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MIXED TYPE BOUNDARY-VALUE PROBLEMS OF SECOND-ORDER DIFFERENTIAL SYSTEMS WITH P-LAPLACIAN

WEIGAO GE, YU TIAN

Abstract. In this article we show the existence of solutions to a mixed boundary-value problem of second-order differential systems with a p-Laplacian. The associated Hamiltonian actions are indefinite and the discussion of the existence of solutions is due to the application of duality principle.

1. Introduction

Second-order differential systems that include the p-Laplacian appear in physical application; see for example [5]. In this article, we study mixed type boundary-value problems of the form

$$\begin{gathered}(\varphi_p(x'))' + \nabla F(t,x) = 0, \quad p \geq 2,\\ x(0) = x'(1) = 0,\end{gathered} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $\varphi_p(x) = (\varphi_p(x_1), \dots, \varphi_p(x_n))^T$ with $\varphi_p(s) = |s|^{p-2}s$ for $s \in \mathbb{R}$, $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is measurable in t for all $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0,1]$. Also we study systems of the form

$$\begin{gathered}(\psi_p(x'))' + \nabla F(t,x) = 0,\ p \geq 2,\\ x(0) = x'(1) = 0,\end{gathered} \tag{1.2}$$

where $\psi_p(x) = |x|^{p-2}x$.

When $\lim_{|x|\to\infty} F(t,x) = -\infty$, it is easy to obtain a solution of (1.1), by using a minimizing sequence of this functional

$$\Phi(x) = \int_0^1 [\Phi_p(x'(t)) - F(t,x(t))]dt,$$

where $\Phi_p(x'(t)) = \sum_{i=1}^n \frac{1}{p}|x_i'|^p$. However, such an approach is not applicable if $\lim_{|x|\to\infty} F(t,x) = +\infty$, since $\Phi(x)$ does not admit maximum, and does not admit minimum. In such a case, for $\alpha > 0$, we set $u = (u_1,u_2) = (x, -\varphi_p(\alpha x'))$ for (1.1),

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and $u = (u_1, u_2) = (x, -\psi_p(\alpha x'))$ for (1.2), $\alpha > 0$. Then (1.1) becomes

$$\begin{gathered} -u_2' + \varphi_p(\alpha)\nabla F(t, u_1) = 0, \\ u_1' + \frac{1}{\alpha}\varphi_q(u_2) = 0, \\ u_1(0) = u_2(1) = 0, \end{gathered}$$

and (1.2) becomes

$$\begin{gathered} -u_2' + \psi_p(\alpha)\nabla F(t, u_1) = 0, \\ u_1' + \frac{1}{\alpha}\psi_q(u_2) = 0, \\ u_1(0) = u_2(1) = 0. \end{gathered}$$

So (1.1) and (1.2) become

$$\begin{gathered} J\dot{u} + \nabla G(t, u) = 0, \\ u_1(0) = u_2(1) = 0, \end{gathered} \tag{1.3}$$

and

$$\begin{gathered} J\dot{u} + \nabla H(t, u) = 0, \\ u_1(0) = u_2(1) = 0, \end{gathered} \tag{1.4}$$

respectively, where

$$\begin{gathered} G(t, u) = \Phi_q(u_2) + \varphi_p(\alpha)F(t, u_1) = \sum_{i=1}^{n} \frac{1}{q\alpha}|u_{2,i}|^q + \varphi_p(\alpha)F(t, u_1), \\ H(t, u) = \widetilde{\Phi}_q(u_2) + \varphi_p(\alpha)F(t, u_1) = \frac{1}{q\alpha}|u_2|^q + \varphi_p(\alpha)F(t, u_1), \end{gathered}$$

with $u_1 = (u_{1,1}, \dots, u_{1,n}), u_2 = (u_{2,1}, \dots, u_{2,n})$, $q = \frac{p}{p-1}$,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. Then $G : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ is measurable in t for all $u \in \mathbb{R}^{2n}$ and continually differentiable in u for a.e. $t \in [0, 1]$. Furthermore, if F is strictly convex in u_1, then G and H are strictly convex in u.

When $n = 1$, different types of BVPs have been studied there is a series of results [1, 2, 3, 4], whereas there are only a few results for the case $n \geq 2$, except periodic boundary value problems in [6, 7].

Let $X = \{u \in C([0, 1], \mathbb{R}^{2n}) : u_1(0) = u_2(1) = 0\}$. For $u \in X$ we construct functionals in the forms

$$\Psi(u) = \int_0^1 [\frac{1}{2}(J\dot{u}, u) + G(t, u)]dt, \tag{1.5}$$

$$\mathcal{K}(u) = \int_0^1 [\frac{1}{2}(J\dot{u}, u) + H(t, u)]dt. \tag{1.6}$$

The Euler equations $\Psi(u)$ and $\mathcal{K}(u)$ are the differential systems in (1.3) and (1.4), respectively. The boundary conditions in (1.3) and (1.4) are given by the definition of X.

Let $u_k(t) = (u_{k,1}(t), u_{k,2}(t)) = (\cos\lambda_k t \cdot c, \sin\lambda_k t \cdot c)$ with $c = (c_1, \dots, c_n)$, $\lambda_k = \frac{(2k+1)\pi}{2}$ and $|c| = 1$. Then

$$\begin{aligned}\frac{1}{2}\int_0^1 (J\dot{u}_k(t), u_k(t))dt &= \frac{\lambda_k}{2}\int_0^1 [-\cos^2\lambda_k t|c|^2 - \sin^2\lambda_k t|c|^2]dt \\ &= -\frac{1}{2}\lambda_k = -\frac{(2k+1)\pi}{4} \to \mp\infty\end{aligned}$$

as $k \to \pm\infty$. So $\Psi(u)$ and $\mathcal{K}(u)$ are neither bounded from below nor from above.

Since $G(t,u)$ is continually differentiable in u and strictly convex with respect to u, we can make Fenchel transform

$$G^*(t,\dot{v}) = \sup_{u\in\mathbb{R}^{2n}}[(\dot{v},u) - G(t,u)]. \tag{1.7}$$

By the transform theory, there is only one u_v for v such that

$$(\dot{v}, u_v) - G(t,u_v) = \sup_{u\in\mathbb{R}^{2n}}[(\dot{v},u) - G(t,u)].$$

Therefore $\dot{v} = \nabla G(t,u_v), u_v = \nabla G^*(t,\dot{v})$ and

$$G(t,u_v) + G^*(t,\dot{v}) = (\dot{v}, u_v).$$

Let $u = u_v$, we have the relations

$$G(t,u) + G^*(t,\dot{v}) = (\dot{v}, u)$$

$$\dot{v} = \nabla G(t,u), \quad u = \nabla G^*(t,\dot{v})$$

and among them any one implies the others.

The same is true for the relations

$$\begin{gathered}H(t,u) + H^*(t,\dot{v}) = (\dot{v}, u) \\ \dot{v} = \nabla H(t,u), \quad u = \nabla H^*(t,\dot{v})\end{gathered}$$

where $H^*(t,\dot{v}) = \sup_{u\in\mathbb{R}^{2n}}[(\dot{v},u) - H(t,u)]$. With the duality we aim to prove the following theorem.

Theorem 1.1. *Suppose $F(t,x)$ is measurable in t for all $x \in \mathbb{R}^n$, strictly convex and lower semicontinuous (l.s.c.) in x for a.e. $t \in [0,1]$ and there are $a \in C(\mathbb{R}^n,\mathbb{R}^+), b \in L^2([0,1],\mathbb{R}^+)$ such that*

$$|\nabla F(t,x)| \le b(t)a(|x|)$$

and there are $\delta > 0$, ($\delta \in (0, \frac{\pi^2}{4})$ if $p = 2$), and $\beta, \gamma \ge 0$ such that

$$-\beta \le F(t,x) \le \frac{\delta}{2}|x|^2 + \gamma.$$

Then (1.1) *has at least one solution.*

Theorem 1.2. *Under the assumptions of Theorem 1.1, system* (1.2) *has at least one solution.*

2. Preliminaries

To prove our main theorems, we use the following propositions.

Proposition 2.1. *Assume $F : [0,1]\times\mathbb{R}^n \to \mathbb{R}$ and $F(t,x)$ is strictly convex in x for all $t\in[0,1]$ and there are $\alpha>0$, $\beta(t),\gamma(t)\ge 0$ such that*
$$-\beta(t) \le F(t,x) \le \frac{\alpha}{2}|x|^2+\gamma(t).$$
Then for $v=\nabla F(t,x)$ it holds that
$$|v| \le 2\alpha(|x|+\beta(t)+\gamma(t))+1, \quad \forall t\in[0,1].$$

Proof. On the one hand, by $v=\nabla F(t,x)\Leftrightarrow F^*(t,v)=(v,x)-F(t,x)$, we have
$$F^*(t,v)\le (v,x)+\beta(t), \quad \forall t\in[0,1]. \tag{2.1}$$
On the other hand,
$$\begin{aligned} F^*(t,v) &= \sup_{x\in\mathbb{R}^N}[(v,x)-F(t,x)] \\ &\ge \sup_{x\in\mathbb{R}^N}[(v,x)-\frac{\alpha}{2}|x|^2-\gamma(t)] = \frac{1}{2\alpha}|v|^2-\gamma(t), \quad \forall t\in[0,1]. \end{aligned} \tag{2.2}$$
By (2.1) and (2.2),
$$|v|^2 \le 2\alpha[(v,x)+\beta(t)+\gamma(t)], \ \ \forall t\in[0,1]. \tag{2.3}$$
If $|v|\le 1$, the result is obvious. If $|v|>1$, by (2.3), $|v|^2\le 2\alpha[|v||x|+\beta(t)|v|+\gamma(t)|v|]$. The result also follows. □

Proposition 2.2. *If $u\in X=\{x\in H^1([0,1],\mathbb{R}^{2n}) : x_1(0)=x_2(1)=0\}$, then*
$$|u|_2^2 \le \frac{4}{\pi^2}|\dot u|_2^2. \tag{2.4}$$

Proof. Let $u=(u_1,u_2)$, $u_1,u_2\in\mathbb{R}^n$. From
$$\begin{gathered} \dot u(t)=\lambda J u(t), \\ u_1(0)=u_2(1)=0, \end{gathered} \tag{2.5}$$
and the expression $e^{\lambda J t}=\cos(\lambda t)I+\sin(\lambda t)J$, we have the set of eigenvalues λ_k of (2.5)
$$\lambda_k=\frac{(2k+1)\pi}{2}, \quad k=0,\pm1,\pm2,\dots.$$
Then for each λ_k, $k=0,1,2,3,\dots$, (2.5) possesses $2n$-dimensional vector space
$$u_k(t)=\begin{pmatrix}\sin(\lambda_k t)C_{1,k}\\ \cos(\lambda_k t)C_{2,k}\end{pmatrix},$$
where $C_{1,k},C_{2,k}\in\mathbb{R}^n$ are arbitrary vectors. Then $u\in X$ can be expressed as
$$u(t)=\begin{pmatrix}\sum_{k=0}^\infty \sin(\lambda_k t)C_{1,k}\\ \sum_{k=0}^\infty \cos(\lambda_k t)C_{2,k}\end{pmatrix}, \quad C_{1,k},C_{2,k}\in\mathbb{R}^n.$$
Then
$$\begin{aligned} |u|_2^2 &= \sum_{k=0}^\infty\Big[\int_0^1 \sin^2\lambda_k t dt\cdot|C_{1,k}|^2+\int_0^1\cos^2\lambda_k t dt\cdot|C_{2,k}|^2\Big] \\ &= \frac12\sum_{k=0}^\infty[|C_{1,k}|^2+|C_{2,k}|^2], \end{aligned}$$

$$|\dot{u}|_2^2 = \frac{1}{2}\sum_{k=0}^{\infty}\lambda_k^2(|C_{1,k}|^2 + |C_{2,k}|^2) \geq \frac{1}{2}\sum_{k=0}^{\infty}\frac{\pi^2}{4}(|C_{1,k}|^2 + |C_{2,k}|^2),$$

and hence (2.4) holds. □

Proposition 2.3. *If $u \in X$, then*

$$\int_0^1 (J\dot{u}, u)dt \geq -\frac{2}{\pi}|\dot{u}|_2^2.$$

Proof. The result follows directly from the calculation

$$\begin{aligned}
\int_0^1 (J\dot{u}, u)dt &\geq -\int_0^1 |J\dot{u}| \cdot |u| dt \\
&\geq -\Big[\int_0^1 |J\dot{u}|^2 dt \cdot \int_0^1 |u|^2 dt\Big]^{1/2} \\
&= -\Big[\int_0^1 |\dot{u}|^2 dt \cdot \frac{4}{\pi^2}\int_0^1 |\dot{u}|^2 dt\Big]^{1/2} \\
&= -\frac{2}{\pi}|\dot{u}|_2^2.
\end{aligned}$$

□

Proposition 2.4. *Under the conditions in Theorem 1.1, we can choose a suitable $\alpha > 0$ so that after the transform $u = (u_1, u_2) = (x, -\varphi_p(\alpha\dot{x}))$, the function $G(t,u)$ in BVP (1.3) satisfies*

$$-\xi \leq G(t,u) \leq \frac{l}{2}|u|^2 + \eta, \tag{2.6}$$

where $\xi, \eta \geq 0, l \in (0, \frac{\pi}{2})$ are appropriate real numbers.

Proof. If $p = 2$, then $\delta \in (0, \pi^2/4)$. Choose $\alpha = 1/\sqrt{\delta}$. One get $G(t,u) = \frac{\sqrt{\delta}}{2}|u_2|^2 + \frac{1}{\sqrt{\delta}}F(t,u_1)$ and

$$-\frac{\beta}{\sqrt{\delta}} \leq G(t,u) \leq \frac{\sqrt{\delta}}{2}(|u_1|^2 + |u_2|^2) + \frac{\gamma}{\sqrt{\delta}}.$$

Let $\xi = \beta/\sqrt{\delta}$, $\eta = \gamma/\sqrt{\delta}$, $l = \sqrt{\delta}$. Obviously $\xi, \eta > 0$, $l \in (0, \frac{\pi}{2})$.

If $p > 2$, then $q \in (1,2)$. Without loss of generality, assume that $\delta > \pi^2/4$. Let $\alpha = (\pi/4\delta)^{q-1}$, then

$$\begin{aligned}
-\varphi_p(\alpha)\beta \leq G(t,u) &\leq \frac{n}{\alpha q}|u_2|^q + \varphi_p(\alpha)F(t,u_1) \\
&\leq \frac{n}{\alpha q}|u_2|^q + \frac{\delta\varphi_p(\alpha)}{2}|u_1|^2 + \varphi_p(\alpha)\gamma \\
&= \frac{n}{\alpha q}|u_2|^q + \frac{\pi}{8}|u_1|^2 + \varphi_p(\alpha)\gamma.
\end{aligned}$$

It follows from $q \in (1,2)$ that there is $M > 0$ such that

$$\frac{n}{\alpha q}|u_2|^q \leq M + \frac{\pi}{8}|u_2|^2.$$

Let $\xi = \varphi_p(\alpha)\beta, \eta = M + \varphi_p(\alpha)\gamma, l = \frac{\pi}{4}$. Then it holds

$$-\xi \leq G(t,u) \leq \frac{l}{2}|u|^2 + \eta.$$

□

For the rest of this article, we assume $G(t,u)$ satisfies (2.6). Similarly we can prove he following result.

Proposition 2.5. *Under the conditions in Theorem 1.2, there is an $\alpha > 0$ such that after the transform $u = (u_1, u_2) = (x, -\psi_p(\alpha\dot{x}))$, the function H in (1.4) satisfies*

$$-\xi \le H(t,u) \le \frac{l}{2}|u|^2 + \eta, \tag{2.7}$$

where $\xi, \eta \ge 0$, $l \in (0, \frac{\pi}{2})$ are some real numbers.

In the Clarke transform $G^*(t,\dot{v}) = \sup_{u\in\mathbb{R}^{2n}}[(\dot{v},u) - G(t,u)]$, $G^*(t,u)$ is convex in u. On the other hand, if

$$\begin{gathered} G_\varepsilon(t,u) = \frac{\varepsilon}{2}(u,u) + G(t,u), \\ G_\varepsilon^*(t,\dot{v}) = \sup_{u\in\mathbb{R}^{2n}}[(\dot{v},u) - G_\varepsilon(t,u)], \end{gathered}$$

then $G_\varepsilon(t,u)$ is strictly convex in u and satisfies $\lim_{|u|\to\infty} \frac{G_\varepsilon(t,u)}{|u|} = \infty$. Hence $G_\varepsilon^*(t,\dot{v})$ is differentiable in $\dot{v}$; i.e., $\nabla G_\varepsilon^*(t,y)$ is continuous in y. This time we have

$$-\xi + \frac{\varepsilon}{2}|u|^2 \le G_\varepsilon(t,u) \le \frac{l+\varepsilon}{2}|u|^2 + \eta \tag{2.8}$$

and

$$G_\varepsilon^*(t,\dot{v}) = \sup_{u\in\mathbb{R}^{2n}}[(\dot{v},u) - G_\varepsilon(t,u)] \ge \sup_{u\in\mathbb{R}^{2n}}[(\dot{v},u) - \frac{l+\varepsilon}{2}|u|^2 - \eta] = \frac{1}{2(l+\varepsilon)}|\dot{v}|^2 - \eta.$$

$$G_\varepsilon^*(t,\dot{v}) \le \frac{1}{2\varepsilon}|\dot{v}|^2 + \xi. \tag{2.9}$$

Let $\dot{v} \in \partial G_\varepsilon(t,u)$. One has

$$G_\varepsilon^*(t,\dot{v}) = (\dot{v},u) - G_\varepsilon(t,u) \le (\dot{v},u) - \frac{\varepsilon}{2}|u|^2 + \xi$$

and

$$\frac{1}{2(l+\varepsilon)}|\dot{v}|^2 - \eta \le |\dot{v}||u| + \xi,$$

which implies

$$|\dot{v}| \le 1 + 2(l+\varepsilon)(|u| + \xi + \eta). \tag{2.10}$$

Similarly for $u \in \partial G_\varepsilon^*(t,\dot{v})$, we have

$$|u| \le 1 + \frac{2}{\varepsilon}(|\dot{v}| + \xi + \eta). \tag{2.11}$$

Let $\varepsilon > 0$ be such that $l + \varepsilon \in (0, \frac{\pi}{2})$. Take in account the boundary-value problem

$$\begin{gathered} J\dot{u} + \nabla G_\varepsilon(t,u) = 0 \\ u_1(0) = u_2(1) = 0, \end{gathered} \tag{2.12}$$

whose functional is

$$\Psi_\varepsilon(u) = \int_0^1 [\frac{1}{2}(J\dot{u},u) + G_\varepsilon(t,u)]dt.$$

Let $v = -Ju$. Then

$$\Psi_\varepsilon(u) = -\frac{1}{2}\int_0^1 (J\dot{u},u)dt + \int_0^1 [(J\dot{u},u) + G_\varepsilon(t,u)]dt$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_0^1 (J\dot{u}, u)dt - \int_0^1 [(\dot{v}, u) - G_\varepsilon(t, u)]dt \\
&= -\int_0^1 [\frac{1}{2}(J\dot{v}, v) + G_\varepsilon^*(t, \dot{v})]dt =: -\mathcal{K}_\varepsilon(v).
\end{aligned}
$$

Proposition 2.6. *Under the conditions in Theorem 1.1, $\mathcal{K}_\varepsilon$ has one critical point $v_\varepsilon \in Y = \{y \in H^1([0,1], \mathbb{R}^{2n}) : y_1(1) = 0, y_2(0) = 0\}$, which minimize the value of $\mathcal{K}_\varepsilon$ and is uniformly bounded below for all $\varepsilon \in (0, \frac{\pi}{2} - l)$. Furthermore $u_\varepsilon = Jv_\varepsilon$ is a solution of BVP (2.12).*

Proof. It follows from

$$G_\varepsilon(t, u) \le \frac{l+\varepsilon}{2}|u|^2 + \eta =: G(u)$$

that

$$G_\varepsilon^*(t, v) \ge G^*(\dot{v}) = \sup_{u \in \mathbb{R}^{2n}} [(\dot{v}, u) - G(u)] = \frac{1}{2(l+\varepsilon)}|\dot{v}|^2 - \eta$$

and then

$$\mathcal{K}_\varepsilon(v) \ge \frac{1}{2}\Big(\frac{1}{l+\varepsilon} - \frac{2}{\pi}\Big)\int_0^1 |\dot{v}(t)|^2 dt - \int_0^1 \eta(t)dt \ge \alpha_0\|\dot{v}\|_2^2 - \eta_0,$$

where $\eta_0 = \int_0^1 \eta(t)dt$, $\alpha_0 = \frac{1}{2}\big(\frac{1}{l+\varepsilon} - \frac{2}{\pi}\big) > 0$. Obviously $\mathcal{K}_\varepsilon(v) \to +\infty$ as $\|\dot{v}\|_2 \to \infty$ and uniformly bounded below. Let

$$\mathcal{K}_{\varepsilon 1}(v) = \frac{1}{2}\int_0^1 (J\dot{v}, v)dt, \quad \mathcal{K}_{\varepsilon 2}(v) = \int_0^1 G_\varepsilon^*(t, \dot{v})dt.$$

Both $\mathcal{K}_{\varepsilon 1}$ and $\mathcal{K}_{\varepsilon 2}$ are weakly lower semi-continuous (w.l.s.c.) imply $\mathcal{K}_\varepsilon$ is w.l.s.c. and then $\mathcal{K}_\varepsilon$ possesses one minimum at some point $v_\varepsilon \in Y$.

At the same time, by $L(t, x, y) = \frac{1}{2}(Jy, x) + G_\varepsilon^*(t, y)$, we have from (2.8) (2.9) and (2.11) that

$$
\begin{gathered}
|L(t, x, y)| \le \frac{1}{2}|x||y| + \frac{1}{2\varepsilon}|y|^2 + \xi, \\
|\nabla_x L(t, x, y)| = \frac{1}{2}|y|, \\
|\nabla_y L(t, x, y)| \le \frac{1}{2}|x| + |\nabla_y G^*(t, \dot{y})| \le \frac{1}{2}|x| + 1 + \frac{2}{\varepsilon}(|\dot{y}| + \xi + \eta),
\end{gathered}
$$

and then $\mathcal{K}_\varepsilon$ is continuously differentiable on Y. As for all $w \in Y$,

$$
\begin{aligned}
\langle \mathcal{K}_\varepsilon'(v), w\rangle &= \int_0^1 \Big[\frac{1}{2}(J\dot{v}_\varepsilon, w) - \frac{1}{2}(Jv_\varepsilon, \dot{w}) + (\nabla G_\varepsilon^*(t, \dot{v}_\varepsilon), \dot{w})\Big] dt \\
&= \int_0^1 (-Jv_\varepsilon + \nabla G_\varepsilon^*(t, \dot{v}_\varepsilon), \dot{w})dt = 0.
\end{aligned}
$$

One gets $Jv_\varepsilon = \nabla G_\varepsilon^*(t, \dot{v}_\varepsilon)$, i.e., $u_\varepsilon = \nabla G_\varepsilon^*(t, \dot{v}_\varepsilon)$. From the duality principle, it holds

$$\dot{v}_\varepsilon = \nabla G_\varepsilon(t, u)$$

and hence

$$-J\dot{u}_\varepsilon = \nabla G_\varepsilon(t, u);$$

i.e.,

$$J\dot{u}_\varepsilon + \nabla G_\varepsilon(t, u) = 0.$$

Clearly $v_\varepsilon \in Y$ implies $u_\varepsilon \in X$. □

Let $\varepsilon \in (0, \frac{\pi}{2} - l)$ and $H_\varepsilon(t,u) = H(t,u) + \frac{\varepsilon}{2}|u|^2$. Consider the system

$$\begin{gathered} J\dot{u} + \nabla H_\varepsilon(t,u) = 0, \\ u_1(0) = u_2(0) = 0. \end{gathered} \tag{2.13}$$

From $v = -Ju$ one has

$$\begin{aligned} \mathcal{K}_\varepsilon(u) &= \int_0^1 [\frac{1}{2}(J\dot{u}, u) + H_\varepsilon(t,u)]dt \\ &= -\int_0^1 [\frac{1}{2}(J\dot{v}, v) + H_\varepsilon^*(t,\dot{v})]dt =: -\Pi_\varepsilon(v), \end{aligned}$$

where $H_\varepsilon^*(t,\dot{v}) = \sup_{u\in\mathbb{R}^{2n}}[(\dot{v},u) - H_\varepsilon(t,u)]$.

The following proposition can be proved in a similar way as Proposition 2.5.

Proposition 2.7. *Under the conditions given in Theorem 1.2,* Π_ε *has one critical point* $v_\varepsilon \in Y = \{y \in H^1([0,1],\mathbb{R}^{2n}) : y_1(1) = 0, y_2(0) = 0\}$*, which minimize the value of* $\mathcal{K}_\varepsilon$ *and is uniformly bounded below for all* $\varepsilon \in (0, \frac{\pi}{2} - l)$*. Furthermore* $u_\varepsilon = Jv_\varepsilon$ *is a solution of* (2.13)*.*

3. Proof of main theorems

Proof of Theorem 1.1. In Proposition 2.5 we have proven that for each $\varepsilon \in (0, \frac{\pi}{2} - l)$, BVP (2.12) has a solution $u_\varepsilon = Jv_\varepsilon$, and $\mathcal{K}_\varepsilon(v_\varepsilon)$ is the minimum of $\mathcal{K}_\varepsilon$ on Y with $\mathcal{K}_\varepsilon(v_\varepsilon) \geq -\eta_0 + \alpha_0\|\dot{v}_\varepsilon\|_2^2$. Furthermore,

$$G(t,u) \leq G_\varepsilon(t,u)$$

implies

$$G_\varepsilon^*(t,\dot{v}) \leq G^*(t,\dot{v}).$$

So

$$\alpha_0\|\dot{v}_\varepsilon\|_2^2 - \eta_0 \leq \mathcal{K}_\varepsilon(v_\varepsilon) \leq \mathcal{K}_\varepsilon(0) = \int_0^1 G^*(t,0)dt = c < \infty$$

and then there is a $c_1 > 0$ such that

$$\|\dot{v}_\varepsilon\|_2^2 < c_1^2,$$

which in turn implies

$$\|\dot{u}_\varepsilon\|_2 = \|J\dot{v}_\varepsilon\|_2 < c_1$$

and there is $c_2 > 0$ such that $\|u_\varepsilon\|_2 < c_2$. Therefore, there is a $c_3 > 0$ such that

$$\|u_\varepsilon\|_X < c_3.$$

Since X is reflexive, there is a sequence $\{u_{\varepsilon_n}\} \subset \{u_\varepsilon : 0 < \varepsilon < \frac{\pi}{2} - l\}$ such that $u_{\varepsilon_n} \rightharpoonup u_0 \in X \subset H^1$ as $\varepsilon_n \to 0$ when $n \to \infty$. Hence

$$u_{\varepsilon_n} \to u_0 \quad \text{uniformly in } C([0,1],\mathbb{R}^{2n}).$$

It follows from $J\dot{u}_{\varepsilon_n}(t) + \nabla G_\varepsilon(t,u_{\varepsilon_n}(t)) = 0$ that

$$J(u_{\varepsilon_n}(t) - u_{\varepsilon_n}(0)) + \int_0^t [\varepsilon_n u_\varepsilon(s) + \nabla G(s,u_{\varepsilon_n}(s))]\,ds = 0$$

and then

$$J(u_0(t) - u_0(0)) + \int_0^t \nabla G(s,u_0(s))ds = 0.$$

Consequently,

$$J\dot{u}_0(t) + \nabla G(t,u_0(t)) = 0$$

and $u_0 \in X$ implies $u_{0,1}(0) = u_{0,2}(1) = 0$. That is to say, $u_0(t)$ is a solution to (1.3). Then $x(t) = u_{0,1}(t)$ is a solution to (1.1). Theorem 1.1 is now proved. $\square$

Theorem 1.2 is proved in a similar way as Theorem 1.1.

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Weigao Ge
Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, China
E-mail address: gew@bit.edu.cn, Phone 86-010-68911627

Yu Tian
School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China
E-mail address: tianyu2992@163.com