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EXISTENCE, UNIQUENESS AND OTHER PROPERTIES OF THE LIMIT CYCLE OF A GENERALIZED VAN DER POL EQUATION

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Abstract. In this article, we study the bifurcation of limit cycles from the linear oscillator $\dot{x} = y$, $\dot{y} = -x$ in the class

$$\dot{x} = y, \quad \dot{y} = -x + \varepsilon y^{p+1}\big(1 - x^{2q}\big),$$

where ε is a small positive parameter tending to 0, $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$. We prove that the above differential system, in the global plane where $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$, has a unique limit cycle. More specifically, the existence of a limit cycle, which is the main result in this work, is obtained by using the Poincaré's method, and the uniqueness can be derived from the work of Sabatini and Villari [6]. We also investigate and some other properties of this unique limit cycle for some special cases of this differential system. Such special cases have been studied by Minorsky [3] and Moremedi et al. [4].

1. Introduction

In this article, we study the second part of Hilbert's 16th problem for a generalized Van der Pol equation. More specifically, we consider the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \varepsilon y^{p+1}\big(1 - x^{2q}\big), \end{aligned} \tag{1.1}$$

where $p \in \mathbb{N}_0$ is even, $q \in \mathbb{N}$ and $0 < \varepsilon \ll 1$. System (1.1) reduces to the Van der Pol equation for $p = 0$ and $q = 1$. Our purpose here is to find an upper bound for the number of limit cycles for system (1.1), depending only on the degree of its polynomials.

System (1.1) is the generalized Van der Pol equation of the form

$$\ddot{x} - \varepsilon(\dot{x})^{p+1}\big(1 - x^{2q}\big) + x = 0, \tag{1.2}$$

where $p \in \mathbb{N}_0$ is even, $q \in \mathbb{N}$ and $0 < \varepsilon \ll 1$. We search to find an upper bound for the number of limit cycles for equation (1.2), depending only on p and q. We prove that the generalized Van der Pol equation (1.2) has a unique limit cycle, and it is simple and stable. We also examine the manner in which the position and size of the limit cycle depend on p and q.

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Several other generalizations of the Van der Pol equation have been considered in the literature. Minorsky [3] has considered a generalized Van der Pol equation of the form

$$\ddot{x} - \varepsilon\dot{x}\big(1 - x^{2q}\big) + x = 0, \tag{1.3}$$

where $q \in \mathbb{N}$ and $0 < \varepsilon \ll 1$. For $q = 1$, equation (1.3) reduces to the Van der Pol equation. For $p = 0$ equations (1.2) and (1.3) are identical. By applying a perturbation method, he showed for (1.3) that the stationary amplitude A_0, to first order in ε, is

$$A_0 = \Big(\frac{\int_0^{2\pi} \sin^2(t)\,dt}{\int_0^{2\pi} \sin^2(t)\cos^{2q}(t)\,dt}\Big)^{1/(2q)}. \tag{1.4}$$

For $q = 1$, 2 and 3, Minorsky found from (1.4) that $A_0 = 2$, 1.68 and 1.53, respectively.

The solution of the generalized Rayleigh equation

$$\ddot{y} - \varepsilon\dot{y}\Big(1 - \frac{1}{2q+1}(\dot{y})^{2q}\Big) + y = 0, \tag{1.5}$$

where $q \in \mathbb{N}$, is closely related to the solution of (1.3). For, if we differentiate (1.5) with respect to t and let $\dot{y} = x$, then x satisfies (1.3). Hence, results for (1.5) can be derived from the corresponding results for (1.3).

Holmes and Rand [2] have examined the qualitative behaviour of the non-linear oscillations governed by a differential equation of the form

$$\ddot{x} + \dot{x}\big(\alpha + \gamma x^2\big) + \beta x + \delta x^3 = 0,$$

where α, β, γ and δ are constants; $\alpha = -1$, $\beta = 1$, $\gamma = 1$ and $\delta = 0$ corresponds to the Van der Pol equation. They investigated the presence of local and global bifurcations and considered their physical significance.

A more general class of equations, containing (1.2) as a special case, has the form

$$\ddot{x} + \dot{x}\phi(x, \dot{x}) + x = 0, \tag{1.6}$$

and was studied in [7] and [8]. They obtained conditions about the existence and uniqueness of limit cycles of (1.6). In general, we observe that the existence and uniqueness theorem for limit cycles of (1.6) proved there does not apply for equation (1.2).

The plan of this paper is as follows. In Section 2 we will make some elementary remarks about small perturbation of a Hamiltonian system. Section 3 will be devoted to study system (1.1).

2. Elementary remarks about small perturbation of a Hamiltonian system

We consider the system

$$\begin{aligned} \dot{x} &= y + \varepsilon f_1(x, y), \\ \dot{y} &= -x + \varepsilon f_2(x, y), \end{aligned} \tag{2.1}$$

where $0 < \varepsilon \ll 1$ and f_1, f_2 are C^1 functions of x and y, which is a perturbation of the linear harmonic oscillator

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x, \end{aligned}$$

which has all the solutions periodic with:

$$x^0(t) = A\cos(t-t_0) \quad \text{and} \quad y^0(t) = -A\sin(t-t_0).$$

In general, the phase curves of (2.1) are not closed and it is possible to have the form of a spiral with a small distance of order ε between neighboring turns. In order to decide if the phase curve approaches the origin or recedes from it, we consider the function (mechanic energy)

$$E(x,y) = \frac{1}{2}\big(x^2+y^2\big).$$

It is easy to compute the derivative of the energy and it is proportional to ε:

$$\frac{d}{dt}E(x,y) = x\dot{x} + y\dot{y} = \varepsilon\big(xf_1(x,y) + yf_2(x,y)\big) =: \varepsilon\dot{E}(x,y). \tag{2.2}$$

We want information for the sign of the quantity

$$\int_0^{T(\varepsilon)} \varepsilon\dot{E}\big(x^\varepsilon(t), y^\varepsilon(t)\big)dt =: \Delta E, \tag{2.3}$$

which corresponds to the change of energy of $(x^\varepsilon(t), y^\varepsilon(t))$ in one complete turn: $y^\varepsilon(0) = y^\varepsilon(T(\varepsilon)) = 0$. Using the theorem of continuous dependence on parameters in ODEs, one can prove the following lemma (see [1]):

Lemma 2.1. *For* (2.3) *we have*

$$\Delta E = \varepsilon\int_0^{2\pi} \dot{E}\big(A\cos(t-t_0), -A\sin(t-t_0)\big)dt + o(\varepsilon). \tag{2.4}$$

Let

$$F(A) := \int_0^{2\pi} \dot{E}\big(x^0(t), y^0(t)\big)dt, \tag{2.5}$$

and we write (2.4) as

$$\Delta E = \varepsilon\Big[F(A) + \frac{o(\varepsilon)}{\varepsilon}\Big].$$

Using the implicit function theorem, one can prove the following theorem, which is the Poincaré's method (see [1]):

Theorem 2.2. *If the function F given by* (2.5), *has a positive simple root A_0, namely*

$$F(A_0) = 0 \quad \textit{and} \quad F'(A_0) \neq 0,$$

then (2.1) *has a periodic solution with amplitude $A_0 + O(\varepsilon)$ for $0 < \varepsilon \ll 1$.*

3. The non-linear equation $\ddot{x} - \varepsilon(\dot{x})^{p+1}\big(1-x^{2q}\big) + x = 0$

In this section, we prove that system (1.1) has a unique limit cycle, and it is simple and stable. We present this main result in Theorem 3.1. In Proposition 3.3 we study the system (1.1), with $p \in \mathbb{N}_0$ is even, $q \in \mathbb{N}$ satisfying $p+2 = 2q$. The system (1.1), in the case where $p=0$ and $q \to \infty$ will be studied in Proposition 3.5 and in the case where $q=1$ and $p \to \infty$ will be studied in Proposition 3.8.

Our main result in this section is given in the following theorem.

Theorem 3.1. *System (1.1), where $p \in \mathbb{N}_0$ is even, $q \in \mathbb{N}$ and $0 < \varepsilon \ll 1$ has the unique limit cycle*

$$x^2 + y^2 = \Big[\frac{(p+2q+2)(p+2q)\dots(2q+2)}{(p+2)p\dots4\cdot2}\frac{2q(2q-2)\dots4\cdot2}{(2q-1)(2q-3)\dots3\cdot1}\Big]^{1/q} + O(\varepsilon),$$

and it is simple and stable.

Proof. From (2.2) we have

$$\dot{E}(x,y) = y^{p+2}\big(1 - x^{2q}\big), \tag{3.1}$$

where $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$. Substituting (3.1) into (2.5), we obtain that

$$F(A) = \int_0^{2\pi} (y^0(t))^{p+2}\big(1 - (x^0(t))^{2q}\big)dt, \tag{3.2}$$

where $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$. Substituting $x^0(t) = A\cos(t-t_0)$ and $y^0(t) = -A\sin(t-t_0)$ into (3.2), and using the assumption that $p \in \mathbb{N}_0$ is even we get

$$F(A) = A^{p+2}\Big[\int_0^{2\pi} \sin^{p+2}(t-t_0)dt - A^{2q}\int_0^{2\pi} \sin^{p+2}(t-t_0)\cos^{2q}(t-t_0)dt\Big]. \tag{3.3}$$

Let

$$c_1 := \int_0^{2\pi} \sin^{p+2}(t-t_0)dt,$$
$$c_2 := \int_0^{2\pi} \sin^{p+2}(t-t_0)\cos^{2q}(t-t_0)dt,$$

where $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$. Using the fact that

$$c_1 = 4\int_0^{\pi/2} \sin^{p+2}(t-t_0)dt,$$

from Proposition 4.2, we obtain

$$c_1 = 2\frac{(p+1)(p-1)\dots3\cdot1}{(p+2)p\dots4\cdot2}\pi.$$

Using the fact that

$$c_2 = 4\int_0^{\pi/2} \sin^{p+2}(t-t_0)\cos^{2q}(t-t_0)dt,$$

from Proposition 4.1, we obtain

$$c_2 = 2\frac{(p+1)(p-1)\dots5\cdot3\cdot1}{(p+2q+2)(p+2q)\dots(2q+2)}\frac{(2q-1)(2q-3)\dots3\cdot1}{2q(2q-2)\dots4\cdot2}\pi.$$

Substituting c_1 and c_2 given as above into (3.3) it follows that

$$\begin{aligned}F(A) = 2\pi A^{p+2}\Big[&\frac{(p+1)(p-1)\dots3\cdot1}{(p+2)p\dots4\cdot2}\\ &- \frac{(p+1)(p-1)\dots5\cdot3\cdot1}{(p+2q+2)(p+2q)\dots(2q+2)}\frac{(2q-1)\dots3\cdot1}{2q\dots4\cdot2}A^{2q}\Big].\end{aligned}$$

Now, for $A > 0$ the polynomial F has the root

$$A = \Big[\frac{(p+2q+2)(p+2q)\dots(2q+2)}{(p+2)p\dots4\cdot2}\frac{2q(2q-2)\dots4\cdot2}{(2q-1)(2q-3)\dots3\cdot1}\Big]^{1/(2q)}.$$

Let

$$A_0 = A_0(p,q) := \Big[\frac{(p+2q+2)(p+2q)\dots(2q+2)}{(p+2)p\dots 4\cdot 2}\,\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\Big]^{1/(2q)}, \tag{3.4}$$

where $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$.

For the derivative of F we have that

$$\begin{aligned} F'(A) = 2\pi A^{p+1}\Big[&\frac{(p+1)(p-1)\dots 3\cdot 1}{p(p-2)\dots 4\cdot 2} \\ &- \frac{(p+1)(p-1)\dots 3\cdot 1}{(p+2q)(p+2q-2)\dots(2q+2)}\,\frac{(2q-1)\dots 3\cdot 1}{2q\dots 4\cdot 2}A^{2q}\Big]. \end{aligned}$$

We compute the derivative of F at A_0 and we get

$$F'(A_0) = -4\pi A_0^{p+1}\frac{(p+1)(p-1)\dots 3\cdot 1}{(p+2)p\dots 4\cdot 2}\cdot q \neq 0,$$

using the assumptions that $p \in \mathbb{N}_0$ is even, $q \in \mathbb{N}$ and $A_0 > 0$. So, from Theorem 2.2, it follows that (1.1) has a limit cycle close to the circle $x^2+y^2 = A_0^2$. Moreover, since $F'(A_0) < 0$, this limit cycle is simple and stable.

Let now prove that the number of limit cycles for system (1.1), with ε small is exactly one. The proof of this can be derived from the work of Sabatini and Villari [6] using Corollary 1 proved there. We first note that the system (1.1) can be written and in the form

$$\begin{gathered} \dot{x} = y - \varepsilon x^{p+1}\big(y^{2q}-1\big), \\ \dot{y} = -x, \end{gathered}$$

where $p \in \mathbb{N}_0$ is even, $q \in \mathbb{N}$ and $0 < \varepsilon \ll 1$. As we already saw, Poincaré's method (see Theorem 2.2) ensures the existence of a limit cycle for (1.1). Since $a = -1, b = 1, G(x) = \frac{x^2}{2}$, one has $G(a) = G(b)$, so the hypotheses of Corollary 1 hold (see [6]), and the system (1.1) has exactly one limit cycle. This completes the proof that (1.1) has exactly one limit cycle.

So, we prove that (1.1) has a unique limit cycle, and it is simple and stable. $\square$

Remark 3.2. The expression (1.4) obtained by Minorsky, is a special case of the expression (3.4) which we found. Indeed, for $p = 0$ it can be verified that (3.4) equals (1.4). This may be done by evaluating the integral in the denominator of (1.4), using the Proposition 4.1 from the appendix.

Proposition 3.3. *System* (1.1)*, with* $p \in \mathbb{N}_0$ *is even,* $q \in \mathbb{N}$ *satisfying* $p+2 = 2q$*, and* $0 < \varepsilon \ll 1$ *has the unique limit cycle* $x^2+y^2 = 4 + O(\varepsilon)$*, and it is simple and stable.*

Proof. From Theorem 3.1 it follows that system (1.1), with $p \in \mathbb{N}_0$ is even, $q \in \mathbb{N}$ and $0 < \varepsilon \ll 1$ has a unique limit cycle, and it is simple and stable. It remains to prove that

$$\Big[\frac{(p+2q+2)(p+2q)\dots(2q+2)}{(p+2)p\dots 4\cdot 2}\,\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\Big]^{1/q} = 4, \tag{3.5}$$

when $p+2 = 2q$.

By the assumption that $p+2 = 2q$ the left-hand side of (3.5) gives

$$\Big[\frac{2^q(2q)(2q-1)(2q-2)\dots(q+2)(q+1)}{(2q-1)(2q-3)(2q-5)\dots 5\cdot 3\cdot 1}\Big]^{1/q} = 2\Big[\frac{2q(2q-1)\dots(q+2)(q+1)}{(2q-1)(2q-3)\dots 5\cdot 3\cdot 1}\Big]^{1/q}.$$

Hence it suffices to show that
$$\Big[\frac{2q(2q-1)(2q-2)\dots(q+2)(q+1)}{(2q-1)(2q-3)(2q-5)\dots 5\cdot 3\cdot 1}\Big]^{1/q}=2.$$

Claim. It is valid that
$$\frac{2q(2q-1)(2q-2)\dots(q+2)(q+1)}{(2q-1)(2q-3)(2q-5)\dots 5\cdot 3\cdot 1}=2^q,\quad \forall q\in\mathbb{N}.$$

Proof. It will be proved by induction on q. For $q=1$, we have $\frac{2}{1}=2^1$, therefore the claim is valid for $q=1$. Supposing that the claim is valid for q, we will prove that it is true and for $q+1$, namely
$$\frac{\big[2(q+1)\big](2q+1)(2q)(2q-1)\dots(q+3)(q+2)}{(2q+1)(2q-1)(2q-3)(2q-5)\dots 5\cdot 3\cdot 1}=2^{q+1}. \tag{3.6}$$
The left-hand side of (3.6) is equal to
$$2(q+1)\frac{2q(2q-1)(2q-2)\dots(q+2)}{(2q-1)(2q-3)\dots 5\cdot 3\cdot 1}=2\cdot 2^q=2^{q+1},$$
which is the right-hand side of (3.6). Therefore, the claim is valid for every $q\in\mathbb{N}$. □

This completes the proof of the proposition. □

Remark 3.4. It is well known that the Van der Pol equation with $0<\varepsilon\ll 1$ has the unique limit cycle $x^2+y^2=4+O(\varepsilon)$, and it is simple and stable. This arises and from Proposition 3.3 with $p=0$ and $q=1$.

In the next proposition, we give a different proof, much more elementary than the proof has been given by Moremedi et al. [4], concerning the decreases of the amplitude of the limit cycle of system (1.1) with $p=0$ and $0<\varepsilon\ll 1$, as q increases.

Proposition 3.5. *System* (1.1)*, with $p=0$, $q\in\mathbb{N}$ and $0<\varepsilon\ll 1$ has a unique limit cycle which is simple, stable and its amplitude decreases monotonically from 2 to 1 as q increases from $q=1$. Therefore, the unique limit cycle of the system* (1.1)*, with $p=0$ has the equation $x^2+y^2=1+O(\varepsilon)$ as $q\to\infty$.*

Proof. From Theorem 3.1 it follows that system (1.1), with $p=0$, $q\in\mathbb{N}$ and $0<\varepsilon\ll 1$ has a unique limit cycle, and it is simple and stable. From (3.4) when $p=0$ it follows that
$$A_0=\Big[\frac{2q+2}{2}\,\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\Big]^{1/(2q)}.$$
Let
$$A_0(q):=\Big[\frac{2q+2}{2}\,\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\Big]^{1/(2q)},\quad q\in\mathbb{N}. \tag{3.7}$$
Clearly, $A_0(1)=2$. In order to prove that the sequence $A_0(q)$, $q\in\mathbb{N}$ given by (3.7) is strictly decreasing we must show that $A_0(q+1)<A_0(q)$ for all $q\in\mathbb{N}$.

We have that
$$\begin{aligned}A_0(q+1)&=\Big[\frac{2q+4}{2}\,\frac{(2q+2)(2q)\dots 4\cdot 2}{(2q+1)(2q-1)\dots 3\cdot 1}\Big]^{\frac{1}{2(q+1)}}\\&=\Big[\frac{2q+4}{2q+1}\Big]^{\frac{1}{2(q+1)}}\Big[\frac{2q+2}{2}\,\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\Big]^{\frac{1}{2q}-\frac{1}{2q(q+1)}}\end{aligned}$$

$$= \big[s(q)\big]^{\frac{1}{2(q+1)}} A_0(q),$$

where

$$s(q) = \frac{2q+4}{2q+1}\Big[\frac{1}{q+1}\frac{(2q-1)(2q-3)\dots 3\cdot 1}{2q(2q-2)\dots 4\cdot 2}\Big]^{1/q}, \quad q \in \mathbb{N}.$$

Now, in order to show that $A_0(q+1) < A_0(q)$, it suffices to show that $s(q) < 1$ for all $q \in \mathbb{N}$. We have that

$$s(q) < \frac{2q+4}{2q+1}\frac{1}{(q+1)^{1/q}}.$$

Claim I. It is valid that

$$\frac{2q+4}{2q+1} \le (q+1)^{1/q}, \quad \forall\, q \in \mathbb{N}. \tag{3.8}$$

Proof. The inequality (3.8) is valid for $q = 1, \dots, 5$, as it can easily be checked. In order to prove (3.8) for $q \in \mathbb{N}$, $q \ge 6$ we will show that

$$1 + \frac{2}{q} < q^{1/q} \Longleftrightarrow \big(1 + \frac{2}{q}\big)^q < q, \quad \forall q \in \mathbb{N},\, q \ge 6. \tag{3.9}$$

One can easily check that the inequality (3.9) is valid for $q = 6$ and 7. Since $\lim_{q\to\infty}\big(1 + \frac{2}{q}\big)^q = \mathrm{e}^2$, in order to prove (3.9) for $q \in \mathbb{N}$, $q \ge 8$, it suffices to show that the sequence $\big(1 + \frac{2}{q}\big)^q$, $q \in \mathbb{N}$, is strictly increasing. Notice that

$$\begin{aligned}
\big(1 + \frac{2}{q}\big)^q < \big(1 + \frac{2}{q+1}\big)^{q+1} &\Longleftrightarrow \frac{q+1}{q+3} < \Big[\frac{q(q+3)}{(q+1)(q+2)}\Big]^q \\
&\Longleftrightarrow 1 - \frac{2}{q+3} < \Big[1 - \frac{2}{(q+1)(q+2)}\Big]^q.
\end{aligned}$$

Now, using Bernoulli's inequality, we have for $q \in \mathbb{N}$ that

$$\Big[1 - \frac{2}{(q+1)(q+2)}\Big]^q \ge 1 - \frac{2q}{(q+1)(q+2)}.$$

Since is valid that

$$\frac{1}{q+3} > \frac{q}{(q+1)(q+2)},$$

the proof that the sequence $\big(1 + \frac{2}{q}\big)^q$, $q \in \mathbb{N}$ is strictly increasing is complete.

So, we have proved the inequality (3.8) for every $q \in \mathbb{N}$. Therefore,

$$s(q) < 1, \quad \forall q \in \mathbb{N},$$

which proves that the sequence $A_0(q)$, $q \in \mathbb{N}$ is strictly decreasing.

Now, note that (3.7) gives

$$A_0(q) = [(q+1)^{1/q}]^{1/2}[(2q+1)^{1/(2q)}]^{1/2}\Big[\frac{1}{2q+1}\Big(\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\Big)^2\Big]^{1/(4q)}. \tag{3.10}$$

□

Claim II. It is valid that

$$\lim_{q\to\infty}\Big[\frac{1}{2q+1}\Big(\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\Big)^2\Big]^{1/(4q)} = 1. \tag{3.11}$$

Proof. From the inequality $0 < \sin t < 1$, $t \in (0, \pi/2)$ (with induction) we have that $\sin^{2q+1} t < \sin^{2q} t < \sin^{2q-1} t$, for every $t \in (0, \pi/2)$ and $q \in \mathbb{N}$. So, we have that

$$\int_0^{\pi/2} \sin^{2q+1} t\, dt < \int_0^{\pi/2} \sin^{2q} t\, dt < \int_0^{\pi/2} \sin^{2q-1} t\, dt. \tag{3.12}$$

Using Proposition 4.2 from the appendix, (3.12) leads to

$$\frac{1 \cdot 3 \dots (2q-1)}{2 \cdot 4 \dots (2q-2)} < \frac{2 \cdot 4 \dots (2q-2)2q}{1 \cdot 3 \dots (2q-3)(2q-1)} \frac{2}{\pi} < \frac{1 \cdot 3 \dots (2q+1)}{2 \cdot 4 \dots 2q}. \tag{3.13}$$

Multiplying (3.13) by

$$\frac{2 \cdot 4 \dots (2q-2)2q}{1 \cdot 3 \dots (2q-1)(2q+1)} \frac{\pi}{2},$$

we get

$$\frac{2q}{2q+1} \frac{\pi}{2} < \frac{1}{2q+1} \Big[\frac{2 \cdot 4 \dots (2q-2)2q}{1 \cdot 3 \dots (2q-3)(2q-1)}\Big]^2 < \frac{\pi}{2}, \tag{3.14}$$

and then the inequality

$$\Big(\frac{2q}{2q+1}\Big)^{1/(4q)} \Big(\frac{\pi}{2}\Big)^{1/(4q)} < \Big[\frac{1}{2q+1}\Big(\frac{2 \cdot 4 \dots (2q-2)2q}{1 \cdot 3 \dots (2q-3)(2q-1)}\Big)^2\Big]^{1/(4q)} < \Big(\frac{\pi}{2}\Big)^{1/(4q)},$$

implies (3.11). □

Using (3.11), from (3.10), we easily obtain $\lim_{q\to\infty} A_0(q) = 1$. The proof of Proposition 3.5 is complete. □

Remark 3.6. The uniqueness of the limit cycle for the system (1.1), with $p = 0$, $q \in \mathbb{N}$ studied in Proposition 3.5 follows and from the fact that the function $\phi(x, y) = -\varepsilon(1 - x^{2q})$ is strictly star-shaped (see [7],[8]).

Remark 3.7. From (3.14) it follows that

$$\lim_{q\to\infty} \frac{1}{2q+1} \Big[\frac{2 \cdot 4 \dots (2q-2)2q}{1 \cdot 3 \dots (2q-3)(2q-1)}\Big]^2 = \frac{\pi}{2},$$

which is the Wallis's product. It is exciting and unexpected how this limit of Wallis appears in the proof of Proposition 3.5.

Proposition 3.8. *System* (1.1)*, with* $p \in \mathbb{N}_0$ *is even,* $q = 1$ *and* $0 < \varepsilon \ll 1$ *has a unique limit cycle which is simple, stable and its amplitude increases monotonically from* 2 *to infinity as* p *increases from* $p = 0$.

Proof. From Theorem 3.1 it follows that system (1.1), with $p \in \mathbb{N}_0$ is even, $q = 1$ and $0 < \varepsilon \ll 1$ has a unique limit cycle, and it is simple and stable. From (3.4) when $q = 1$ it follows that

$$A_0 = \Big[\frac{(p+4)(p+2)p \dots 6 \cdot 4}{(p+2)p \dots 4 \cdot 2} \cdot \frac{2}{1}\Big]^{1/2} = (p+4)^{1/2}.$$

Let $A_0(p) := (p+4)^{1/2}$, $p \in \mathbb{N}_0$ is even. Clearly, $A_0(0) = 2$. Obviously $A_0(p) < A_0(p+1)$, for all $p \in \mathbb{N}_0$ is even and $A_0(p) \to \infty$ as $p \to \infty$ and so the proof is complete. □

Remark 3.9. We make now an observation on the type of the bifurcation phenomenon of limit cycles encountered in Proposition 3.8. Not the "large amplitude limit cycle" is encountered in Proposition 3.8 but the "medium amplitude limit cycle". For given p the limit cycle of (1.1), with $q = 1$, has a finite limiting radius and therefore is called "medium amplitude limit cycle". When increasing p also the radius of the limiting circle increases; in particular when $p \to \infty$ then the limiting radius also tends to ∞. The "large amplitude limit cycle" would disappear at ∞ when the bifurcation parameter ε tends to 0.

4. Appendix

Here we list some important formulas used in Section 3 (see [5]).

Proposition 4.1. *For each m, $n \in \mathbb{N}$ and even,*

$$\int_0^{\pi/2} \sin^m(t)\cos^n(t)\,dt = \frac{(m-1)(m-3)\dots 5\cdot 3\cdot 1}{(m+n)(m+n-2)\dots(n+2)}\,\frac{(n-1)(n-3)\dots 3\cdot 1}{n(n-2)\dots 4\cdot 2}\,\frac{\pi}{2}.$$

Proposition 4.2. *For each $n \in \mathbb{N}$*

$$\int_0^{\pi/2} \sin^{2n-1}(t)\,dt = \frac{2\cdot 4\dots(2n-2)}{1\cdot 3\dots(2n-1)},$$
$$\int_0^{\pi/2} \sin^{2n}(t)\,dt = \frac{1\cdot 3\dots(2n-1)}{2\cdot 4\dots 2n}\,\frac{\pi}{2}.$$

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