\big]\Big) ds\Big] \\ &= \int_\tau^t \mathbb{E}\Big[\frac{\partial g}{\partial s}(s, X(s)) + \frac{\partial g}{\partial x}(s, X(s))AX(s) \\ &\quad + \frac{1}{2}\operatorname{Tr}\big[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(s, X(s))(BQ^{1/2})\big]\Big] ds. \end{aligned}$$

The above equality is valid for all $t \in [\tau, T]$. Therefore

$$\mathbb{E}\Big[\frac{\partial g}{\partial t}(t, X(t)) + \frac{\partial g}{\partial x}(t, X(t))AX(t) + \frac{1}{2}\operatorname{Tr}\big[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(t, X(t))(BQ^{1/2})\big]\Big] = 0.$$

Rewrite this equality at the origin point (τ, x), as

$$\mathbb{E}\Big[\frac{\partial g}{\partial t}(\tau, x) + \frac{\partial g}{\partial x}(\tau, x)Ax + \frac{1}{2}\operatorname{Tr}\big[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(\tau, x)(BQ^{1/2})\big]\Big] = 0;$$

that is,

$$\mathbb{E}[\frac{\partial g}{\partial t}(\tau, x)] + \mathbb{E}[\frac{\partial g}{\partial x}(\tau, x)Ax] + \frac{1}{2}\mathbb{E}\big[\operatorname{Tr}[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(\tau, x)(BQ^{1/2})]\big] = 0.$$

Note that Ax does not depend on ω, thus

$$\mathbb{E}\big[\frac{\partial g}{\partial x}(\tau, x)Ax\big] = \mathbb{E}\big[\frac{\partial g}{\partial x}(\tau, x)\big]Ax.$$

Using fact that mappings $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$ are independent of the variable ω, and that the expectation $\mathbb{E}$ is an integral of the variable ω, we conclude that all these operators commute with the operator $\mathbb{E}$. Furthermore, according to the interpretation of trace given in section 2, it also commutes with the operator $\mathbb{E}$ by following arguments:

$$\begin{aligned} \mathbb{E}\big[\operatorname{Tr}\big[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(\tau, x)(BQ^{1/2})\big]\big] &= \mathbb{E}\Big[\sum_{k=1}^\infty \langle (BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(\tau, x)(BQ^{1/2})e_j, e_j\rangle\Big] \\ &= \big\langle \mathbb{E}\big[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(\tau, x)(BQ^{1/2})e_j\big], e_j\big\rangle \\ &= \operatorname{Tr}\mathbb{E}\big[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(\tau, x)(BQ^{1/2})\big]. \end{aligned}$$

Note

$$\mathbb{E}[g(\tau, x)] = \mathbb{E}[\mathbb{E}^{\tau,x}[h(X(T))]] = \mathbb{E}\big[\mathbb{E}[h(X(T))]\big] = \mathbb{E}[h(X(T))] = g(\tau, x).$$

Hence we obtain

$$\frac{\partial g}{\partial \tau}(\tau, x) + \frac{\partial g}{\partial x}(\tau, x)Ax + \frac{1}{2}\operatorname{Tr}\big[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(\tau, x)(BQ^{1/2})\big] = 0. \tag{3.10}$$

Varying $\tau \in [0,T]$ we obtain (3.10) for all pairs $\{(\tau,x) : x = X(\tau)\}$. It remains to see that

$$g(T,x) := \mathbb{E}^{T,x}[h(X(T))] = h(X(T))|_{x=X(T)} = h(x),$$

which completes the proof. □

Remark 3.6. Conditions on the function $g(t,x) = E^{t,x}[h(X(T))]$ for having all derivatives in the differential equation (1.5) are not usually supplied by applications. In the case of function h being not enough smooth to ensure the conditions, some type of generalized problem (1.5) have to be considered.

In [9] in finite dimensional case and [4] in infinite dimensional case it was shown that $\{R_t, t \geq 0\}$ ($R_t h(x) := \mathbb{E}^{0,x}[h(X(t))]$) forms a semigroup. According to this the Feynman-Kac theorem may be proved using the semigroup techniques. Probably it will allow to weaken the conditions and it would be the subject of our future research.

Remark 3.7. As we pointed out in the previous section $Q = I$ in the case of cylindrical Wiener process. Further we will not focus on this fact and write equations with operator Q for both types of Wiener processes.

In the previous theorem we deduced backward Kolmogorov problem. It is not some special feature of the relationship, we show that the probability characteristic $\tilde{g}$ leads to the forward Kolmogorov problem.

Theorem 3.8. *Consider the stochastic Cauchy problem* (1.4) *where A is the generator of a C_0-semigroup in a Hilbert space H, $B \in \mathcal{L}(\mathbb{H}, H)$ in the case of Q-Wiener process W and $B \in \mathcal{L}_{\mathrm{HS}}(\mathbb{H}, H)$ in the case of cylindrical Wiener process W. Define $\tilde{g}(t,x) := \mathbb{E}^{0,x}[h(X(t))] : [0,T] \times H \to \mathbb{R}$, h is a measurable function from H to $\mathbb{R}$. Suppose $\mathbb{E}^{t,x}|h(X(T))| < \infty$, derivatives $\frac{\partial \tilde{g}}{\partial t}$, $\frac{\partial \tilde{g}}{\partial x}$ and $\frac{\partial^2 \tilde{g}}{\partial x^2}$ exist for all pairs (t,x). Then $\tilde{g}(t,x)$ is a solution of the infinite dimensional forward Cauchy problem*

$$\frac{\partial \tilde{g}}{\partial t}(t,x) = \frac{\partial \tilde{g}}{\partial x}(t,x)Ax + \frac{1}{2}\operatorname{Tr}[(BQ^{1/2})^* \frac{\partial^2 \tilde{g}}{\partial x^2}(t,x)(BQ^{1/2})] = 0, \quad \tilde{g}(0,x) = h(x). \tag{3.11}$$

Proof. Let $s \in [0,T]$ be defined as $T = t + s$. Using (3.7), homogeneity in time of diffusion process X, and definitions of g and $\tilde{g}$ we have

$$\tilde{g}(t,x) = \mathbb{E}^{0,x}[h(X(t))] = \mathbb{E}^{0,x}[h(X(T-s))] = \mathbb{E}^{s,x}[h(X(T))] = g(s,x).$$

So we have $\frac{\partial \tilde{g}}{\partial t} = -\frac{\partial g}{\partial s}$ and

$$\frac{\partial \tilde{g}}{\partial s}(s,x) = \frac{\partial \tilde{g}}{\partial x}(s,x)Ax + \frac{1}{2}\operatorname{Tr}\left[(BQ^{1/2})^* \frac{\partial^2 \tilde{g}}{\partial x^2}(s,x)(BQ^{1/2})\right] = 0.$$

Boundary condition $g(T,x) = h(x)$ can be written in the form $\tilde{g}(0,x) = h(x)$ that completes the proof. □

4. From deterministic to stochastic

The statement in the opposite direction is also valid.

Theorem 4.1. *Let g be the solution of the deterministic Cauchy problem* (1.5)

$$\frac{\partial g}{\partial t}(t,x) + \frac{\partial g}{\partial x}(t,x)Ax + \frac{1}{2}\operatorname{Tr}\left[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(t,x)(BQ^{1/2})\right] = 0, \quad g(T,x) = h(x),$$

where A is the generator of a C_0-semigroup in a Hilbert space H, $B \in \mathcal{L}(\mathbb{H}, H)$ or $B \in \mathcal{L}_{\mathtt{HS}}(\mathbb{H}, H)$. Suppose process X be a weak solution of the stochastic Cauchy problem (1.4)

$$dX(t) = AX(t)dt + BdW(t), \quad t \in [0,T], \quad X(0) = \xi.$$

where W is an $\mathbb{H}$-valued Q-Wiener process in the case of $B \in \mathcal{L}(\mathbb{H}, H)$ and W is a cylindrical Wiener process in the case of $B \in \mathcal{L}_{\mathtt{HS}}(\mathbb{H}, H)$. Then the equality $g(t,x) = \mathbb{E}^{t,x}[h(X(T))]$ between solutions of Cauchy problems (1.5) *and* (1.4) *takes place.*

Proof. Let X be a solution of (1.4). Fix some $\tau \in [0,T]$ and consider the stochastic Cauchy problem with the initial data $x = X(\tau)$:

$$dX(t) = AX(t)dt + BdW(t), \quad t \in [\tau,T], \quad X(\tau) = x.$$

As we pointed out earlier the solution of this problem is the constriction of the solution (1.4) on the segment $[\tau, T]$. Applying the Ito formula in a Hilbert space to $g(t, X(t))$, $t \in [\tau, T]$, where g is the solution for the Cauchy problem (1.5), we have

$$\begin{aligned} g(T, X(T)) = g(\tau, X(\tau)) + \int_\tau^T \frac{\partial g}{\partial x}(s, X(s))BdW(s) + \int_\tau^T \Big(\frac{\partial g}{\partial s}(s, X(s)) \\ + \frac{\partial g}{\partial x}(s, X(s))AX(s) + \frac{1}{2}\operatorname{Tr}\big[(BQ^{1/2})^* \frac{\partial^2 g}{\partial x^2}(s, X(s))(BQ^{1/2})\big]\Big)ds. \end{aligned}$$

Since g is the solution for the Cauchy problem (1.5), we have

$$g(T, X(T)) = g(\tau, X(\tau)) + \int_\tau^T \frac{\partial g}{\partial x}(s, X(s))BdW(s).$$

Take the mathematical expectation both sides of this equation. From the definition of Ito integral (via the approximation in the mean square by step processes) and properties of the Q-Wiener process, we obtain $\mathbb{E}[\int_\tau^T \frac{\partial g}{\partial x}(s, X(s))BdW(s)] = 0$. Therefore,

$$\mathbb{E}^{\tau,x}[g(T, X(T))] = \mathbb{E}^{\tau,x}[g(\tau, X(\tau))].$$

Rewrite both sides of this equation. According the fact T is the end point of time for the Cauchy problem (1.5) we obtain the reformulation of the left side for this equation

$$\mathbb{E}^{\tau,x}[g(T, X(T))] = \mathbb{E}^{\tau,x}[h(X(T))].$$

On the other hand we may rewrite the right side of this equation

$$\mathbb{E}^{\tau,x}[g(\tau, X(\tau))] = \mathbb{E}^{\tau,x}[g(\tau, x)] = g(\tau, x).$$

As a result we have

$$\mathbb{E}^{\tau,x}[h(X(T))] = g(\tau, x).$$

Varying τ we have this equality on the segment $[0,T]$; this completes the proof. □

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