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SUPERSTABILITY OF DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

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Abstract. In this article, we establish the superstability of differential equations of second order with boundary conditions or with initial conditions as well as the superstability of differential equations of higher order with initial conditions.

1. Introduction

In 1940, Ulam [28] posed a problem concerning the stability of functional equations: "Give conditions in order for a linear function near an approximately linear function to exist."

A year later, Hyers [7] gave an answer to the problem of Ulam for additive functions defined on Banach spaces: Let X_1 and X_2 be real Banach spaces and $\varepsilon > 0$. Then for every function $f : X_1 \to X_2$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \quad (x, y \in X_1),$$

there exists a unique additive function $A : X_1 \to X_2$ with the property

$$\|f(x) - A(x)\| \leq \varepsilon \quad (x \in X_1).$$

After Hyers's result, many mathematicians have extended Ulam's problem to other functional equations and generalized Hyers's result in various directions (see [4, 8, 12, 22]). A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation $\varphi\big(f, y, y', \dots, y^{(n)}\big) = 0$ has the Hyers-Ulam stability if for given $\varepsilon > 0$ and a function y such that

$$\big|\varphi\big(f, y, y', \dots, y^{(n)}\big)\big| \leq \varepsilon,$$

there exists a solution y_0 of the differential equation such that $|y(t) - y_0(t)| \leq K(\varepsilon)$ and $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$.

Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [18, 19]). Thereafter, Alsina and Ger published their paper [1], which handles the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$: If a differentiable function $y(t)$ is a solution of the inequality

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$|y'(t) - y(t)| \le \varepsilon$ for any $t \in (a, \infty)$, then there exists a constant c such that $|y(t) - ce^t| \le 3\varepsilon$ for all $t \in (a, \infty)$.

Those previous results were extended to the Hyers-Ulam stability of linear differential equations of first order and higher order with constant coefficients in [17, 26, 27] and in [16], respectively. Furthermore, Jung has also proved the Hyers-Ulam stability of linear differential equations (see [9, 10, 11]). Rus investigated the Hyers-Ulam stability of differential and integral equations using the Gronwall lemma and the technique of weakly Picard operators (see [24, 25]). Recently, the Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied by using the method of integral factors (see [15, 29]). The results given in [10, 15, 17] have been generalized by Cimpean and Popa [3] and by Popa and Raşa [20, 21] for the linear differential equations of nth order with constant coefficients. Furthermore, the Laplace transform method was recently applied to the proof of the Hyers-Ulam stability of linear differential equations (see [23]).

In 1979, Baker, Lawrence and Zorzitto [2] proved a new type of stability of the exponential equation $f(x + y) = f(x)f(y)$. More precisely, they proved that if a complex-valued mapping f defined on a normed vector space satisfies the inequality $|f(x + y) - f(x)f(y)| \le \delta$ for some given $\delta > 0$ and for all x, y, then either f is bounded or f is exponential. Such a phenomenon is called the superstability of the exponential equation, which is a special kind of Hyers-Ulam stability. It seems that the results of Găvruţa, Jung and Li [5] are the earliest one concerning the superstability of differential equations.

In this paper, we prove the superstability of the linear differential equations of second order with initial and boundary conditions as well as linear differential equations of higher order in the form of (3.14) with initial conditions.

First of all, we give the definition of superstability with initial and boundary conditions.

Definition 1.1. Assume that for any function $y \in C^n[a, b]$, if y satisfies the differential inequality

$$|\varphi\big(f, y, y', \dots, y^{(n)}\big)| \le \epsilon$$

for all $x \in [a, b]$ and for some $\epsilon \ge 0$ with initial(or boundary) conditions, then either y is a solution of the differential equation

$$\varphi\big(f, y, y', \dots, y^{(n)}\big) = 0 \tag{1.1}$$

or $|y(x)| \le K\epsilon$ for any $x \in [a, b]$, where K is a constant not depending on y explicitly. Then, we say that (1.1) has superstability with initial (or boundary) conditions.

2. Preliminaries

Lemma 2.1. *Let $y \in C^2[a, b]$ and $y(a) = 0 = y(b)$, then*

$$\max |y(x)| \le \frac{(b-a)^2}{8} \max |y''(x)|.$$

Proof. Let $M = \max\{|y(x)| : x \in [a, b]\}$. Since $y(a) = 0 = y(b)$, there exists $x_0 \in (a, b)$ such that $|y(x_0)| = M$. By Taylor's formula, we have

$$y(a) = y(x_0) + y'(x_0)(x_0 - a) + \frac{y''(\xi)}{2}(x_0 - a)^2,$$

$$y(b) = y(x_0) + y'(x_0)(b - x_0) + \frac{y''(\eta)}{2}(b - x_0)^2;$$

thus

$$|y''(\xi)| = \frac{2M}{(x_0 - a)^2}, \quad |y''(\eta)| = \frac{2M}{(b - x_0)^2}.$$

In the case $x_0 \in (a, \frac{a+b}{2}]$, we have

$$\frac{2M}{(x_0 - a)^2} \geq \frac{2M}{(b - a)^2/4} = \frac{8M}{(b - a)^2};$$

In the case $x_0 \in [\frac{a+b}{2}, b)$, we have

$$\frac{2M}{(x_0 - b)^2} \geq \frac{2M}{(b - a)^2/4} = \frac{8M}{(b - a)^2}.$$

So

$$\max |y''(x)| \geq \frac{8M}{(b - a)^2} = \frac{8}{(b - a)^2} \max |y(x)|.$$

Therefore,

$$\max |y(x)| \leq \frac{(b - a)^2}{8} \max |y''(x)|.$$

□

Lemma 2.2. *Let $y \in C^2[a, b]$ and $y(a) = 0 = y'(a)$, then*

$$\max |y(x)| \leq \frac{(b - a)^2}{2} \max |y''(x)|.$$

Proof. By Taylor formula, we have

$$y(x) = y(a) + y'(a)(x - a) + \frac{y''(\xi)}{2}(x - a)^2.$$

We have $(x - a)^2 \leq (b - a)^2$. Therefore,

$$y(x) \leq \frac{y''(\xi)}{2}(b - a)^2.$$

Thus

$$\max |y(x)| \leq \frac{(b - a)^2}{2} \max |y''(x)|.$$

□

Theorem 2.3 ([5])**.** *Consider the differential equation*

$$y''(x) + \beta(x)y(x) = 0 \tag{2.1}$$

with boundary conditions

$$y(a) = 0 = y(b), \tag{2.2}$$

where $y \in C^2[a, b]$, $\beta(x) \in C[a, b]$, $-\infty < a < b < +\infty$. If $\max |\beta(x)| < 8/(b - a)^2$. Then (2.1) *has the superstability with boundary conditions* (2.2).

Theorem 2.4 ([5])**.** *Consider the differential equation* (2.1) *with initial conditions*

$$y(a) = 0 = y'(a), \tag{2.3}$$

where $y \in C^2[a, b]$, $\beta(x) \in C[a, b]$, $-\infty < a < b < +\infty$. If $\max |\beta(x)| < 2/(b - a)^2$. Then (2.1) *has the superstability with initial conditions* (2.3).

3. Main results

In the following theorems, we investigate the superstability of the differential equation
$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{3.1}$$
with boundary conditions
$$y(a) = 0 = y(b) \tag{3.2}$$
or initial conditions
$$y(a) = 0 = y'(a), \tag{3.3}$$
where $y \in C^2[a, b]$, $p \in C^1[a, b]$, $q \in C^0[a, b]$, $-\infty < a < b < +\infty$.

Theorem 3.1. *If*
$$\max\{|q(x) - \frac{1}{2}p'(x) - \frac{p^2(x)}{4}|\} < 8/(b - a)^2. \tag{3.4}$$
Then (3.1) *has the superstability with boundary conditions* (3.2).

Proof. Suppose that $y \in C^2[a, b]$ satisfies the inequality
$$|y''(x) + p(x)y'(x) + q(x)y(x)| \le \epsilon \tag{3.5}$$
for some $\epsilon > 0$. Let
$$u(x) = y''(x) + p(x)y'(x) + q(x)y(x), \tag{3.6}$$
for all $x \in [a, b]$, and define $z(x)$ by
$$y(x) = z(x) \exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big). \tag{3.7}$$
By a substitution (3.7) in (3.6), we obtain
$$z''(x) + \big(q(x) - \frac{1}{2}p'(x) - \frac{p^2(x)}{4}\big)z(x) = u(x) \exp\Big(\frac{1}{2}\int_a^x p(\tau)d\tau\Big).$$
Then it follows from inequality (3.5) that
$$\begin{aligned} \big|z''(x) + \big(q(x) - \frac{1}{2}p'(x) - \frac{p^2(x)}{4}\big)z(x)\big| &= |u(x)exp(\frac{1}{2}\int_a^x p(\tau)d\tau)| \\ &\le \exp\Big(\frac{1}{2}\int_a^x p(\tau)d\tau\Big)\epsilon. \end{aligned}$$
From (3.2) and (3.7) we have
$$z(a) = 0 = z(b). \tag{3.8}$$
Define $\beta(x) = q(x) - \frac{1}{2}p'(x) - \frac{p^2(x)}{4}$, then it follows from (3.4) and by Lemma 2.1,
$$\begin{aligned} &\max|z(x)| \\ &\le \frac{(b-a)^2}{8}\max|z''(x)| \\ &\le \frac{(b-a)^2}{8}[\max|z''(x) + \beta(x)z(x)| + \max|\beta(x)|\max|z(x)|] \\ &\le \frac{(b-a)^2}{8}\max\Big\{\exp\Big(\frac{1}{2}\int_a^x p(\tau)d\tau\Big)\Big\}\epsilon + \frac{(b-a)^2}{8}\max|\beta(x)|\max|z(x)|. \end{aligned}$$
Obviously, $\max\{exp(\frac{1}{2}\int_a^x p(\tau)d\tau)\} < \infty$ on $[a, b]$. Hence, there exists a constant $K > 0$ such that $|z(x)| \le K\epsilon$ for all $x \in [a, b]$.

Moreover, $\max\{\exp(-\frac{1}{2}\int_a^x p(\tau)d\tau)\} < \infty$ on $[a,b]$ which implies that there exists a constant $\tilde{K} > 0$ such that

$$\begin{aligned}
|y(x)| &= \big|z(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big)\big| \\
&\le \max\Big\{\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big)\Big\}K\epsilon \\
&\le \tilde{K}\epsilon.
\end{aligned}$$

Thus (3.1) has superstability stability with boundary conditions (3.2). □

As in Theorem 2.4, we can prove the following theorem.

Theorem 3.2. *If*

$$\max\{q(x) - \frac{1}{2}p'(x) - \frac{p^2(x)}{4}\} < 2/(b-a)^2.$$

Then (3.1) *has superstability stability with initial conditions* (3.3).

Now, as examples, we investigate the superstability of the differential equation

$$\alpha(x)y''(x) + \beta(x)y'(x) + \gamma(x)y(x) = 0 \tag{3.9}$$

with boundary conditions

$$y(a) = 0 = y(b) \tag{3.10}$$

and initial conditions

$$y(a) = 0 = y'(a), \tag{3.11}$$

where $y \in C^2[a,b]$, $\alpha, \beta, \gamma \in C^1[a,b]$, $-\infty < a < b < +\infty$ and $\alpha(x) \neq 0$ on $[a,b]$.

Theorem 3.3. *(1) If*

$$\max\{\frac{\gamma(x)}{\alpha(x)} - \frac{1}{2}(\frac{\beta(x)}{\alpha(x)})' - \frac{1}{4}(\frac{\beta(x)}{\alpha(x)})^2\} < 8/(b-a)^2,$$

then (3.9) *has superstability with boundary conditions* (3.10).

(2) If

$$\max\{\frac{\gamma(x)}{\alpha(x)} - \frac{1}{2}(\frac{\beta(x)}{\alpha(x)})' - \frac{1}{4}(\frac{\beta(x)}{\alpha(x)})^2\} < 2/(b-a)^2,$$

then (3.9) *has superstability with initial conditions* (3.11).

Corollary 3.4. *(1) If*

$$\max\{\frac{l(x)}{k(x)} - \frac{1}{2}\frac{d}{dx}\frac{k'(x)}{k(x)} - \frac{(k'(x)/k(x))^2}{4}\} < 8/(b-a)^2,$$

then

$$\frac{d}{dx}[k(x)y'(x)] + l(x)y(x) = 0 \tag{3.12}$$

has superstability with boundary conditions (3.10)*, where* $k \in C^1[a,b]$*,* $k(x) \neq 0$ *on* $[a,b]$ *and* $l \in C^0[a,b]$.

(2) If

$$\max\{\frac{l(x)}{k(x)} - \frac{1}{2}\frac{d}{dx}\frac{k'(x)}{k(x)} - \frac{(k'(x)/k(x))^2}{4}\} < 2/(b-a)^2,$$

then (3.12) *has superstability with initial conditions* (3.11).

Example 3.5. The differential equation

$$y''(x) + 2y'(x) + y(x) = 0 \tag{3.13}$$

has the superstability with boundary conditions (3.10) on any closed bounded interval $[a, b]$ and the superstability with initial conditions (3.11) on any closed bounded interval $[a, b]$.

In the following theorem, we investigate the stability of differential equation of higher order of the form

$$y^{(n)}(x) + \beta(x)y(x) = 0 \tag{3.14}$$

with initial conditions

$$y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0, \tag{3.15}$$

where $n \in \mathbb{N}^+$, $y \in C^n[a, b]$, $\beta \in C^0[a, b]$, $-\infty < a < b < +\infty$.

Theorem 3.6. *If* $\max|\beta(x)| < \frac{n!}{(b-a)^n}$. *Then* (3.14) *has the superstability with initial conditions* (3.15).

Proof. For every $\epsilon > 0$, $y \in C^2[a, b]$, if $|y^{(n)}(x) + \beta(x)y(x)| \le \epsilon$ and $y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0$. Similarly to the proof of Lemma 2.2,

$$y(x) = y(a) + y'(a)(x-a) + \cdots + \frac{y^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{y^{(n)}(\xi)}{n!}(x-a)^n.$$

Thus

$$|y(x)| = \left|\frac{y^{(n)}(\xi)}{n!}(x-a)^n\right| \le \max|y^{(n)}(x)|\frac{(b-a)^n}{n!}$$

for every $x \in [a, b]$; so, we obtain

$$\begin{aligned}\max|y(x)| &\le \frac{(b-a)^n}{n!}[\max|y^{(n)}(x) + \beta(x)y(x)|] + \frac{(b-a)^n}{n!}\max|\beta(x)y(x)| \\ &\le \frac{(b-a)^n}{n!}\epsilon + \frac{(b-a)^n}{n!}\max|\beta(x)|\max|y(x)|.\end{aligned}$$

Let $\eta = \frac{(b-a)^n}{n!}\max|\beta(x)|$, $K = \frac{(b-a)^n}{n!(1-\eta)}$. It is easy to see that

$$|y(x)| \le K\epsilon.$$

Hence (3.14) has superstability with initial conditions (3.15). □

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