Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 225, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SOLUTION OF AN INITIAL-VALUE PROBLEM FOR PARABOLIC EQUATIONS VIA MONOTONE OPERATOR METHODS

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Abstract. We study a general initial-value problem for parabolic equations in Banach spaces, by using a monotone operator method. We provide sufficient conditions for the existence of solution to such problem.

1. Introduction

Over the years, the theory of differential equations has been well investigated and consequently the methods developed for their solution are strongly related to the particular equation, see for instance [3]. In [6], Evans furnishes a comprehensive study on this topic. In this paper, referring to heat equation variants, we consider the existence of solutions for the following initial-value problem for parabolic equations:

$$\begin{gathered} u_t(x,t) = q(x,t,u(x,t)) + k u_{xx}(x,t), \quad x \in \mathbb{R},\ t \in (0,T], \\ u(x,0) = \varphi(x), \quad x \in \mathbb{R}, \end{gathered} \tag{1.1}$$

where we assume that $q : \mathbb{R} \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\varphi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, φ and φ' are bounded and $k, T > 0$.

A function $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ is a solution of the parabolic equation (1.1) if:

(S1) $u \in C(\mathbb{R} \times [0,T])$;
(S2) $u_t, u_{xx} \in C(\mathbb{R} \times [0,T])$;
(S3) u is bounded in $\mathbb{R} \times [0,T]$;
(S4) $u_t(x,t) = q(x,t,u(x,t)) + u_{xx}(x,t)$ for all $(x,t) \in \mathbb{R} \times [0,T]$.

The existence and uniqueness of solutions for general initial-value problems on some (possibly small) interval have been studied extensively, see for instance [2, 12] and the references therein. Practically, researchers are interested in establishing how large this interval might be and how solutions of initial-value problems change when the differential equation or initial conditions are perturbed. In this direction, some results are obtained by using simple notions and techniques of fixed point theory. For instance, the well-known Banach's contraction principle is an important tool for studying the existence and uniqueness of fixed points of certain

2000 *Mathematics Subject Classification.* 35K15, 47H07, 47H10.
Key words and phrases. Banach spaces; metric spaces; monotone operators; parabolic equations; transitive relations.
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Submitted April 18, 2014. Published October 22, 2014.

self-mappings in metric spaces. Also, it provides a constructive method to find those fixed points. Finally, various applications to matrix equations, ordinary differential equations, and integral equations were presented by using this principle and its generalizations and extensions, see for instance [1, 5, 9, 11].
Thus, we give a generalization of a fixed point theorem for monotone mappings, due to Gordji et al [8], in the setting of complete metric spaces endowed with a transitive relation. Then, by combining our result with Green's function formalism, we discuss the existence of solution for the initial-value problem (1.1) via monotone operator methods.

2. Preliminaries

Let X be a nonempty set. In the sequel $\mathcal{M}$ denotes a transitive relation on X, that is, $\mathcal{M}$ is a subset of $X \times X$ such that $(x, z) \in \mathcal{M}$ whenever $(x, y), (y, z) \in \mathcal{M}$. Let $f : X \to X$ be a mapping and $\mathcal{M}$ a subset of $X \times X$. The set $\mathcal{M}$ is f-invariant if $(fx, fy) \in \mathcal{M}$ whenever $(x, y) \in \mathcal{M}$.

Example 2.1. Let $\preceq$ be a partial order on X such that $(X, \preceq)$ is a partially ordered set. Then
$$\mathcal{M} = \{(x, y) \in X \times X : x \preceq y\}$$
is a transitive relation on X. Also if $f : X \to X$ is a nondecreasing mapping, then the set $\mathcal{M}$ is f-invariant.

Now, we present a generalization of Banach's contraction principle, which will be extended in this paper. This result is due to Geraghty [7] and was proved by using the following class of functions.

Definition 2.2. Let Γ denote the class of functions $\beta : [0, +\infty) \to [0, 1)$ which satisfy the condition:
$$\beta(t_n) \to 1 \quad \text{implies} \quad t_n \to 0, \text{ as } n \to +\infty.$$

Geraghty proved the following result.

Theorem 2.3. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Assume that there exists $\beta \in \Gamma$ such that*
$$d(fx, fy) \le \beta(d(x, y))d(x, y)$$
for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and, for any choice of the initial point $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = fx_{n-1}$ for each $n \in \mathbb{N}$ converges to the point z.

On the same line of research, Gordji et al [8] proved existence and uniqueness results. Before giving a comprehensive theorem, we recall the following class of control functions.

Definition 2.4. Let Ψ denote the class of functions $\psi : [0, +\infty) \to [0, +\infty)$ which satisfy the following conditions:

(i) ψ is nondecreasing;
(ii) ψ is continuous;
(iii) $\psi(t) = 0$ if and only if $t = 0$.

Note that, unlike Gordji et al [8], we do not assume that the function ψ is subadditive. Then, we have:

Theorem 2.5 ([8, Theorems 2.2-2.3]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping such that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that there exist $\beta \in \Gamma$ and subadditive $\psi \in \Psi$ such that*

$$\psi(d(fx, fy)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$$

for all $x, y \in X$ with $x \succeq y$. Assume that either f is continuous or X is such that if an increasing sequence $\{x_n\}$ converges to x, then $x_n \preceq x$ for each $n \in \mathbb{N}$. Then f has a fixed point. Besides, if for all $x, y \in X$, there exists $z \in X$ which is comparable to x and y, then f has a unique fixed point in X.

For our further use, we state also the following lemma.

Lemma 2.6. *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X such that:*

$$\lim_{n\to+\infty} d(x_{n+1}, x_n) = 0.$$

If $\{x_{2n}\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers, with $m_k < n_k$, such that the following four sequences $\{d(x_{2m_k}, x_{2n_k})\}$, $\{d(x_{2m_k}, x_{2n_k+1})\}$, $\{d(x_{2m_k-1}, x_{2n_k})\}$, $\{d(x_{2m_k-1}, x_{2n_k+1})\}$ tend to ϵ as $k \to +\infty$.

Notice that assertions similar to Lemma 2.6 (see, e.g. [10]) were used (and proved) in the course of proofs of some fixed point theorems in various papers.

3. Main Results

Next, we give two existence results with and without continuity hypothesis.

Let $f : X \to X$ be a mapping and denote

$$M(x, y) := \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(fx, y)]\right\}$$

for all $x, y \in X$. In the first theorem, we use the continuity hypothesis of f.

Theorem 3.1. *Let (X, d) be a complete metric space endowed with a transitive relation $\mathcal{M}$ on X and $f : X \to X$ be a mapping. Assume that the following conditions hold:*

(i) *there exist $\beta \in \Gamma$ and $\psi \in \Psi$ such that*

$$\psi(d(fx, fy)) \leq \beta(M(x, y))\psi(M(x, y)) \tag{3.1}$$

for all $(x, y) \in \mathcal{M}$ with $x \neq y$;

(ii) *there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{M}$;*

(iii) *$\mathcal{M}$ is f-invariant;*

(iv) *f is continuous.*

Then f has a fixed point.

Proof. Let $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{M}$. We consider the sequence $\{x_n\}$ defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n-1} = x_n$ for some $n \in \mathbb{N}$, then x_{n-1} is a fixed point of f and the existence of a fixed point is proved. Now, we suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. From $(x_0, x_1) = (x_0, fx_0) \in \mathcal{M}$, since $\mathcal{M}$ is f-invariant, we deduce $(x_1, x_2) = (fx_0, fx_1) \in \mathcal{M}$. This implies

$$(x_{n-1}, x_n) \in \mathcal{M} \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Now, using (3.1) with $x = x_{n-1}$ and $y = x_n$, we have

$$\psi(d(x_n, x_{n+1})) = \psi(d(fx_{n-1}, fx_n)) \le \beta(M(x_{n-1}, x_n))\psi(M(x_{n-1}, x_n)), \tag{3.3}$$

where

$$\begin{aligned}
&M(x_{n-1}, x_n)\\
&= \max\{d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_n), \frac{1}{2}[d(x_{n-1}, fx_n) + d(fx_{n-1}, x_n)]\}\\
&= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}d(x_{n-1}, x_{n+1})\}\\
&= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.
\end{aligned}$$

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, by (3.3), we have

$$\psi(d(x_n, x_{n+1})) \le \beta(d(x_n, x_{n+1}))\,\psi(d(x_n, x_{n+1})) < \psi(d(x_n, x_{n+1}))$$

which is a contradiction. Then $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. By (3.3), we obtain

$$\psi(d(x_n, x_{n+1})) \le \beta(d(x_{n-1}, x_n))\psi(d(x_{n-1}, x_n)) < \psi(d(x_{n-1}, x_n)). \tag{3.4}$$

Thus $\{d(x_{n-1}, x_n)\}$ is a decreasing sequence of nonnegative numbers and hence there exists

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = r \ge 0.$$

Assume $r > 0$. Since $\psi(d(x_{n-1}, x_n)) \neq 0$ for all $n \in \mathbb{N}$, from (3.4) we deduce

$$\frac{\psi(d(x_n, x_{n+1}))}{\psi(d(x_{n-1}, x_n))} \le \beta(d(x_{n-1}, x_n)) \le 1 \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

Letting $n \to +\infty$ in (3.5), by the continuity of the function ψ, we obtain

$$\lim_{n\to+\infty} \beta(d(x_{n-1}, x_n)) = 1.$$

On the other hand, since $\beta \in \Gamma$, we have $\lim_{n\to+\infty} d(x_{n-1}, x_n) = 0$ and so $r = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. This implies that $\{x_{2n}\}$ is not a Cauchy sequence. Since $\mathcal{M}$ is a transitive relation, from $(x_{n-1}, x_n) \in \mathcal{M}$ for all $n \in \mathbb{N}$, we deduce that $(x_m, x_n) \in \mathcal{M}$ for all $m, n \in \mathbb{N}$ with $m < n$. If ϵ, $\{m_k\}$ and $\{n_k\}$ are as in Lemma 2.6, using (3.1) with $x = x_{2m_k-1}$ and $y = x_{2n_k}$ obviously we can assume that $x_{2m_k-1} \neq x_{2n_k}$, it follows that

$$\psi(d(x_{2m_k}, x_{2n_k+1})) \le \beta(M(x_{2m_k-1}, x_{2n_k}))\psi(M(x_{2m_k-1}, x_{2n_k})) \tag{3.6}$$

where

$$\begin{aligned}
M(x_{2m_k-1}, x_{2n_k}) = \max\Big\{&d(x_{2m_k-1}, x_{2n_k}), d(x_{2m_k-1}, x_{2m_k}), d(x_{2n_k}, x_{2n_k+1}),\\
&\frac{1}{2}[d(x_{2m_k-1}, x_{2n_k+1}) + d(x_{2m_k}, x_{2n_k})]\Big\}.
\end{aligned}$$

Then, for $k \to +\infty$, we obtain $M(x_{2m_k-1}, x_{2n_k}) \to \epsilon$.

We can assume $\psi(M(x_{2m_k-1}, x_{2n_k})) > 0$ for all $k \in \mathbb{N}$. From (3.6), we have

$$\frac{\psi(d(x_{2m_k}, x_{2n_k+1}))}{\psi(M(x_{2m_k-1}, x_{2n_k}))} \le \beta(M(x_{2m_k-1}, x_{2n_k})) \le 1$$

for all $k \in \mathbb{N}$. Now, letting $k \to +\infty$ in the previous inequality, by the continuity of the function ψ and Lemma 2.6, we obtain

$$\lim_{k\to+\infty} \beta(M(x_{2m_k-1}, x_{2n_k})) = 1.$$

Since $\beta \in \Gamma$, we have

$$\lim_{k\to\infty} M(x_{2m_k-1}, x_{2n_k}) = 0$$

a contradiction and hence $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, there exists $z \in X$ such that $\lim_{n\to+\infty} x_n = z$.

If f is a continuous mapping, then

$$z = \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} fx_n = fz$$

and hence $fz = z$, that is, z is a fixed point of f. □

In the next theorem, we omit the continuity hypothesis of f.

Theorem 3.2. *Let (X, d) be a complete metric space endowed with a transitive relation $\mathcal{M}$ on X and $f : X \to X$ be a mapping. Assume that the following conditions hold:*

(i) *there exist $\beta \in \Gamma$ and $\psi \in \Psi$ such that*
$$\psi(d(fx, fy)) \le \beta(M(x, y))\psi(M(x, y)) \tag{3.7}$$
for all $(x, y) \in \mathcal{M}$ with $x \neq y$;

(ii) *there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{M}$;*

(iii) *$\mathcal{M}$ is f-invariant;*

(iv) *if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in \mathcal{M}$ for all $n \in \mathbb{N}$ and $x_n \to z \in X$ as $n \to +\infty$, then $(x_n, z) \in \mathcal{M}$ for all $n \in \mathbb{N}$.*

Then f has a fixed point.

Proof. Following the proof of Theorem 3.1, we know that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Then, there exists $z \in X$ such that $x_n \to z$ as $n \to +\infty$. On the other hand, from (3.2) and the hypothesis (iv), we have

$$(x_n, z) \in \mathcal{M} \quad \text{for all } n \in \mathbb{N}.$$

We assume that $z \neq fz$. From $x_n \neq x_{n+1}$ follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \neq z$ for all $n \in \mathbb{N}$. Using (3.7) with $x = x_{n_k}$ and $y = z$, we get

$$\psi(d(fx_{n_k}, fz)) \le \beta(M(x_{n_k}, z))\psi(M(x_{n_k}, z)) < \psi(M(x_{n_k}, z)) \tag{3.8}$$

where

$$\begin{aligned} &M(x_{n_k}, z) \\ &= \max\big\{d(x_{n_k}, z), d(x_{n_k}, fx_{n_k}), d(z, fz), \frac{1}{2}[d(x_{n_k}, fz) + d(z, fx_{n_k})]\big\} \\ &= \max\big\{d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}), d(z, fz), \frac{1}{2}[d(x_{n_k}, z) + d(z, fz) + d(z, x_{n_k+1})]\big\}. \end{aligned}$$

Since $d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}) \to 0$ as $k \to +\infty$, for k great enough, we have

$$M(x_{n_k}, z) = d(z, fz).$$

Thus from (3.8), we obtain

$$\psi(d(fx_{n_k}, fz)) \le \beta(d(z, fz))\psi(d(z, fz)) \tag{3.9}$$

for k great enough. Letting $k \to +\infty$ in (3.9), by the continuity of the function ψ, we have

$$\psi(d(z, fz)) \le \beta(d(z, fz))\psi(d(z, fz)) < \psi(d(z, fz))$$

Therefore, $fz = z$; this completes the proof. □

Thus, by using Theorems 3.1 and 3.2, we are able to establish the existence of a fixed point. Next step is to give sufficient conditions for obtaining uniqueness. Precisely, we will consider the following hypothesis:

(U1) For all $(x,y) \notin \mathcal{M}$ there exists $z \in X$ such that $(x,z),(y,z) \in \mathcal{M}$ and
$$\lim_{n\to+\infty} d(f^{n-1}z, f^n z) = 0.$$

Theorem 3.3. *Adding condition* (U1) *to the hypotheses of Theorem 3.1 (resp. Theorem 3.2) we obtain uniqueness of the fixed point of f.*

Proof. Suppose that x and y, with $x \neq y$, are two fixed points of f. If $(x,y) \in \mathcal{M}$, using (3.1) we have
$$\begin{aligned}\psi(d(x,y)) &= \psi(d(fx,fy)) \le \beta(M(x,y))\psi(M(x,y))\\ &= \beta(d(x,y))\psi(d(x,y)) < \psi(d(x,y)),\end{aligned}$$
which is a contradiction and hence $x = y$. If $(x,y) \notin \mathcal{M}$, from (U1), there exists $z \in X$ such that $(x,z),(y,z) \in \mathcal{M}$. Put $z_n = f^n z$ for all $n \in \mathbb{N}$. Since $\mathcal{M}$ is f-invariant, we have $(x,z_n),(y,z_n) \in \mathcal{M}$ for all $n \in \mathbb{N}$. Now, using (3.1) for all $n \in \mathbb{N}$ such that $z_n \neq \{x,y\}$, we obtain
$$\psi(d(x,z_{n+1})) = \psi(d(fx,fz_n)) \le \beta(M(x,z_n))\psi(M(x,z_n)), \tag{3.10}$$
$$\psi(d(y,z_{n+1})) = \psi(d(fy,fz_n)) \le \beta(M(y,z_n))\psi(M(y,z_n)). \tag{3.11}$$

Step 1. Assume that there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to x$. If $\{z_{n_k}\}$ has a subsequence that converges to y, in this case we can assume that $z_{n_k} \to y$, then from $d(x,y) \le d(x,z_{n_k}) + d(z_{n_k},y)$ letting $k \to +\infty$, we obtain $d(x,y) = 0$, that is $x = y$. Now, we assume that $d(y,z_{n_k}) > 0$ for all $k \in \mathbb{N}$. From (3.11), we have
$$\frac{\psi(d(y,z_{n_k+1}))}{\psi(M(y,z_{n_k}))} \le \beta(M(y,z_{n_k})) \le 1 \tag{3.12}$$
for all $k \in \mathbb{N}$, where
$$\begin{aligned}M(y,z_{n_k}) &= \max\{d(y,z_{n_k}), d(y,fy), d(z_{n_k},fz_{n_k}), \frac{1}{2}[d(y,fz_{n_k}) + d(z_{n_k},fy)]\}\\ &= \max\{d(y,z_{n_k}), d(z_{n_k},z_{n_k+1}), \frac{1}{2}[d(y,z_{n_k+1}) + d(z_{n_k},y)]\}.\end{aligned}$$
Using the continuity of the function ψ, letting $k \to +\infty$ in (3.12), we obtain
$$\lim_{k\to+\infty} \beta(M(y,z_{n_k})) = 1 \quad \text{which implies} \quad d(y,x) = \lim_{k\to+\infty} M(y,z_{n_k}) = 0$$
a contradiction and hence $x = y$. The same holds if there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to y$.

Step 2. We consider the case that there exist $\epsilon > 0$ and $n(\epsilon) \in \mathbb{N}$ such that $d(x,z_n) \ge \epsilon$ for all $n \ge n(\epsilon)$. From condition (U1),
$$\begin{aligned}M(x,z_n) &= \max\{d(x,z_n), d(x,fx), d(z_n,fz_n), \frac{1}{2}[d(x,fz_n) + d(z_n,fx)]\}\\ &= \max\{d(x,z_n), d(z_n,z_{n+1}), \frac{1}{2}[d(x,z_{n+1}) + d(z_n,x)]\}\end{aligned}$$
and (3.10), we deduce that $M(x,z_n) = d(x,z_n)$ for n great enough. Consequently, by (3.10), the sequence $\{d(x,z_n)\}$ for n great enough is decreasing and hence

$d(x, z_n) \to r \geq 0$. Assume $r > 0$. Using the continuity of the function ψ, letting $n \to +\infty$ in (3.10), we get

$$\lim_{n\to+\infty} \beta(M(x, z_n)) = 1 \quad \text{which implies} \quad r = \lim_{n\to+\infty} M(x, z_n) = 0$$

a contradiction and hence $r = 0$. Similarly, one can prove that $d(y, z_n) \to 0$ and hence $d(x, y) = 0$, that is, $x = y$. □

Proceeding as in the proof of Theorems 3.1 and 3.3, we obtain the following theorem; to avoid repetitions the details are omitted.

Theorem 3.4. *Let (X, d) be a complete metric space endowed with a transitive relation $\mathcal{M}$ on X and $f : X \to X$ be a mapping. Assume that the following conditions hold:*

(i) *there exist $\beta \in \Gamma$ and $\psi \in \Psi$ such that*
$$\psi(d(fx, fy)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$$
for all $(x, y) \in \mathcal{M}$ with $x \neq y$;

(ii) *there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{M}$;*

(iii) *$\mathcal{M}$ is f-invariant;*

(iv) *f is continuous.*

Then f has a fixed point. In addition, the fixed point is unique provided that

(v) *for all $(x, y) \notin \mathcal{M}$ there exists $z \in X$ such that $(x, z), (y, z) \in \mathcal{M}$.*

Proceeding as in the proof of Theorems 3.2 and 3.3, we obtain the following theorem.

Theorem 3.5. *Let (X, d) be a complete metric space endowed with a transitive relation $\mathcal{M}$ on X and $f : X \to X$ be a mapping. Assume that the following conditions hold:*

(i) *there exist $\beta \in \Gamma$ and $\psi \in \Psi$ such that*
$$\psi(d(fx, fy)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$$
for all $(x, y) \in \mathcal{M}$ with $x \neq y$;

(ii) *there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{M}$;*

(iii) *$\mathcal{M}$ is f-invariant;*

(iv) *if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in \mathcal{M}$ for all $n \in \mathbb{N}$ and $x_n \to z \in X$ as $n \to +\infty$, then $(x_n, z) \in \mathcal{M}$ for all $n \in \mathbb{N}$.*

Then f has a fixed point. In addition, the fixed point is unique provided that

(v) *for all $(x, y) \notin \mathcal{M}$ there exists $z \in X$ such that $(x, z), (y, z) \in \mathcal{M}$.*

Notice that Theorems 3.4 and 3.5 are generalizations of Theorem 2.5 of Gordji et al [8]. In fact, we get Theorem 2.5 if we choose the set $\mathcal{M}$ as in Example 2.1. Also, from Theorem 3.4, we deduce the result of Geraghty (Theorem 1) if we choose $\mathcal{M} = X \times X$ and $\psi(t) = t$.

4. Initial-value problem for parabolic equations

In this section, we adapt the calculations in [8] to our situation. Precisely, by using Theorem 3.5, we study the existence of solution for the initial-value problem

(1.1). It is well-know that this problem, see [12], is equivalent to the integral equation:

$$u(x,t)=\int_0^t d\tau\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}}\,q(\xi,\tau,u(\xi,\tau))d\xi+\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-\xi)^2}{4kt}}}{\sqrt{4\pi kt}}\,\varphi(\xi)d\xi \quad (4.1)$$

for all $x\in\mathbb{R}$ and $t\in(0,T]$.

Here, we consider the Banach space $(\Omega,\|\cdot\|)$, where

$$\Omega=\{v(x,t):v,v_x\in C(\mathbb{R}\times[0,T])\text{ and }\|v\|<+\infty\},$$
$$\|v\|=\sup_{x\in\mathbb{R},t\in[0,T]}|v(x,t)|+\sup_{x\in\mathbb{R},t\in[0,T]}|v_x(x,t)|.$$

Clearly $(\Omega,\|\cdot\|)$ with the metric d given by

$$d(u,v)=\sup_{x\in\mathbb{R},t\in[0,T]}|u(x,t)-v(x,t)|+\sup_{x\in\mathbb{R},t\in[0,T]}|u_x(x,t)-v_x(x,t)|$$

is a complete metric space. Also the set Ω can be naturally endowed with the partial order:

$$\text{for all } u,v\in\Omega,\ \ u\preceq v\iff\ u(x,t)\le v(x,t)\text{ for any }x\in\mathbb{R}\text{ and }t\in[0,T].$$

Now we consider a monotone nondecreasing sequence $\{v_n\}\subseteq\Omega$ converging to $v\in\Omega$, for all $x\in\mathbb{R}$ and $t\in[0,T]$. This means that

$$v_1(x,t)\le v_2(x,t)\le v_3(x,t)\le\cdots v_n(x,t)\le\cdots\le v(x,t)$$

for all $x\in\mathbb{R}$ and $t\in[0,T]$. Therefore, condition (iv) of Theorem 3.5 holds true, by choosing the set $\mathcal{M}$ as in Example 2.1.

Our theorem in this section links the existence of a solution for the initial-value problem (1.1) to the existence of a fixed point for an integral operator. The reader is referred to the paper of Aronson and Serrin [4] for further discussion of hypotheses below.

Theorem 4.1. *Assume that the following conditions hold:*

(a) *for any $c>0$ with $|u|<c$, the function $q(x,t,u)$ is bounded and uniformly Hölder continuous in x and in t for each compact subset of $\mathbb{R}\times[0,T]$;*
(b) *there exists a constant $c_q\le(T+2\pi^{-1/2}T^{1/2})^{-1}$ such that*

$$0\le q(x,t,u(x,t))-q(x,t,v(x,t))\le c_q[1-e^{-\min\{d(u,v),1\}}]$$

for all $(u,v)\in\mathbb{R}\times\mathbb{R}$ with $v\preceq u$;
(c) *there exists $u_0\in\Omega$ such that*

$$u_0(x,t)\le\int_0^t d\tau\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}}q(\xi,\tau,u_0(\xi,\tau))d\xi+\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi kt}}\varphi(\xi)d\xi(x,t)$$

for all $x\in\mathbb{R}$ and $t\in(0,T]$.

Then the initial-value problem (1.1) *has at least a solution.*

Proof. As said above, the initial-value problem (1.1) is equivalent to the integral equation (4.1) for all $x\in\mathbb{R}$ and $t\in(0,T]$. Then, the initial-value problem (1.1) possesses a solution if and only if the integral equation (4.1) has a solution u satisfying certain properties. Roughly speaking, this solution can be seen as fixed point of an integral operator, so that we can apply our fixed point theorems to get it. To this aim, we define the operator $f : \Omega \to \Omega$ by:

$$(fu)(x,t) = \int_0^t d\tau \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}}\, q(\xi,\tau,u(\xi,\tau))d\xi + \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4kt}}}{\sqrt{4\pi kt}}\, \varphi(\xi)d\xi$$

for all $x \in \mathbb{R}$ and $t \in [0,T]$.

We will show that f satisfies all the requirements of Theorem 3.5, by choosing the set $\mathcal{M}$ as in Example 2.1. We have already remarked at the beginning of this section that condition (iv) of Theorem 3.5 holds true.

Now, by condition (b) we deduce trivially that the operator f is nondecreasing. In fact, for all $u, v \in \Omega$ with $v \preceq u$, from

$$q(x,t,u(x,t)) \ge q(x,t,v(x,t))$$

for all $x \in \mathbb{R}$ and $t \in (0,T]$, we obtain

$$\begin{aligned}
(fu)(x,t) &= \int_0^t d\tau \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}}\, q(\xi,\tau,u(\xi,\tau))d\xi + \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4kt}}}{\sqrt{4\pi kt}}\, \varphi(\xi)d\xi \\
&\ge \int_0^t d\tau \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}}\, q(\xi,\tau,v(\xi,\tau))d\xi + \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4kt}}}{\sqrt{4\pi kt}} \varphi(\xi)d\xi \\
&= (fv)(x,t)
\end{aligned}$$

for all $x \in \mathbb{R}$ and $t \in (0,T]$. Then, f is nondecreasing and, in view of Example 2.1, this implies that $\mathcal{M}$ is f-invariant, that is, condition (iii) of Theorem 3.5 holds. Clearly, from assertion (c), $u_0 \preceq fu_0$ and hence condition (ii) of Theorem 3.5 holds true.

Now we only need to show that f satisfies the contractive condition in Theorem 3.5. In fact, we get

$$\begin{aligned}
&|(fu)(x,t) - (fv)(x,t)| \\
&\le \int_0^t d\tau \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}} |q(\xi,\tau,u(\xi,\tau)) - q(\xi,\tau,v(\xi,\tau))| d\xi \\
&\le c_q[1 - e^{-\min\{d(u,v),1\}}] \int_0^t d\tau \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}}\, d\xi \\
&\le c_q[1 - e^{-\min\{d(u,v),1\}}]T
\end{aligned}$$

for all $x \in \mathbb{R}$ and $t \in (0,T]$. Analogous reasoning shows that

$$\begin{aligned}
&\Big|\frac{\partial(fu)(x,t)}{\partial x} - \frac{\partial(fv)(x,t)}{\partial x}\Big| \\
&\le c_q[1 - e^{-\min\{d(u,v),1\}}] \int_0^t d\tau \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\Big(\frac{1}{\sqrt{4\pi(t-\tau)}}\, e^{-\frac{(x-\xi)^2}{4(t-\tau)}}\Big) d\xi \\
&\le c_q[1 - e^{-\min\{d(u,v),1\}}] 2\pi^{-1/2}T^{1/2}.
\end{aligned}$$

By combining the obtained results, we deduce that

$$d(fu,fv) \le c_q(T + 2\pi^{-1/2}T^{1/2})(1 - e^{-\min\{d(u,v),1\}}) \le 1 - e^{-\min\{d(u,v),1\}},$$

which further gives us

$$d(fu,fv) \le \frac{1-e^{-\min\{d(u,v),1\}}}{\min\{d(u,v),1\}} d(u,v).$$

Therefore, condition (i) of Theorem 3.5 holds true with $\psi(s) = s$, and $\beta(s) = \frac{e^{-\min\{s,1\}}-1}{-\min\{s,1\}}$ for $s > 0$ and $\beta(0) = \frac{1}{2}$. Thus, we can apply Theorem 3.5 to conclude that f has a fixed point and hence the initial-value problem (1.1) has a solution. $\square$

Acknowledgements. This research is supported by Università degli Studi di Palermo (Local University Project ex 60%). The second author is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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