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EXISTENCE OF SOLUTIONS TO NONLINEAR PARABOLIC UNILATERAL PROBLEMS WITH AN OBSTACLE DEPENDING ON TIME

NABILA BELLAL

Abstract. Using the penalty method, we prove the existence of solutions to nonlinear parabolic unilateral problems with an obstacle depending on time. To find a solution, the original inequality is transformed into an equality by adding a positive function on the right-hand side and a complementary condition. This result can be seen as a generalization of the results by Mokrane in [11] where the obstacle is zero.

1. Introduction

The main purpose of this article is to prove the existence of a solution to a nonlinear parabolic inequality of obstacle type. Our problem is associated to a second-order nonlinear operator of Leray-Lions type. We prove that actually the solution satisfies an equation with a modification of the right-hand side by a positive function and a complementary condition. This result can be seen as a generalization of the result obtained Mokrane [11] when the obstacle is zero.

Statement of the problem. Let Ω be a bounded Lipschitz open set of $\mathbb{R}^N$ with boundary $\partial\Omega$ and T a positive real number. Set $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$. Given functions u_0 and ψ we look for a solution u to the problem

$$\frac{\partial u}{\partial t} + A(u) + g(u, Du) - f = \mu \quad \text{in } Q = \Omega \times]0,T[, \tag{1.1}$$

$$u \geq \psi, \quad \mu \geq 0, \quad \mu(u-\psi) = 0 \quad \text{in } Q, \tag{1.2}$$

$$u(x,t) = 0 \quad \text{on } \Sigma, \tag{1.3}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega. \tag{1.4}$$

Here A is a Leray-Lions operator from $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual, f belongs to $L^{p'}(Q)$ and $g(x,t,u,Du)$ is a nonlinear term, the prototype of which is $u|Du|^q$ with $q < p-1$, we suppose that $p > 2$.

When g is equal to zero, the corresponding result has been proved e.g. in [8]. On the other hand, the equation associated with the unilateral problem (1.1), (1.3), (1.4) (i.e. the case where $\mu = 0$ in (1.1), the conditions (1.2) being omitted) has

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been solved in [5]. Here we extend Mokrane's result [11], by utilizing different techniques. For $\psi = 0$, [11] proved the existence of a solution.

Considered just as an equation (without obstacle) or as a variational inequality this problem, or very similair ones with various types of hypotheses on the operator A (or the function $a(x,t,s,\xi)$ see below), g and the data have been addressed by several authors, [1, 2, 9].

For some of these results, an extra condition on the form $a(x,t,s,.)$ applied to the positive part on any test function is added. It seems for us that it is more interesting and realistic, to avoid this condition, and replace it by an extra regularity condition on the obstacle. Moreover these authors did not deal with the existence of the function μ and the complementary condition $\mu(u-\psi) = 0$ in Q.

In this article we use a regularization-penalization procedure and a compactness result analogous to the ones introduced [11], and some other different techniques.

This article is organised as follows. The first part is devoted to the hypotheses and the setting of the main result. In the second one we proceed by the regularization-penalisation method. We construct a one parameter family of solutions and prove some estimates on these approximate solutions. In the third part we prove the convergence of an extracted subsequence of this family, to a solution of our problem.

2. Hypotheses and the main result

Let Ω be a bounded subset of $\mathbb{R}^N$, with Lipschitz boundary $\partial\Omega, Q$ be $\Omega\times]0,T[$ for a given T, $0 < T < \infty$ and $\Sigma = \partial\Omega\times]0,T[$. Let p and p' be fixed with $\frac{1}{p} + \frac{1}{p'} = 1$, $2 < p < \infty$, $W_0^{1,p}(\Omega)$ is the usual Sobolev space equipped with the L^p norm of the gradients. Let A be a nonlinear operator from $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$ of Leray-Lions type defined by

$$A(u) = -\operatorname{div}(a(x,t,u,Du)),$$

where $a(x,t,s,\xi)$ is a Carathéodory function such that

$$\begin{gathered} a(x,t,s,\xi) \le \beta[|s|^{p-1} + |\xi|^{p-1} + k(x,t)], \quad k(x,t) \in L^{p'}(Q),\ \beta > 0 \\ [a(x,t,s,\xi) - a(x,t,s,\eta)][\xi - \eta] > 0, \quad \forall \xi \neq \eta \\ a(x,t,s,\xi)\xi \ge \alpha|\xi|^p, \quad \alpha > 0. \end{gathered} \tag{2.1}$$

Let $g(x,t,u,Du)$ be a nonlinear lower order term having growth of order q, ($q < p-1$) with respect to $|Du|$ and of order m ($1 < m < p-q$) with respect to $|u|$ and satisfying a sign condition. To be more precise we assume that g is a Carathéodory function such that

$$|g(x,t,s,\xi)| \le b(|s|)(h(x,t) + |\xi|^q) \tag{2.2}$$

where $1 < q < p-1$, $h \in L^\infty(Q)$, and $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, nonnegative increasing function, having growth of order m, ($1 < m < p-q$) with respect to $|u|$:

$$b(|u|) \le \rho + |u|^m, \quad \rho > 0,\ 1 < m < p-q; \tag{2.3}$$

$$g(x,t,s,\xi)s \ge 0 \quad \forall (x,t,s,\xi) \in \Omega \times \mathbb{R}^2 \times \mathbb{R}^N. \tag{2.4}$$

We have the following assumptions on u_0, f and ψ:

$$u_0 \in L^2(\Omega), \tag{2.5}$$

$$f \in L^{p'}(Q), \tag{2.6}$$

$$\psi \in L^p(0,T;W^{1,p}(\Omega)) \quad \text{with } \psi \le 0 \text{ on } \Sigma, \tag{2.7}$$
$$\psi(0) \le u_0 \quad \text{a.e. in } \Omega, \tag{2.8}$$
$$\psi^+ \in L^\infty(Q), \tag{2.9}$$
$$\frac{\partial \psi}{\partial t} \in L^{p'}(Q) \tag{2.10}$$

Also we assume a complementary condition on a and ψ,

$$\operatorname{div}(a(x,t,u,D\psi) \in L^{p'}(Q) \quad \text{for } u \in L^p(0,T,W_0^{1,p}(\Omega)) \tag{2.11}$$

and is bounded in $L^{p'}(Q)$ on bounded sets of $L^p(0,T,W_0^{1,p}(\Omega))$.

Our main result is the following.

Theorem 2.1. *Under assumptions* (2.1)–(2.10) *there exist at least one pair of functions u and μ which are a solution of* (1.1)–(1.4) *and satisfy*

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)), \tag{2.12}$$
$$\frac{\partial u}{\partial t} = \lambda_1 + \lambda_2 \quad \textit{with } \lambda_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega)),\ \lambda_2 \in L^1(Q), \tag{2.13}$$
$$u \ge \psi \quad \textit{in } Q, \tag{2.14}$$
$$\mu \in L^{p'}(Q), \tag{2.15}$$
$$\mu \ge 0, \tag{2.16}$$
$$g(x,t,u,Du) \in L^1(Q) \quad \textit{and } ug(x,t,u,Du) \in L^1(Q), \tag{2.17}$$
$$\frac{\partial u}{\partial t} + A(u) + g(x,t,u,Du) - f = \mu \quad \textit{in } Q, \tag{2.18}$$
$$\mu(u-\psi) = 0 \quad \textit{in } Q, \tag{2.19}$$
$$u \in C^0(0,T;W^{-1,r}(\Omega)) \quad \textit{for } r < \inf(p, \frac{p}{p-1}, \frac{N}{N-1}), \tag{2.20}$$
$$u(x,0) = u_0(x) \quad \textit{in } \Omega. \tag{2.21}$$

3. Proof of the Theorem 2.1

3.1. **Approximate solutions.** For $\varepsilon > 0$, we define

$$g_\varepsilon(x,t,s,\xi) = \frac{g(x,t,s,\xi)}{1+\varepsilon|g(x,t,s,\xi)|} \tag{3.1}$$

and we denote by u_ε the solution of the approximate and penalized problem

$$\begin{gathered}
\frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}(a(x,t,u_\varepsilon,Du_\varepsilon)) + g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon) \\
- \frac{1}{\varepsilon^{p-1}}|(u_\varepsilon-\psi)^-|^{p-2}(u_\varepsilon-\psi)^- = f, \quad \text{in } Q, \\
u_\varepsilon(x,0) = u_0(x), \quad x \in \Omega, \\
u_\varepsilon = 0 \text{ on } \Sigma, \\
u_\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega))
\end{gathered} \tag{3.2}$$

which has a weak solution by the classical result of Lions [10], Donati [8], where v^- denotes the negative part of v, i.e. $v^- = \sup(0,-v)$, for any function v.

The function u_ε is a solution of (3.2) in the following sense:

$$
\begin{gathered}
u_\varepsilon \in L^p(]0,T[,W_0^{1,p}(\Omega)) \cap \mathcal{C}([0,T],L^2(\Omega)),\\
\frac{\partial u_\varepsilon}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega)), \quad u_\varepsilon(x,0)=u_0(x),
\end{gathered}
$$

$$
\begin{aligned}
&\int_0^T \langle \frac{\partial u_\varepsilon}{\partial t}, v\rangle dt + \int_Q a(x,t,u_\varepsilon,Du_\varepsilon)Dv\,dx\,dt + \int_Q g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)v\,dx\,dt\\
&- \frac{1}{\varepsilon^{p-1}}\int_Q ((u_\varepsilon-\psi)^-)^{p-2}(u_\varepsilon-\psi)^- v\,dx\,dt\\
&= \int_Q fv\,dx\,dt, \quad \forall v\in L^p(]0,T[,W_0^{1,p}(\Omega))
\end{aligned} \tag{3.3}
$$

3.2. $L^p(0,T;W_0^{1,p}(\Omega))$ **- estimate of** u_ε**.** Recall that since $\psi \in L^p(]0,T[,W^{1,p}(\Omega))$, $p>2$ and $\frac{\partial\psi}{\partial t}\in L^{p'}(Q)$, we have $\psi\in W^{1,p'}(Q)$. From this and by a slight modifaction of the [14, Lemma 1.1], we deduce that $\frac{\partial\psi^+}{\partial t}\in L^{p'}(Q)$ and $(u_\varepsilon-\psi^+)$ is a possible test function. We use it in (3.3).

Multiplying (3.2) by the test function $(u_\varepsilon-\psi^+)$ we get, denoting by $\langle,\rangle$ the duality pairing between $W_0^{1,p}(\Omega)$ and its dual

$$
\begin{aligned}
&\int_0^t \langle \frac{\partial(u_\varepsilon-\psi^+)}{\partial t}, u_\varepsilon-\psi^+\rangle dt' + \int_0^t\int_\Omega a(x,t',u_\varepsilon,Du_\varepsilon)D(u_\varepsilon-\psi^+)\,dx\,dt'\\
&+\int_0^t\int_\Omega g_\varepsilon(x,t',u_\varepsilon,Du_\varepsilon)(u_\varepsilon-\psi^+)\,dx\,dt'\\
&-\frac{1}{\varepsilon^{p-1}}\int_0^t\int_\Omega |(u_\varepsilon-\psi)^-|^{p-2}(u_\varepsilon-\psi)^-(u_\varepsilon-\psi^+)\,dx\,dt'\\
&=\int_0^t\int_\Omega (f-\frac{\partial\psi^+}{\partial t})(u_\varepsilon-\psi^+)\,dx\,dt'.
\end{aligned} \tag{3.4}
$$

which implies

$$
\begin{aligned}
&\frac12\|u_\varepsilon(t)-\psi^+(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega a(x,t',u_\varepsilon,Du_\varepsilon)Du_\varepsilon\,dx\,dt'\\
&+\int_0^t\int_\Omega u_\varepsilon g_\varepsilon(x,t',u_\varepsilon,Du_\varepsilon)\,dx\,dt'\\
&+\frac{1}{\varepsilon^{p-1}}\int_0^t \|(u_\varepsilon-\psi)^-(t')\|^p_{L^p(\Omega)}dt' + \frac{1}{\varepsilon^{p-1}}\int_0^t\int_\Omega |(u_\varepsilon-\psi)^-|^{p-1}\psi^-\,dx\,dt'\\
&=\frac12\|(u_0-\psi^+(0))\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega (f-\frac{\partial\psi^+}{\partial t})u_\varepsilon\,dx\,dt' - \int_0^t\int_\Omega (f-\frac{\partial\psi^+}{\partial t})\psi^+\,dx\,dt'\\
&\quad+\int_0^t\int_\Omega a(x,t',u_\varepsilon,Du_\varepsilon)D\psi^+\,dx\,dt' + \int_0^t\int_\Omega \psi^+ g_\varepsilon(x,t',u_\varepsilon,Du_\varepsilon)\,dx\,dt'\,.
\end{aligned} \tag{3.5}
$$

Using the conditions (2.1), (2.2), (2.3), (2.4), (2.9), Poincaré and Hölder inequalities we obtain

$$
\begin{aligned}
&\int_Q |a(x,t,u_\varepsilon,Du_\varepsilon)D\psi^+|\,dx\,dt \\
&\le \beta\int_Q |u_\varepsilon|^{p-1}|D\psi^+|\,dx\,dt+\beta\int_Q |Du_\varepsilon|^{p-1}|D\psi^+|\,dx\,dt+\int_Q |k(x,t)|\,|D\psi^+|\,dx\,dt \\
&\le \theta\int_Q |Du_\varepsilon|^p\,dx\,dt+M_1+M_2,
\end{aligned}
\tag{3.6}
$$

and

$$
\Big|\int_Q \psi^+ g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)\,dx\,dt\Big|\le 3\theta\int_0^t |Du_\varepsilon|^p_{L^p(\Omega)}\,dt'+M_3,
$$

where θ is any positive real number and M_1, M_2 and M_3 depend on the data θ and T.

By (2.1), we obtain

$$
\int_0^t\int_\Omega a(x,t',u_\varepsilon,Du_\varepsilon)Du_\varepsilon\,dx\,dt'\ge\alpha\int_0^t\int_\Omega |Du_\varepsilon|^p\,dx\,dt'=\alpha\int_0^t\|Du_\varepsilon\|^p_{L^p(\Omega)}dt'.
\tag{3.7}
$$

Moreover, since $f,\frac{\partial\psi^+}{\partial t}\in L^{p'}(Q)$ and $u_0\in L^2(\Omega)$ we deduce from (2.9) and Hölder inequality that

$$
\begin{aligned}
&\int_0^t\int_\Omega(f-\frac{\partial\psi^+}{\partial t})u_\varepsilon\,dx\,dt'-\int_0^t\int_\Omega(f-\frac{\partial\psi^+}{\partial t})\psi^+\,dx\,dt'+\frac12\|(u_0-\psi^+(0))\|^2_{L^2(\Omega)} \\
&\le M_4+\theta\int_0^t\|Du_\varepsilon\|^p_{L^p(\Omega)}dt'.
\end{aligned}
\tag{3.8}
$$

Now we deduce from (3.5) and inequalities (3.6), (3.7) and (3.8) that

$$
\begin{aligned}
&\frac12\|u_\varepsilon(t)-\psi^+(t)\|^2_{L^2(\Omega)}+(\alpha-5\theta)\int_0^t\|u_\varepsilon\|^p_{W_0^{1,p}(\Omega)}dt' \\
&+\int_0^t\int_\Omega u_\varepsilon g_\varepsilon(x,t',u_\varepsilon,Du_\varepsilon)\,dx\,dt'+\frac{1}{\varepsilon^{p-1}}\int_0^t\|(u_\varepsilon-\psi)^-(t')\|^p_{L^p(\Omega)}dt' \\
&+\frac{1}{\varepsilon^{p-1}}\int_0^t\int_\Omega|(u_\varepsilon-\psi)^-|^{p-2}(u_\varepsilon-\psi)^-\psi^-\,dx\,dt' \\
&\le M_1+M_2+M_3+M_4.
\end{aligned}
\tag{3.9}
$$

Choosing θ small enough (for example $\theta=\frac{\alpha}{10}$) it results that

$$
\|u_\varepsilon\|_{L^p(0,T;W_0^{1,p}(\Omega))}\le C_1,
\tag{3.10}
$$

$$
\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}\le C_2,
\tag{3.11}
$$

$$
\int_Q u_\varepsilon g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)\,dx\,dt\le C_3.
\tag{3.12}
$$

Note that θ, M_i and C_i denote nonnegative constants which do no depend on ε. Then by extracting a subsequence also denoted by u_ε, we see that there exists

$$
u_\varepsilon\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(0,T;L^2(\Omega))
\tag{3.13}
$$

such that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{3.14}$$

$$u_\varepsilon \rightharpoonup u \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)) \tag{3.15}$$

Then (2.12) is proved.

3.3. **$L^p(Q)$-estimate of $\frac{(u_\varepsilon-\psi)^-}{\varepsilon}$.** The equation (3.2) can be written as

$$\begin{aligned} &\frac{\partial(u_\varepsilon-\psi)}{\partial t} - \operatorname{div}[(a(x,t,u_\varepsilon,Du_\varepsilon)-a(x,t,u_\varepsilon,D\psi))] + g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)\\ &- \frac{1}{\varepsilon^{p-1}}|(u_\varepsilon-\psi)^-|^{p-2}(u_\varepsilon-\psi)^- \\ &= f - \frac{\partial\psi}{\partial t} + \operatorname{div}(a(x,t,u_\varepsilon,D\psi)), \quad \text{in } Q. \end{aligned} \tag{3.16}$$

Multiplying (3.16) by the test function $-\frac{(u_\varepsilon-\psi)^-}{\epsilon}$, we obtain

$$\begin{aligned} &-\frac{1}{\varepsilon}\int_0^T \big\langle \frac{\partial(u_\varepsilon-\psi)}{\partial t}, (u_\varepsilon-\psi)^-\big\rangle dt\\ &-\frac{1}{\varepsilon}\int_Q [(a(x,t,u_\varepsilon,Du_\varepsilon)-a(x,t,u_\varepsilon,D\psi))]D(u_\varepsilon-\psi)^-\,dx\,dt\\ &-\frac{1}{\varepsilon}\int_Q (u_\varepsilon-\psi)^- g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)\,dx\,dt + \frac{1}{\varepsilon^p}\int_Q |(u_\varepsilon-\psi)^-|^p\,dx\,dt\\ &= -\frac{1}{\varepsilon}\int_0^T \big\langle f - \frac{\partial\psi}{\partial t} + \operatorname{div}(a(x,t,u_\varepsilon,D\psi)), (u_\varepsilon-\psi)^-\big\rangle dt. \end{aligned} \tag{3.17}$$

Using (2.6), (2.10), (2.11), we have $f-\frac{\partial\psi}{\partial t}+\operatorname{div}(a(x,t,u_\varepsilon,D\psi)) \in L^{p'}(0,T;L^{p'}(\Omega))$, then using Young inequality the right hand side of (3.17) is absorbed by the fourth term of the left hand side. On the set where $u_\varepsilon \le \psi$, thanks to the strict monotony, the second term is non negative.

Concerning the third term of (3.17), we can rewrite it in the form

$$\begin{aligned} I = &-\frac{1}{\varepsilon}\int_{\{u_\varepsilon\le\psi, u_\varepsilon<0\}} (u_\varepsilon-\psi)^- g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)\,dx\,dt\\ &-\frac{1}{\varepsilon}\int_{\{0\le u_\varepsilon\le\psi\}} (u_\varepsilon-\psi)^- g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)\,dx\,dt = I_1+I_2, \end{aligned}$$

by the sign condition on g, I_1 is non negative.

For I_2 using the growth condition on g,h,b and ψ^+, we can easily obtain two positive constants K_1 and K_2 such that $|g(x,t,u_\varepsilon,Du_\varepsilon)| \le K_1 + K_2|Du_\varepsilon|^q$. Then I_2 can be estimated as follows

$$\begin{aligned} |I_2| &\le K_1\int_{\{0\le u_\varepsilon\le\psi\}} \frac{(u_\varepsilon-\psi)^-}{\varepsilon}\,dx\,dt + K_2\int_{\{0\le u_\varepsilon\le\psi\}} |Du_\varepsilon|^q\frac{(u_\varepsilon-\psi)^-}{\varepsilon}\,dx\,dt\\ &= A_1 + A_2. \end{aligned}$$

It is clear that $|A_1| \le C\|\frac{(u_\varepsilon-\psi)^-}{\varepsilon}\|_{L^p(Q)}$. For A_2 we use (3.10) and Hölder inequality to obtain

$$A_2 = K_2\int_{\{0\le u_\varepsilon\le\psi\}} |Du_\varepsilon|^q\frac{(u_\varepsilon-\psi)^-}{\varepsilon}\,dx\,dt$$

$$\le K_2 \int_{\{0\le u_\varepsilon\le\psi\}} \Big(|Du_\varepsilon|^{qr}\Big)^{\frac{1}{r}} \Big(\big(\frac{(u_\varepsilon-\psi)^-}{\varepsilon}\big)^{r'}\Big)^{\frac{1}{r'}}\,dx\,dt$$

with $\frac{1}{r}+\frac{1}{r'}=1$. Choosing r such that $qr=p$ and thus $r'=\frac{p}{p-q}$, one has $A_2 \le C\|\frac{(u_\varepsilon-\psi)^-}{\varepsilon}\|_{L^{r'}(Q)}$. Since $q<p-1$ and thus $r'<p$, we get $|A_2|\le C\|\frac{(u_\varepsilon-\psi)^-}{\varepsilon}\|_{L^p(Q)}$. Therefore, we obtain

$$\|\frac{(u_\varepsilon-\psi)^-}{\varepsilon}\|^p_{L^p(Q)} \le C \tag{3.18}$$

From (3.18) we infer that

$$(u_\varepsilon-\psi)^- \to 0 \quad \text{strongly in } L^p(Q) \tag{3.19}$$

and thus

$$u\ge\psi \quad \text{a.e. on } Q \tag{3.20}$$

which proves (2.14).

3.4. **Equi-integrability of $g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)$.** Now we adapt a method of [15] to prove the equi-integrability of $g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)$. For $\delta>0$, define the sets

$$\begin{gathered} F_\delta=\{(x,t)\in Q:|u|\le\delta\},\\ G_\delta=\{(x,t)\in Q:|u|>\delta\}. \end{gathered}$$

Using the estimates (3.10) on u_ε, the conditions (2.2), (2.3) and (2.4), for any measurable subset $E\subset Q$, we have

$$\begin{aligned}
&\int_E |g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)|\,dx\,dt\\
&= \int_{E\cap F_\delta} |g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)|\,dx\,dt + \int_{E\cap G_\delta} |g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)|\,dx\,dt\\
&\le \int_{E\cap F_\delta} (\rho+|u_\varepsilon|^m)(h(x,t)+|Du_\varepsilon|^q)\,dx\,dt + \frac{1}{\delta}\int_{E\cap G_\delta} u_\varepsilon g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)\,dx\,dt\\
&\le (\rho+\delta^m)\int_E (h(x,t)+|Du_\varepsilon|^q)\,dx\,dt + \frac{1}{\delta}\int_E u_\varepsilon g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)\,dx\,dt\\
&\le (\rho+\delta^m)(\|h\|_{L^\infty(Q)}|E| + C_1^{q/p}(|E|)^{1-\frac{q}{p}}) + \frac{1}{\delta}C_3.
\end{aligned} \tag{3.21}$$

From (3.21), by choosing first δ sufficiently large and the measure of E sufficiently small, we deduce that

$$g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon) \text{ is equi-integrable.} \tag{3.22}$$

Note also that (3.21) with $E=Q$ implies

$$g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon) \text{ is bounded in } L^1(Q). \tag{3.23}$$

3.5. **Almost pointwise convergence of u_ε and Du_ε.** From (3.2) we can write $\frac{\partial u_\varepsilon}{\partial t}=\lambda_1^\varepsilon+\lambda_2^\varepsilon$, with $\lambda_2^\varepsilon=g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)$. Since u_ε is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ (see (3.10) and $\frac{(u_\varepsilon-\psi)^-}{\varepsilon}$ is bounded in $L^p(Q)$ (see (3.18)) we deduce from (3.23) that

$$\frac{\partial u_\varepsilon}{\partial t}=\lambda_1^\varepsilon+\lambda_2^\varepsilon \tag{3.24}$$

with λ_1^ε bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and λ_2^ε bounded in $L^1(Q)$.

Since $g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)$ is equi-integrable in $L^1(Q)$ we can extract subsequences (still denoted by λ_1^ε and λ_2^ε) such that

$$\lambda_1^\varepsilon \rightharpoonup \lambda_1 \quad \text{weakly in } L^{p'}(0,T;W^{-1,p'}(\Omega)), \tag{3.25}$$

$$\lambda_2^\varepsilon \rightharpoonup \lambda_2 \quad \text{weakly in } L^1(Q) \tag{3.26}$$

This implies

$$\frac{\partial u}{\partial t} = \lambda_1 + \lambda_2 \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q) \tag{3.27}$$

which proves (2.13).

From (3.24) and the estimate (3.10) on u_ε we have

$$\begin{gathered} u_\varepsilon \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ with } \tfrac{\partial u_\varepsilon}{\partial t} \text{ bounded in} \\ L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(0,T;L^1(\Omega)) \subset L^1(0,T;W^{-1,r}(\Omega)) \\ \text{for all } r < \inf\{\tfrac{N}{N-1}, \tfrac{p}{p-1}\}. \end{gathered} \tag{3.28}$$

Since $W_0^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{-1,r}(\Omega)$ for $p > r$, the first injection being compact, a lemma of Aubin's type (see eg. [13, corollary 4]) implies that

$$u_\varepsilon \to u \quad \text{strongly in } L^p(0,T;L^p(\Omega)) \tag{3.29}$$

which also implies that at least for a subsequence; still denoted by u_ε,

$$u_\varepsilon \to u \quad \text{a.e in } Q. \tag{3.30}$$

Then we apply a compactness result due to Boccardo and Murat [5, 6], and more precisely [6, Theorem 4.3 and Remark 4.1]. Since u_ε is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and since

$$\frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}(a(x,t,u_\varepsilon,Du_\varepsilon)) = \lambda_1^\varepsilon + \lambda_2^\varepsilon \text{is bounded in } L^{p'}(Q) + L^1(Q), \tag{3.31}$$

in view of the approximation $g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)$ which is weakly compact in $L^1(Q)$ see (3.22), (3.23) and (3.18), we have (for a subsequence)

$$Du_\varepsilon \to Du \quad \text{strongly in } L^q(Q) \forall q < p, \tag{3.32}$$

which implies

$$Du_\varepsilon \to Du \quad \text{a.e in } Q. \tag{3.33}$$

3.6. **Passing to the limit in the equation.** Using (3.1) and

$$g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon) \to g(x,t,u,Du) \quad \text{a.e in } Q, \tag{3.34}$$

which follows from (3.30), (3.33) and (3.22), we deduce, by Vitali's theorem, that

$$g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon) \to g(x,t,u,Du) \quad \text{strongly in } L^1(Q). \tag{3.35}$$

Moreover since $u_\varepsilon g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon) \geq 0$ a.e. in Q and by (3.12), Fatou's lemma implies

$$ug(x,t,u;Du) \text{ belongs to } L^1(Q). \tag{3.36}$$

which completes the proof of (2.17).

Similarly since u_ε is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ (see (3.10)) and since u_ε and Du_ε tends to u and Du a.e in Q we have

$$a(x,t,u_\varepsilon,Du_\varepsilon) \rightharpoonup a(x,t,u,Du) \quad \text{weakly in } L^{p'}(Q). \tag{3.37}$$

Since $\frac{(u_\varepsilon-\psi)^-}{\varepsilon}$ is bounded in $L^p(Q)$ (see (3.18))

$$\frac{1}{\varepsilon^{p-1}}|(u_\varepsilon-\psi)^-|^{p-2}(u_\varepsilon-\psi)^- \rightharpoonup \mu \quad \text{weakly in } L^{p'}(Q) \tag{3.38}$$

and we have $\mu \in L^{p'}(Q)$, $\mu \geq 0$ which proves (2.15), (2.16). Therefore we can pass to the limit in each term of (3.2) and thus prove that equation (2.18) holds.

Let us now prove (2.19); i.e,

$$\mu \cdot (u-\psi) = 0 \quad \text{a.e. in } Q\,.$$

This follows from the equality

$$\frac{1}{\varepsilon^{p-1}}|(u_\varepsilon-\psi)^-|^{p-2}(u_\varepsilon-\psi)^-(u_\varepsilon-\psi) = -\varepsilon\Big|\frac{(u_\varepsilon-\psi)^-}{\varepsilon}\Big|^p$$

since u_ε tends to u strongly in $L^p(Q)$ (see (3.29)) while $\frac{1}{\varepsilon^{p-1}}|(u_\varepsilon-\psi)^-|^{p-2}(u_\varepsilon-\psi)^-$ tends weakly to μ in $L^{p'}(Q)$ and $\frac{(u_\varepsilon-\psi)^-}{\varepsilon}$ is bounded in $L^p(Q)$.

3.7. **Initial condition.** To complete the proof of the Theorem it remains to prove that (2.20) and (2.21) hold. We first prove that for $r < \inf\{\frac{N}{N-1}, \frac{p}{p-1}\}$

$$u_\varepsilon \to u \quad \text{strongly in } C^0(0,T;W^{-1,r}(\Omega)). \tag{3.39}$$

This allows us to pass to the limit in $u_\varepsilon(x,0) = u_0(x)$ and implies that u satisfies the initial condition.

Recalling that $g_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon)$ converges in the strong topology of $L^1(Q)$, (see (3.35)) we can improve (3.24) to

$$\frac{\partial u_\varepsilon}{\partial t} = \lambda_1^\varepsilon + \lambda_2^\varepsilon \tag{3.40}$$

with λ_1^ε bounded in the space $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and λ_2^ε relatively compact in $L^1(0,T;L^1(\Omega))$. Since

$$W^{-1,p'}(\Omega) + L^1(\Omega) \subset W^{-1,r}(\Omega), \tag{3.41}$$

for all $h > 0$ we have

$$\begin{aligned} &\|u_\varepsilon(t+h) - u_\varepsilon(t)\|_{W^{-1,r}(\Omega)} \\ &= \|\int_t^{t+h} (\lambda_1^\varepsilon + \lambda_2^\varepsilon)dt'\|_{W^{-1,r}(\Omega)} \\ &\leq C\int_t^{t+h} \|\lambda_1^\varepsilon\|_{W^{-1,p'}(\Omega)}dt' + C\int_t^{t+h}\|\lambda_2^\varepsilon\|_{L^1(\Omega)}dt' \\ &\leq Ch^{\frac{1}{p}}\|\lambda_1^\varepsilon\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + C\|\lambda_2^\varepsilon\|_{L^1(t,t+h;L^1(\Omega))}, \end{aligned} \tag{3.42}$$

which in view of (3.40) implies that the function u_ε is uniformly equicontinuous in $C^0(0,T;W^{-1,r}(\Omega))$. Since u_ε is bounded in $L^\infty(0,T;L^2(\Omega))$, (see (3.11)) we deduce from Ascoli's theorem (see, eg [13, Lemma 1]) that u_ε is relatively compact in $C^0(0,T;W^{-1,r}(\Omega))$ which proves (3.39).

Remarks. In this article, we assumed that $p > 2$, and realized that does not seem to be easy extending this method for the case $p < 2$.

It seems difficult to avoid a supplementary condition on ψ like (2.11). A similar condition is assumed for example in [8, hypotheses (9), (10)]. The condition (2.11) can be seen as follows: let us define for $u \in L^p(0,T,W_0^{1,p}(\Omega))$ the function $G = f - \frac{\partial \psi}{\partial t} + \operatorname{div} a(x,t,u,D\psi)$. The hypotheses on a, ψ are set in order to have $G \in L^{p'}(Q)$. In the case where a is independent of u, this is essentially a regularity condition on the obstacle ψ. If a depends on u, then with suitable condition on the derivative of $a(x,t,s,\xi)$ with respect to x, s, ξ one can see that(2.11) is satisfied by a function a of the form $a(x,t,s,\xi) = b(x,t,s)|\xi|^{p-2}\xi$.

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Université 20 août 1955, Skikda, Algeria
Université Badji Mokhtar Faculté des Sciences B.P. 12, Annaba, 23000, Algeria
E-mail address: nabilabellal@yahoo.fr