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CRITICAL POINTS AND CURVATURE IN CIRCULAR CLAMPED PLATES

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ABSTRACT. In this article we investigate some qualitative properties of the solutions of the classical linear model for clamped plates on circular domains, under constant sign external loads. In particular we prove that inside the circle there are at most a finite number of critical points, which in turn rules out the existence of critical curves. We also study the curvature of the level curves of the solutions, and we prove that the curvature function is continuous up to the border, even though the gradient of the solutions vanishes on the border circle.

1. INTRODUCTION

The classical model for the deflection u of a clamped plate under an external load is given by

$$\begin{gathered} \Delta^2 u = f \quad \text{in } B, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B, \end{gathered} \tag{1.1}$$

where Δ^2 is the biharmonic operator, B is a planar domain, f is the density of the external load and $\frac{\partial u}{\partial \nu}$ is the outward normal derivative of u at the boundary ∂B.

It is well known that problem (1.1) possesses exactly one solution $u \in C^4(B) \cap C^2(\bar{B})$ provided $f \in C(\bar{B})$, see for example [10]. Moreover, for some special domains the Green function of problem (1.1) has been explicitly computed, as it is the case of the disk ([5]) and the limaçon ([7]). However, in contrast with existence, uniqueness and regularity, the geometric properties of the solutions seem to be not so well documented. Unlike second order elliptic operators, the maximum principle does not hold for the clamped plate problem, and as a consequence, the *sign preserving property* (SPP) for a domain B associated to problem (1.1),

$$f \geq 0 \text{ in } B \quad \text{implies} \quad u \geq 0 \text{ in } B,$$

does not hold for general domains. The famous 1908 Boggio-Hadamard conjecture claims that the SPP applies on convex domains. Boggio [5] proved in 1905 that the SPP holds for circular domains. However, in 1949 Duffin [8] proved the conjecture to be false for infinitely long rectangles and in 1951 a result due to Garabedian [9] showed that the SPP does not hold for eccentric enough ellipses. Since then,

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many other counterexamples have been given in the literature. A strikingly simple, explicit example, of a function u that changes sign and satisfies (1.1), with $f > 0$ and B a certain ellipse, was provided by Shapiro [16] in 1994.

The critical set $\mathbb{K}$ of a function u is the set of its critical points, that is to say, the points with vanishing gradient. The description of $\mathbb{K}$ can be a key step in the way to describe the geometric behavior of solutions u to (1.1). To the authors' knowledge, there are few results regarding the structure of this set. In a 1994 paper Soranzo [17] proved that if B is a disk and f is a radially symmetric, nonnegative, nonzero function, then, inside of B, the only critical point of the solution u to (1.1) is the center of B. We also refer to the results of Grunau and Sweers [11], who proved that when B is a disk, the solution u to (1.1) possesses no minima in the interior of B, provided that f is a nonzero, nonnegative function.

In a recent article two of the present authors [3], studied the critical set of solutions to equations that model the deflection of membranes fixed at the border, subject to the action of a constant sign analytical external force (see also [1]). According to this work the critical set associated to the solutions of the corresponding second order elliptic boundary-value problem, is made up of finitely many critical points and finitely many Jordan critical curves. Although the authors think that analogue results should hold for the biharmonic problem (1.1), at least for domains where the SPP holds, a proof is not known to them. In this paper we follow techniques similar to the ones employed in [3], to show that when B is a disk, f is a nonzero, nonnegative, real analytic function, and u is a solution to (1.1), the critical set of u, inside of B, is made up of finitely many isolated critical points. According to [11], critical points should be either maxima or saddle points.

We remark that the analyticity condition on f grants that the solution u to (1.1) is analytic (see for example [15]), which in turn allows us to apply some general results obtained in previous works (see [4], [3] and [2]), concerning the structure of the critical set of certain analytical functions.

We also analyze the nodal sets of the directional derivatives of the solution to (1.1), in order to show that, whenever the density of the external load f is real analytic, the curvature function on the level curves of u can be extended to the border of the disk, even though ∇u vanishes there. Further, we present an explicit example showing that when f changes sign, it is possible that the curvature function fails to be continuous on the border. We do not know of any previous analysis regarding the curvature of level sets of solutions to problem (1.1).

2. Radial solutions and critical points to the clamped plate equation

Let us summarize some known results concerning the qualitative properties of the solution to problem (1.1), under the assumption that f is a continuous, nonzero, nonnegative function.

Theorem 2.1 (Boggio [5], 1904). *Sign preserving property holds when B is a disk.*

Theorem 2.2 (Grunau-Sweers [11], 2001). *If f is a nonzero, nonnegative function, the solution to* (1.1) *possesses no local minima when B is a disk.*

In contrast with the above results, the following property holds for more general domains.

Lemma 2.3. *If the SPP holds for a domain B whose boundary satisfies the interior ball condition, and f is a nonzero, nonnegative function, then the solution u to problem* (1.1) *satisfies $\Delta u|_{\partial B} \geq 0$.*

Proof. Let us assume that $\Delta u(q) < 0$ at some point $q \in \partial B$. Let D be a disk inside of B, tangent to ∂B at q and such that $\Delta u < 0$ on D. Since the SPP holds, it follows that $u \geq 0$ on B. Hopf's boundary point lemma then implies that $\frac{\partial u}{\partial \nu}(q) < 0$, which contradicts the fact that $\frac{\partial u}{\partial \nu} = 0$ on ∂B. □

However when B is a disk and u is a radial solution, the inequality in Lemma 2.3 can be proved to be strict. This and some other properties of radial solutions are summarized in Lemma 2.4 and Corollary 2.5, which for the most part follow [17, Proposition 1].

Lemma 2.4. *Let $B \equiv B_\rho(0)$ be the disk of radius ρ centered at the origin and let $u \in C^4(B) \cap C^2(\bar{B})$, be a radial nontrivial solution of* (1.1)*, with f a nonzero, nonnegative function. Then u is strictly positive in B, $w(|x|) = u(x)$ satisfies $w' < 0$ on $(0, \rho)$, $\Delta u(0) < 0$, and $\Delta u|_{\partial B} > 0$.*

Proof. The first three statements correspond to Proposition 1 in [17]. To prove that $\Delta u|_{\partial B} > 0$, notice that if u is a radially symmetric function, and $w(|x|) = u(x)$, then $\Delta u = \frac{1}{r}(rw')'$ so that, if $W = \frac{1}{r}\left(rw'\right)'$, it follows, according to equation (51) in [17], that $W'(r) \geq 0$ on $(0, \rho)$. Moreover, by Lemma 2.3, $W(\rho) \geq 0$. However $W(\rho) = 0$ would imply $W(r) \leq 0$ on $(0, \rho)$, and, as $u = 0$ on ∂B, the maximum principle yields $u > 0$ on B. In that case, $\frac{\partial u}{\partial \nu} < 0$ on ∂B according to Hopf boundary point Lemma, thus contradicting the boundary condition $\frac{\partial u}{\partial \nu} = 0$. □

As an incidental consequence of Lemma 2.4 we can fully describe the critical set of any radially symmetric solution to problem (1.1) on a disk, as stated in the following corollary.

Corollary 2.5. *If f is a nonzero, nonnegative function, the center of the disk is the only interior critical point of a radially symmetric solution to* (1.1).

To finish, we quote a result that characterizes the critical set of any semi-Morse function. Following [3], we say that a critical point of a function v is semi–Morse, if the Hessian matrix of v does not vanish at that point. A function is termed semi–Morse if all of its critical points are semi-Morse. The structure of the critical set of analytical semi-Morse functions was studied in [4] and [3].

Lemma 2.6 ([3, Lemma 2])**.** *Let $B \subset \mathbb{R}^2$ be a planar domain with smooth boundary ∂B, and let v be a real analytic, semi–Morse function defined on an open neighborhood of $\bar{B}$. If all of the critical points of v belong to B, then the critical set of v is made up of finitely many isolated critical points, and finitely many regular analytic Jordan curves.*

In the next section we will show that solutions u to (1.1) are semi-Morse provided B is a disk. Notice that this result heavily relies on the SPP exhibited by the domain.

3. Isolated critical points

Given a solution u to problem (1.1) on the unit disk, we can follow Grunau and Sweers (see [11]) and consider the Möebius transformations on the unit complex disk B

$$h_a(x) = \frac{a - x}{1 - \bar{a}x},$$

where $x \in B$, $a \in B$, in order to define the function v as

$$v(x) = \frac{1}{|h'_a(x)|}\, u(h_a(x)). \tag{3.1}$$

We notice that

$$a\Delta v(x) = \frac{\Delta(u \circ h_a)(x)}{|h'_a(x)|} + u(h_a(x))\, \Delta\Big(\frac{1}{|h'_a(x)|}\Big) + 2\,\nabla\Big(\frac{1}{|h'_a(x)|}\Big) \cdot \nabla(u \circ h_a)(x).$$

On the other hand, taking into account that h_a is a conformal map, we have that

$$\Delta(u \circ h_a)(x) = |h'_a(x)|^2\, \Delta u(h_a(x)).$$

Therefore in the particular case that $h_a(x)$ turns to be a critical point of u, it follows that

$$\Delta v(x) = |h'_a(x)|\, \Delta u(h_a(x)) + u(h_a(x))\, \Delta\Big(\frac{1}{|h'_a(x)|}\Big). \tag{3.2}$$

Grunau and Sweers [11, Lemma 1] showed that v satisfies:

$$\begin{gathered} \Delta^2 v = |h'_a|^3 (f \circ h_a) \quad \text{in } B, \\ v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial B. \end{gathered} \tag{3.3}$$

Moreover, they also showed (see [11], equations (6) and (7)), that the function

$$w(x) \equiv \frac{1}{2\pi} \int_{|z|=1} v(|x|\, z)\, ds(z), \quad x \in B, \tag{3.4}$$

solves the equation

$$\begin{gathered} \Delta^2 w = g \quad \text{in } B, \\ w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial B, \end{gathered} \tag{3.5}$$

where

$$g(x) = \frac{1}{2\pi} \int_{|z|=1} |h'_a|^3\, f \circ h_a(|x|\, z)\, ds(z).$$

The function w can be seen as the radial average of v. Since w is radial, we can consider it as a single variable function, $w(r) = w(x)$, for $r = |x|$.

Next we show that when B is a disk, interior critical points of solutions to (1.1) turn to be semi-Morse.

Lemma 3.1. *Let B be the unit disk, and let u be a solution to* (1.1)*, with f a nonnegative, nonzero function. If $a \in B$ is a critical point of u then $\Delta u(a) < 0$.*

Proof. A straightforward computation shows that

$$w''(r) = \frac{1}{2\pi} \int_{|z|=1} H_v(rz)\, z \cdot z\, ds(z),$$

where v and w are defined by (3.1) and (3.4) respectively, and H_v stands for the Hessian matrix of v. Therefore

$$w''(0) = \frac{1}{2\pi} \int_{|z|=1} H_v(0)\, z \cdot z\, ds(z) = \frac{1}{2}\, \Delta v(0).$$

Notice that, as w is a radial solution of the boundary-value problem (3.5), then, according to Lemma 2.4, we have $\Delta w(0) < 0$. Since $\Delta v(0) = 2\,w''(0)$, it follows that $\Delta v(0) < 0$.

Now, taking into account (3.2), and the fact $h_a(0) = a$, it follows that in the case a is a critical point of u,

$$\Delta v(0) = (1 - |a|^2)\Delta u(a) + \frac{4\,|a|^2}{1 - |a|^2}\,u(a).$$

The conclusion follows from the above identity, given that, as B has the sign preservation property (Theorem 2.1), $u(a) \geq 0$. $\square$

It is known that locally the critical set of an analytic, semi-Morse function is either an isolated point or a curve (see [4], [3], [2]). The above lemma proves that u is semi–Morse in the interior of B, thus yielding the local structure of the critical set in the interior of B. We will prove that curves of critical points are precluded and that critical points do not accumulate on the boundary of B.

Lemma 3.2. *Let B be the unit disk and let $a \in B$. If u is a radial solution to* (1.1) *(with f a nonnegative, nonzero function), and $v(x)$ is given by* (3.1)*, then $\Delta v > 0$ on ∂B.*

Proof. Notice that every point on ∂B is critical for u, and that ∂B is invariant under h_a. The result now follows from (3.2), given that u vanishes on ∂B, and, by Lemma 2.4, $\Delta u(h_a(x)) > 0$. $\square$

Theorem 3.3. *If B is a disk, solutions u to* (1.1) *have no critical curves, whenever f is an analytic, nonnegative, nonzero function.*

Proof. Let us assume that Γ is a curve of critical points of u. We know from Lemma 2.6 that Γ is smooth, so let τ and η be respectively tangent and normal unitary vectors to Γ at some given point p. We can see that if $r = r(t)$ parametrizes Γ, $u(r(t))$ is constant, from which it follows that

$$H_u(r(t))\,r'(t)\cdot r'(t) + \nabla u\,(r(t))\cdot r''(t) = 0.$$

Moreover, as Γ is a curve of critical points, we conclude that $H_u(p)\,\tau\cdot\tau = 0$, and $\Delta u(p) = H_u(p)\,\eta\cdot\eta$. From Lemma 3.1 it then follows that Γ is a curve of local maxima, therefore there must be a minimum of u inside of Γ, thus contradicting Theorem 2.2. $\square$

We remark that Theorem 3.3 does not hold if f changes sign. Let us define

$$u(x_1, x_2) = (1 - x_1^2 - x_2^2)^2(1 + 8x_1^2 + 8x_2^2)^2.$$

A direct calculation shows that u satisfies Problem 1.1 in the unitary disk with $f = \Delta^2 u = 192\left(-5 + 24x_1^2 + 24x_2^2\right)$ changing sign in B. However it can be seen that the circle $x_1^2 + x_2^2 = \frac{1}{4}$ is a critical curve of u inside of B.

Lemma 3.4. *If B is a disk, u is a solution to* (1.1)*, and f is nonnegative and nonzero, then $\Delta u|_{\partial B} > 0$.*

Proof. According to Lemma 2.3 we already know that $\Delta u|_{\partial B} \geq 0$. Let us suppose there exists a point $q \in \partial B$ such that $\Delta u(q) = 0$. Now, as f is a nonzero, nonnegative function, we might fix a point $a \in B$ such that $f(a) > 0$, and for $\epsilon > 0$,

$\epsilon f(a) < 1$ given, define the radial function

$$f_\epsilon(x) = \begin{cases} \epsilon f(a) - |x| & \text{if } |x| < \epsilon f(a), \\ 0 & \text{if } |x| \geq \epsilon f(a). \end{cases}$$

However, as h_a maps neighborhoods of 0 into neighborhoods of a, we can choose ϵ small enough to guarantee that $f_\epsilon \circ h_a$ vanishes outside a small neighborhood of a, and

$$f \geq |h_a'|^3 (f_\epsilon \circ h_a) \quad \text{in } B.$$

We consider now the solution w to the boundary-value problem

$$\Delta^2 w = f_\epsilon \quad \text{in } B,$$
$$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial B,$$

and set $v(x) = \frac{1}{|h_a'|}\, w(h_a(x))$. We notice that v satisfies (3.3), if we replace f by f_ϵ. It follows that $z \equiv u - v$ satisfies

$$\Delta^2 z \geq 0 \quad \text{in } B,$$
$$z = \frac{\partial z}{\partial \nu} = 0 \quad \text{on } \partial B.$$

Now, as $\Delta u(q) = 0$ and according to Lemma 3.2, $\Delta v(q) > 0$, we would have $\Delta z(q) < 0$, thus yielding a contradiction, as by Lemma 2.3 $\Delta z \geq 0$ in ∂B. $\square$

Alternatively, we could resort to the well known Boggio's formula for the Green function of problem (1.1) in the disk, to prove the above Lemma in a more direct way, though we have opted for this indirect approach.

Lemmas 3.1 and 3.4 show that analytic solutions to problem (1.1) are semi-Morse functions, hence the critical set should be as described by Lemma 2.6. Further, as we state in the next theorem, the critical set do not include critical curves.

Theorem 3.5. *If B is a disk, f is an analytic, nonnegative, nonzero function, and u is a solution to* (1.1)*, the set of critical points of u inside of B is made up of finitely many points. Moreover critical points are either maxima or saddle points.*

Proof. Lemmas 3.1 and 2.6, and Theorem 3.3, grants that critical points of a solution u to (1.1) are isolated. To prove that there are only finitely many critical points in the interior of B, it only remains to prove that they do not accumulate on the boundary. To see this, let us assume that (a_k) is a sequence of critical points in B, such that $a_k \to a$, with a a point in the the boundary of B. However this would imply that $\Delta u(a_k) \to \Delta u(a)$ and $\Delta u(a) \leq 0$, given that by Lemma 3.1 $\Delta u(a_k) < 0$, thus contradicting Lemma 3.4. The fact that critical points are either maxima or saddle points is just a consequence of a Grunau and Sweers result (see Theorem 2.2). $\square$

Example 3.6. Let B denote the disk of radius 1 centered at the origin. Define

$$u_1(x_1, x_2) = \frac{19(1 - x_1^2 - x_2^2)^2}{64(100 - 180x_1 + 81x_1^2 + 81x_2^2)}$$

and set $u_2(x_1, x_2) = u_1(-x_1, x_2)$. A straightforward calculation shows that $u = u_1 + u_2$ satisfies $\Delta^2 u \geq 0$ in B and $u = \frac{\partial u}{\partial \nu} = 0$ on ∂B. Moreover u possesses exactly three critical points inside B: a saddle at $(0,0)$ and two maxima at $(-p, 0)$ and $(p, 0)$, where $p^2 = 73/54 - \sqrt{30761}/162$. Figure 1 pictures the critical points of

u, as well as its level curves. Notice that the curvature of the level curves behaves nicely near the boundary. This turns to be the case for solutions of (1.1), whenever f is nonzero, nonnegative, as we show in the next section. However this behavior is not granted if $f = \Delta^2 u$ does not have a constant sign, as we will see later (see Example 4.5).

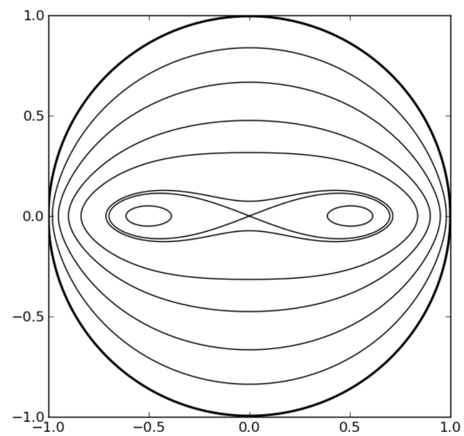

FIGURE 1. Critical set and level curves of the function u given in Example 3.6. Notice the curvature of the level sets near the border circle.

4. Curvature of the level sets

In this section we will show that the function giving the curvature of the level curves associated to the solution u to (1.1), can be extended to the boundary of the disk. This claim is non-obvious since at points of non-vanishing gradient, the curvature is given by

$$\kappa(x) = -\frac{H_u\,\theta(x)\cdot\theta(x)}{|\nabla u(x)|}, \quad \theta(x) = J\frac{\nabla u(x)}{|\nabla u(x)|}, \tag{4.1}$$

with J the $-\pi/2$ rotation matrix. According to Theorem 3.5, if f is analytic the above formula makes sense for all but finitely many points in B. However, as ∇u vanishes on the border circle, it is not clear that this formula can be continuously extended to that curve.

We say that f in problem 1.1 is real analytic in $\bar{B}$ if there exists an analytic extension of f to an open set including $\bar{B}$. It is well known that in that case the solution u to (1.1) can be analytically extended to an open domain including $\bar{B}$ (see [15]).

The techniques in this section heavily rely on the description of the nodal sets of the directional derivatives of the solution u to (1.1). Following [2], given a direction θ, the derivative of u in the θ direction will be denoted u_θ; that is to say,

$u_\theta(x) = \nabla u(x) \cdot \theta$. The nodal set of u_θ is now defined to be the set N_θ where u_θ vanishes; i.e.,

$$N_\theta = \{x \in \bar{B} : u_\theta(x) = 0\}.$$

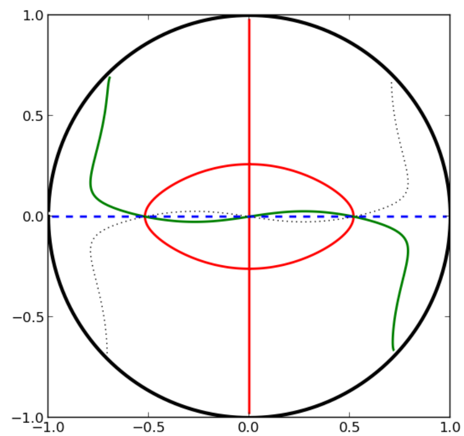

FIGURE 2. Nodal lines N_θ for $\theta = 0$ (red), $\frac{\pi}{4}$ (green), $\frac{\pi}{2}$ (blue) and $\frac{3\pi}{4}$ (gray), associated to the function u in Example 3.6 (points in the border circle belong to all N_θ).

If for a point $x \in N_\theta$, $\nabla u_\theta(x) \neq 0$, the nodal set N_θ is locally a curve, and given that $\nabla u_\theta(x) = H_u(x)\,\theta$, it follows that this curve satisfies the ODE

$$z' = JH_u(z)\theta, \tag{4.2}$$

with initial data $z(0) = x$. On the other hand when x is a critical point of u_θ, the local structure of N_θ can be quite involved. However, if x is a Morse point of u_θ (that is, when the Hessian matrix of u_θ is non singular at x), the local structure of N_θ can be easily obtained from elementary Morse Theory. In particular, if $\det JH_{u_\theta}(x) < 0$, N_θ must be locally homeomorphic to the level set of the function $f(x, y) = xy$ at the origin. It follows that, locally, N_θ is the union of the stable and unstable manifolds of (4.2) at the equilibrium x. Our task now is to compute the eigenvalues and the eigenspaces of $JH_{u_\theta}(x)$, in order to approximate the stable and unstable manifolds of (4.2) at x. To start with, for a given $v \in C^2\left(\bar{B}\right)$ and $\theta \in S^1$ we denote

$$H_{v,\theta}(x) = D(H_v(x)\theta), \tag{4.3}$$

D being the standard derivative of a $\mathbb{R}^2$ value function. For the reader's convenience we write the full expression of $H_{v,\theta}$ for $x = (x_1, x_2)$ and $\theta = (\cos\theta, \sin\theta)$ (notice that we use of the same letter to denote the direction θ and its argument):

$$H_{v,\theta}(x) = \begin{pmatrix} \cos\theta\, v_{x_1x_1x_1} + \sin\theta\, v_{x_1x_1x_2} & \cos\theta\, v_{x_1x_1x_2} + \sin\theta\, v_{x_1x_2x_2} \\ \cos\theta\, v_{x_1x_1x_2} + \sin\theta\, v_{x_1x_2x_2} & \cos\theta\, v_{x_1x_2x_2} + \sin\theta\, v_{x_2x_2x_2} \end{pmatrix}.$$

The matrix $H_{v,\theta}(x)$ turns out to be symmetric and to satisfy the commuting property

$$H_{v,\theta}(x)\alpha = H_{v,\alpha}(x)\theta, \quad \alpha, \theta \in S^1.$$

Now, given a solution u to (1.1), we write $\mathcal{H}(x)$ and $\mathcal{H}_\theta(x)$ instead of $H_u(x)$ and $H_{u,\theta}(x)$. With this simplified notation we have

$$H_{u_\theta}(x) = \mathcal{H}_\theta(x).$$

Lemma 4.1. *Let f in Problem* (1.1) *be a nonzero, nonnegative function, that is real analytic in $\bar{B}$. Let $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi)$, be the standard parametrization of ∂B by arc length and set $\theta(t) = \gamma'(t)$. Then for all $t \in [0, 2\pi)$, setting $x = \gamma(t)$ and $\theta = \theta(t)$, we have*

$$\nabla(\det \mathcal{H}(x)) = \Delta u(x) \mathcal{H}_\theta(x)\theta \tag{4.4}$$

$$\mathcal{H}_\theta(x)\,\theta = \Delta u(x) J\theta, \tag{4.5}$$

$$\det \mathcal{H}_\theta(x) = -(\Delta u(x))^2. \tag{4.6}$$

Proof. Recall that if α and β are orthogonal directions, and A is any 2×2 symmetric matrix,

$$\det A = (A\alpha \cdot \alpha)\,(A\beta \cdot \beta) - (A\alpha \cdot \beta)^2 \ \text{ and } \ \operatorname{Tr} A = A\alpha \cdot \alpha + A\beta \cdot \beta.$$

Therefore, if $A = \mathcal{H}(x)$, we can deduce that

$$\nabla(\det \mathcal{H}(x)) = (\mathcal{H}(x)\beta \cdot \beta)\mathcal{H}_\alpha(x)\alpha + (\mathcal{H}(x)\alpha \cdot \alpha)\mathcal{H}_\beta(x)\beta - (2\mathcal{H}(x)\alpha \cdot \beta)\mathcal{H}_\alpha(x)\beta.$$

Now, given that $\nabla u(\gamma(t)) = 0$ for $t \in [0, 2\pi)$, we have

$$\mathcal{H}(\gamma(t))\theta(t) = 0. \tag{4.7}$$

Therefore, if we set $x = \gamma(t)$, $\alpha = \theta(t) = \theta$ and $\beta = J\theta$, we would have

$$\nabla(\det \mathcal{H}(x)) = (\mathcal{H}(x)\beta \cdot \beta)\mathcal{H}_\alpha(x)\alpha.$$

Moreover, given that $\Delta u(x) = \operatorname{Tr} \mathcal{H}(x) = \mathcal{H}(x)\beta \cdot \beta$, equation (4.4) follows.

We can now derive (4.7) to obtain (4.5), and multiplying this last one by $\mathcal{H}_\theta(x)J$, we deduce that

$$\det(\mathcal{H}_\theta(x))J\theta = -(\Delta u(x))^2 J\theta,$$

and the final claim follows. $\square$

Taking into account (4.7), we notice that points in ∂B, satisfy $\det \mathcal{H}(x) = 0$. As a consequence of Lemma 4.1 it follows that, locally, the the nodal set of $\det \mathcal{H}(x)$ coincides with ∂B. We had already noticed that N_θ is related to the stable and unstable manifolds associated to (4.2). Next we are going to take advantage of the well known geometric regularity of these manifolds (see for example [14]), to further study the structure of the nodal set N_θ.

Lemma 4.2. *Let f be a real analytic, nonzero, nonnegative function, defined on $\bar{B}$, let ∂B be counterclockwise oriented, and denote by $\mathcal{H}$ the Hessian matrix of the solution u to problem* (1.1)*. Then, if x is a point in ∂B and θ is the unitary tangent vector to ∂B at x, it follows that x is a saddle equilibrium of*

$$z' = J\mathcal{H}(z)\theta, \tag{4.8}$$

and ∂B coincides with the stable manifold of (4.8) *at x. Moreover, there is an open neighborhood V of ∂B, such that for every point $z \in V \cap B$, there exists exactly one point $x \in \partial B$, such that z belongs to the unstable manifold of* (4.8) *at x, and the backward orbit of z stays in V.*

Proof. Let x be a point on ∂B, and notice that $J\mathcal{H}_\theta(x) = JH_{u_\theta}(x)$ is the linearization of (4.8), at x.

According to (4.7) x is an equilibrium of (4.8), and, taking into account (4.6), x turns to be a saddle. Moreover, by (4.5)

$$J\mathcal{H}_\theta(x)\,\theta = -\Delta u(x)\,\theta,$$

hence, recalling Lemma 3.4, $-\Delta u(x)$ is a negative eigenvalue, with θ as associated eigenvector. Furthermore, we had already noticed that at every saddle of (4.8), the union of the stable and the unstable manifolds equals N_θ. Given that $\partial B \subset N_\theta$, the stable manifold at x must coincide with ∂B.

Now, (4.6) implies that $\Delta u(x)$ is the positive eigenvalue at x. On the other hand, as $\mathcal{H}_\theta(x)^2 = \operatorname{Tr}(\mathcal{H}_\theta(x))\mathcal{H}_\theta(x) - \det \mathcal{H}_\theta(x)\, I$ and, by equation (4.5), $\mathcal{H}_\theta(x)^2\theta = \Delta u(x)\mathcal{H}_\theta(x)J\theta$, we readily obtain

$$J\mathcal{H}_\theta(x)\,J\theta = -\operatorname{Tr}\left(\mathcal{H}_\theta(x)\right)\theta + \Delta u(x)J\theta. \tag{4.9}$$

Thus, as we already know how the matrix $J\mathcal{H}_\theta(x)$ transforms the orthogonal basis $\{\theta, J\theta\}$, it is not difficult to find the eigenspace associated to the eigenvalue $\Delta u(x)$. In fact a straightforward calculation shows that

$$J\mathcal{H}_\theta(x)\alpha = \Delta u(x)\alpha, \quad \text{with} \quad \alpha = -\operatorname{Tr}\left(\mathcal{H}_\theta(x)\right)\theta + 2\Delta u(x)J\theta. \tag{4.10}$$

Notice now that (4.8) defines a system of differential equations depending on the parameter θ. We know that given a direction θ there exists a point $x_\theta \in \partial B$ such that θ is tangent to ∂B at x_θ. Furthermore, x_θ is a saddle of the system associated to θ, having unstable manifold U_θ.

As it is known (see for instance [6]), the unstable manifold depends smoothly on θ, so that, for fixed θ_0 we might locally parametrize U_θ by a C^∞ function $w(t,\theta)$, defined on an open neighborhood $(-\epsilon,\epsilon)\times(\theta_0-\delta,\theta_0+\delta)$, satisfying $w(0,\theta) = x_\theta$. Moreover, w can be chosen so that $\frac{\partial w}{\partial t}(0,\theta_0) = \alpha_{\theta_0}$, where α_{θ_0} is given by (4.10), with $\theta = \theta_0$ and $x = x_{\theta_0}$. As $x_\theta = J\theta$ we can readily compute the Jacobian determinant of w at $(0,\theta_0)$ to find out it is equal to $-2\Delta u(x_{\theta_0})$. It follows that w defines a local diffeomorphism, onto an open neighborhood of x_{θ_0}. Notice that, according to Theorem 3.5, this neighborhood can be granted to be small enough so that does not contain critical points of u. Next, we can choose finitely many of these sets to cover ∂B, and let V be the union of these neighborhoods.

It should be clear now that for every point $z \in V \cap B$ there exists θ such that $z \in U_\theta$ and the backward orbit of z under (4.8) stays in V. Moreover, z cannot belong to U_{θ_0} if θ_0 is noncollinear with θ, given that U_θ and U_{θ_0} only could meet at interior critical points of the solution u. However, it is still possible that z belongs to $U_{-\theta}$, but the associated backward orbit could not stay in V. $\square$

We had already noticed that ∇u does not define a normal direction on ∂B, given that points on this curve are critical. The next lemma allows us to define a vector function that coincides with the direction of ∇u in points near the border of the circle, and with the inward normal vector at boundary points.

Lemma 4.3. *Let f be a real analytic, nonzero, nonnegative function, defined on $\bar{B}$, let u be the solution to problem* (1.1)*, and let θ_x denote the counterclockwise oriented unitary tangent vector at $x \in \partial B$. There exists $\epsilon > 0$ such that*

$$\theta(x) = \begin{cases} J\frac{\nabla u(x)}{|\nabla u(x)|}, & 1-\epsilon \le |x| < 1, \\ \theta_x, & |x| = 1, \end{cases}$$

is continuous on $1-\epsilon \le |x| \le 1$.

Proof. Let $x \in \partial B$ be fixed and let θ_x be the unitary tangent vector to ∂B at x. Consider now a sequence (z_n) in B such that $z_n \to x$. According to Lemma 4.2, for z_n close enough to ∂B, there exists a point $x_n \in \partial B$, such that $z_n \in U_{\theta_n}$, with θ_n the counterclockwise oriented tangent vector at x_n. However, as $U_{\theta_n} \subset N_{\theta_n}$, it follows that $J\nabla u(z)$ points in the θ_n direction at every point $z \in U_{\theta_n}$. Thus, we have

$$J\frac{\nabla u(z_n)}{|\nabla u(z_n)|} = \theta_n,$$

and given that $\theta_n \to \theta$, the conclusion follows. □

Now we turn to the question of the continuity of the curvature function near ∂B.

Theorem 4.4. *If u solves problem* (1.1)*, with f a real analytic, nonzero, nonnegative function defined on $\bar{B}$, then for all $x \in \partial B$, the curvature $\kappa(z)$ defined in* (4.1) *satisfies*

$$\lim_{z\to x, z\in B} \kappa(z) = 1.$$

Proof. Let x be a point on ∂B and let θ be the (counterclockwise oriented) tangent vector to ∂B at x, α the unstable direction at x. According to the generalized L'Hopital rule in [13]

$$\lim_{z\to x} -\frac{\mathcal{H}(z)\theta\cdot\theta}{|\nabla u(z)|} = \lim_{z\to x} -\frac{D_\alpha\left(\mathcal{H}(z)\theta\cdot\theta\right)}{D_\alpha\left(|\nabla u(z)|\right)}, \tag{4.11}$$

whenever the latter limit exists. However

$$\frac{D_\alpha\left(\mathcal{H}(z)\theta\cdot\theta\right)}{D_\alpha\left(|\nabla u(z)|\right)} = \frac{\mathcal{H}_\theta(z)\alpha\cdot\theta}{\mathcal{H}(z)\frac{\nabla u(z)}{|\nabla u(z)|}\cdot\alpha},$$

and as by Lemma 4.3

$$\lim_{z\to x} \frac{\nabla u(z)}{|\nabla u(z)|} = -J\theta,$$

it follows that the limit in (4.11) exists. Moreover, as a consequence of (4.7), $\mathcal{H}(x)J\theta = \Delta u(x)J\theta$, so that, taking (4.5) into account, we conclude that this limit equals 1. □

We note that without the hypothesis about f, Theorem 4.4 may fail to be true. The next example shows that if f changes sign, it could happen that the curvature be discontinuous on ∂B.

Example 4.5. Let

$$u(x_1,x_2) = (1-x_1^2-x_2^2)^2\Big((x_1-1)^2 + \frac{19}{10}(x_1-1)x_2 + x_2^2\Big)$$

A straightforward calculation shows that u satisfies (1.1) with f given by

$$f(x_1,x_2) = \frac{32}{5}\left(-10-60x_1-57x_2+114x_1x_2+90x_1^2+90x_2^2\right)$$

It can also be shown that $u \ge 0$ on B and that f changes sign in B. Some of the level curves of u can be seen in Figure 3: it can be adverted that there are points arbitrarily close to the boundary point $(1,0)$ where the curvature is negative, notwithstanding the fact that on ∂B, the curvature equals 1.

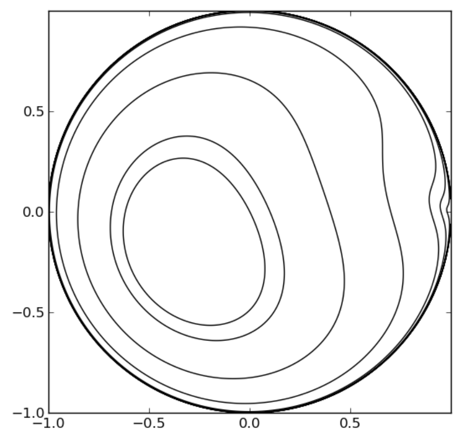

FIGURE 3. Level curves of function u in Example 4.5.

Conclusions. We have shown that on a disk, critical sets of analytic solutions to problem (1.1) have the same structure as for the analogous membrane deflection problem. The analyticity assumption is rather strong, however it seems difficult to avoid as our proofs heavily depend on the structure of the critical set of semi–Morse functions, that was developed in an analytical setting (see [4]). Certainly the analyticity, or the lack of it, affects the structure of the critical set, at least in the case of solutions of second order elliptic equations, as shown by a trivial example like

$$u(x,y) = \begin{cases} 1, & \text{if } x^2+y^2 \le \frac{1}{4}, \\ 1 - \big(2\sqrt{x^2+y^2} - 1\big)^3, & \text{if } \frac{1}{4} < x^2+y^2 \le 1. \end{cases}$$

In the above case $\Delta u \ge 0$ in B, $u = 0$ on ∂B, but the critical set is the whole disk $x^2 + y^2 \le \frac{1}{4}$, instead of being a discrete set of points. In light of Corollary 2.5 an analogue behavior for radially symmetric solutions for the clamped plate model is not possible, and the authors are not aware of examples shedding light on the peculiarities exhibited by nonanalytic solutions to the problem (1.1).

On the other hand, techniques in this paper are tailored for disks, since our work is grounded in the use of Möbius transformation defined on a disk. Nevertheless, for the membrane deflection problem it is known that on simply connected planar domains the critical set reduces to finitely many interior critical points. We expect that at least some of the conclusions of Theorems 3.5 and 4.4 can be generalized for domains satisfying the SPP. In a domain where the SPP does not hold, it is clear that the last statement of Theorem 3.5 is not true. Actually, even for a constant external force in (1.1), solutions of this model can change sign depending on the domain, as shown by Grunau and Sweers in a recent paper [12].

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References

[1] J. Arango. Uniqueness of critical points for semi-linear Dirichlet problems in convex domains. *Electron. J. Differential Equations*, 2005, No. 43, 5 pp. (electronic), 2005.

[2] J. Arango and A. Gómez. Critical points of solutions to elliptic problems in planar domains. *Commun. Pure Appl. Anal.*, 10(1): 327–338, January 2011.

[3] J. Arango and A. Gómez. Critical points of solutions to quasilinear elliptic problems. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 75(11): 4375–4381, 2012.

[4] J. Arango and O. Perdomo. Morse theory for analytic functions on surfaces. *J. Geom.*, 84 (1–2): 13–22, 2005.

[5] T. Boggio. Sulle funzioni di green d´ordinem. *Rend. Circ. Mat. Palermo*, pages 97–135, 1905.

[6] X. Cabré, E. Fontich, and R. de la Llave. The parameterization method for invariant manifolds III: overview and applications. *J. Diff. Equations*, 218: 444–515, 2005.

[7] A. Dall'Acqua and G. Sweers. The clamped-plate equation for the Limacon. *Ann. Mat. Pura Appl. (4)*, 184(3): 361–374, 2005.

[8] R. J. Duffin. On a question of Hadamard concerning super-biharmonic functions. *J. Math. Physics*, 27: 253–258, 1949.

[9] P. R. Garabedian. A partial differential equation arising in conformal mapping. *Pacific J. Math.*, 1:485–524, 1951.

[10] F. Gazzola, H. C. Grunau, and G. Sweers. *Polyharmonic boundary value problems. Positivity preserving and nonlinear higher order elliptic equations in bounded domains*. Berlin: Springer, 2010.

[11] H. C. Grunau and G. Sweers. Nonexistence of local minima of supersolutions for the circular clamped plate. *Pacific J. Math*, 198: 437–442, 2001.

[12] H. C. Grunau and G. Sweers. In any dimension a "clamped plate" with a uniform weight may change sign. *Nonlinear Anal.*, 97: 119–124, 2014.

[13] G. R. Lawlor. A L'hospital's rule for multivariable functions. 2012. http://arxiv.org/abs/1209.0363. Accessed 13 May 2014.

[14] J. D. Meiss. *Differential dynamical systems*, volume 14 of *Mathematical Modeling and Computation*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007.

[15] C. B. Morrey. On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. II. Analyticity at the boundary. *Amer. J. Math.*, 80: 219–237, 1958.

[16] H. S. Shapiro and M. Tegmark. An elementary proof that the biharmonic Green function of an eccentric ellipse changes sign. *SIAM Rev.*, 36(1): 99–101, 1994.

[17] R. Soranzo. A priori estimates and existence of positive solutions of a superlinear polyharmonic equation. *Dyn. Syst. Appl.*, 3(4): 465–487, 1994.

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