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RANDOM ATTRACTORS IN H^1 FOR STOCHASTIC TWO DIMENSIONAL MICROPOLAR FLUID FLOWS WITH SPATIAL-VALUED NOISES

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Abstract. This work studies the long-time behavior of two-dimensional micropolar fluid flows perturbed by the generalized time derivative of the infinite dimensional Wiener processes. Based on the *omega-limit compactness* argument as well as some new estimates of solutions, it is proved that the generated random dynamical system admits an H^1-random attractor which is compact in H^1 space and attracts all tempered random subsets of L^2 space in H^1 topology. We also give a general abstract result which shows that the continuity condition and absorption of the associated random dynamical system in H^1 space is not necessary for the existence of random attractor in H^1 space.

1. Introduction

The micropolar fluid model is a qualitative generalization of the well-known Navier-Stokes model in the sense that it takes into account the microstructure of fluid [23]. It was introduced by Eringen [17] as an important model to describe a class of non-Newtonian fluid motion with micro-rotational effects and inertia involved.

Let $\mathcal{O} \subset \mathbb{R}^2$ be a smooth bounded domain. This paper is concerned with the micropolar fluid flows driven by the time-space additive noises

$$\begin{gathered}
\nabla \cdot v = 0 \quad \text{on } \mathcal{O} \times \mathbb{R}^+, \\
\frac{dv}{dt} - (\nu + \kappa)\Delta v - 2\kappa \nabla \times V + \nabla \pi + v.\nabla v = f + \dot{W}_1 \quad \text{on } \mathcal{O} \times \mathbb{R}^+, \\
\frac{dV}{dt} - \gamma \Delta V + 4\kappa V - 2k \nabla \times v + v.\nabla V = g + \dot{W}_2 \quad \text{on } \mathcal{O} \times \mathbb{R}^+,
\end{gathered} \tag{1.1}$$

associated with the hard wall boundary condition

$$v = 0 \quad \text{on } \partial\mathcal{O} \times \mathbb{R}^+, \quad V = 0 \quad \text{on } \partial\mathcal{O} \times \mathbb{R}^+, \tag{1.2}$$

and the initial value condition

$$v(x,0) = v_0, \quad V(x,0) = V_0, \tag{1.3}$$

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with a velocity vector field $v = v(x,t) = (v_1(x,t), v_2(x,t)) \in \mathbb{R}^2$, a scalar microrotation field $V = V(x,t) \in \mathbb{R}$, a scalar pressure $\pi = \pi(x,t) \in \mathbb{R}$. In equations (1.1), the constants $\nu > 0$, $\kappa \geq 0$, $\gamma > 0$ (ν is usually called the Newtonian viscosity, γ and κ are the microrotation viscosity coefficients), and $f(x) = (f_1(x), f_2(x))$ and $g(x)$ denote the exterior body force and the moment, respectively. Moreover, $W_1(t)$ and $W_2(t)$ are independent two-sided real-valued Wiener processes with values in appropriate function spaces specified later. In addition, Δ is the Laplacian on $\mathcal{O}$ and

$$\nabla \times v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \nabla \cdot v = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}, \quad \nabla \times V = \Big(\frac{\partial V}{\partial x_2}, -\frac{\partial V}{\partial x_1}\Big).$$

There is a large volume of literature on the mathematical theory of the autonomous or non-autonomous micropolar fluid model; see, e.g., [20, 23, 24, 25, 29, 16, 8, 9]. Especially, for this two dimensional autonomous model, Lukaszewicz [23] proved the existence of L^2-global attractor in a bounded domain; Dong and Chen [16] established the existence of L^2-global attractor in some unbounded domains; Chen et al [8] proved that the L^2-global attractor was compact in the space H^2 based on the notion of the so-called Kuratowski measure of noncompactness of a bounded set [33]. As for the non-autonomous model, Zhao et al [29] proved the existence of H^1-uniform attractor in an unbounded Poincaré domains by utilizing the energy method originated from [24]; Chen [10] considered the non-homogeneous micropolar fluid flows and obtained the existence of L^2-pullback attractor in a Lipschitz bounded domain by energy equation method; Chen et al [9] and Lukaszewicz and Tarasińska [25] obtained the existence of H^1-pullback attractors in a bounded domain from a viewpoint of measuring noncompactness [33], respectively.

It is well known that the random attractor, which was initiated by [26, 14], is an appropriate notion to describe the long-time behavior of the solution of stochastic partial differential equation. The applications cover a wide range of concrete differential equations; see, recently, [28, 30, 31, 22, 3] and the references cited there. Such a attractor, which generalizes non-trivially the global attractors well developed (see, e.g., [27, 2]), is a compact invariant random set which attracts every orbit in the state space. It is uniquely determined by attracting deterministic compact sets of phase space [13].

The goal of this article is to prove the existence of random attractors of the micropolar fluid model (1.1)–(1.3) in H^1 space with irregular and spatially valued noise. On account of the irregularity of solutions in H^1 space, the Sobolev compact imbedding method is unavailable in the proof the compactness of the random attractor. To achieve our study, we utilize the technique developed in [31, 32] to surmount this obstacle. Specifically, the notion of omega-limits compactness, which was initiated in [22] and [21] in the framework of RDS, is successfully employed. The main advantage of this technique is that we need not to estimate the solutions in functional spaces of higher regularity to show the existence of compact random absorbing set which does not work in this case [18].

To solve our problem, we first prove an abstract result. We show that for the bi-space (X, Z) with a sequence uniqueness (see section 2 Hypothesis A), the random attractor on the space X is a random attractor on the space Y only if the RDS φ possess omega-limit compactness on the space Y. The continuity (even quasi-continuity [22]), absorption of the associated random dynamical systems on Z is not necessary, see Theorem 2.3. This result is new even in deterministic case, see [33, 22].

The outline of this article is as follows. Section 2 presents some basic facts needed for further considerations, including some notions and an abstract result about random attractors and the appropriate spaces and operators. In section 3 we recall the Ornstein-Uhlenbeck process and its regular hypothesis and then give the main conclusion of this study. Section 4 is the proof of our main conclusion.

2. Preliminaries

This section contains some background material which we will use in further discussion.

2.1. **Random dynamical systems and an abstract result.** In this subsection, we list some appropriate concepts and tools from the theory of random dynamical systems (RDSs) and obtain a abstract result. For more details the readers may refer to [1, 11, 14].

Let $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ be two completely separable Banach spaces with Borel σ-algebras $\mathcal{B}(X)$ and $\mathcal{B}(Z)$, respectively.

A random dynamical system on a Banach space X is a family of measurable mappings φ incorporated a metric dynamical system (MDS) θ, where the metric dynamical system θ is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a group $\theta_t, t \in \mathbb{R}$, of measure preserving transformations of $(\Omega, \mathcal{F}, \mathbb{P})$, and the family of measurable mappings $\varphi : \mathbb{R}^+ \times \Omega \times X \to X$; $(t, \omega, x) \mapsto \varphi(t, \omega)x$ satisfies the cocycle property

$$\varphi(0, \omega) = \mathrm{id}, \quad \varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega),$$

for all $s, t \in \mathbb{R}^+$. We will denote this RDS by the simple notation φ. An RDS φ is continuous in the meaning that the mappings $\varphi(t, \omega) : X \to X$ are continuous in X for all $t \in \mathbb{R}^+$ and .

A random set $D = \{D(\omega)\}_{\omega\in\Omega}$ is a family of closed subsets of X indexed by $\omega \in \Omega$ such that for every $x \in X$, the mapping $\omega \mapsto d_X(x, D(\omega))$ is measurable with respect to $\mathcal{F}$, where for the nonempty sets $A, B \in 2^X$,

$$d_X(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|_X,$$

and in particular $d_X(x, B) = d_X(\{x\}, B)$.

A random variable $R \in \mathbb{R}^+$ over a MDS θ is tempered if

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \log^+ R(\theta_t\omega) = 0, \tag{2.1}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. Note that (2.1) is equivalent to

$$\lim_{t\to\pm\infty} e^{-\lambda|t|} R(\theta_t\omega) = 0 \quad \text{for any } \lambda > 0,$$

see [19, 6]. A random set $D = \{D(\omega)\}_{\omega\in\Omega} \in 2^X$ is called tempered if $R(\omega) = \sup_{x\in D(\omega)} \|x\|_X$ is a tempered random variable.

Let $\mathcal{D}_X$ and $\mathcal{D}_Z$ denote the collection of all tempered random subsets of X and Z, respectively. In addition, we assume that range $\varphi(X) \subseteq Z$. In the following we recall the basic concepts about bi-space random attractor; see [2, 30].

Definition 2.1. (1) A random set $K_Z \in \mathcal{D}_Z$ is called an (X,Z)-random absorbing set for the RDS φ over a MDS θ if for every $D \in \mathcal{D}_X$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists an $T = T(D,\omega) > 0$ such that for all $t \geq T$,
$$\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega) \subseteq K_Z(\omega).$$

(2) A compact random set $\mathcal{A}_Z \in 2^Z$ is said to be an (X,Z)-random attractor for the RDS φ over a MDS θ if the invariance property
$$\varphi(t,\omega)\mathcal{A}_Z(\omega) = \mathcal{A}_Z(\theta_t\omega)$$
is satisfied for all $t \geq 0$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, and if in addition, the pullback attracting property
$$\lim_{t\to\infty} d_Z(\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega), \mathcal{A}_Z(\omega)) = 0$$
holds for every $D \in \mathcal{D}_X$ and $\mathbb{P}$-a.e. $\omega \in \Omega$.

Definition 2.2. An RDS φ over an MDS θ is said to be (X,Z)-omega-limit compact if for every $\varepsilon > 0$ and $D \in \mathcal{D}_X$, there exists $T = T(\varepsilon, D, \omega)$ such that for all $t \geq T$,
$$k\Big(\cup_{t\geq T}\, \varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)\Big) \leq \varepsilon, \quad \mathbb{P}\text{-a.e. } \omega \in \Omega,$$
where $k(B)$ is the Kuratowski measure of non-compactness of a bounded subset $B \subset Z$ defined by
$$k(B) = \inf\{d > 0 : B \text{ admits a finite cover by sets of diameter} \leq d\}.$$

Hypothesis (A1). Assume that the bi-space (X,Z) satisfies the sequence limits uniqueness, in the sense that for every bounded sequence $\{x_n\}_n \subset X \cap Z$ such that $x_n \to x$ in X and $x_n \to y$ in Z, respectively, then we have $x = y$. The nested relation between X and Z is unknown except that $\varphi(X) \subseteq Z$.

Theorem 2.3. *Assume that the bi-space* (X,Z) *satisfies* (A1)*, and* φ *is a continuous RDS on* X *over a MDS* θ*. If there exists an* (X,X)*-random absorbing set* K *for* φ *and* φ *is* (X,X)*-omega-limit compact, then the random set* $\mathcal{A}_X$*,*
$$\mathcal{A}_X(\omega) = \cap_{s\geq 0}\overline{\cup_{t\geq s}\varphi(t,\theta_{-t}\omega)K(\theta_{-t}\omega)}^X, \quad \omega \in \Omega, \tag{2.2}$$
is a unique (X,X)*-random attractor for* φ *in* X*, where* $\overline{B}^X$ *denotes the closure of* B *with respect to the* X*-norm.*

Furthermore, if φ *is* (X,Z)*-omega-limit compact then the random set* $\mathcal{A}_Z$*,*
$$\mathcal{A}_Z(\omega) = \cap_{s\geq 0}\overline{\cup_{t\geq s}\varphi(t,\theta_{-t}\omega)K(\theta_{-t}\omega)}^Z, \quad \omega \in \Omega, \tag{2.3}$$
is a unique (X,Z)*-random attractor for* φ*. In addition,* $\mathcal{A}_X = \mathcal{A}_Z \in \mathcal{D}_X$*.*

Proof. The first part is the same as [22, Theorem 4.1], so we omit the proof. We prove the second result. First, (2.3) makes sense by our assumption that $\varphi(t,\omega)X \subseteq Z$. We show that $\mathcal{A}_Z$ is a random attractor in the space Z, that is, $\mathcal{A}_Z$ satisfies the compact, attracting and invariant property.

By [22, Lemma 2.5(v)] and the omega-limit compactness of φ in Z, we have
$$k\Big(\overline{\cup_{t\geq T}\varphi(t,\theta_{-t}\omega)K(\theta_{-t}\omega)}^Z\Big) = k\Big(\cup_{t\geq T}\, \varphi(t,\theta_{-t}\omega)K(\theta_{-t}\omega)\Big) \to 0 \ \text{ as } T \to \infty.$$
At the same time, $\overline{\cup_{t\geq T}\varphi(t,\theta_{-t}\omega)K(\theta_{-t}\omega)}^Z$ is norm-closed in Z. Then thanks to the nested property of the Kuratowski measure of non-compactness (see [22, Lemma 2.5 (iv)]), we know that $\mathcal{A}_Z$ is nonempty and compact as required.

Furthermore by a similar argument as in [14, 30, 22] we can show that $\mathcal{A}_Z$ possesses (X, Z)-attracting property.

By the definitions of formula (2.2)–(2.3) and the omega-limit compactness of φ in X and Z, it is easy to show that $\mathcal{A}_X = \mathcal{A}_Z$. Thus $\mathcal{A}_Z$ is invariant since $\mathcal{A}_X$ is invariant. $\square$

Remark 2.4. In applications, one can choose $X = L^2$ and $Z = L^p$ $(p > 2)$ or H^1 with bounded or unbounded spatial domain. Note that (L^2, L^p) and (L^2, H^1) satisfy (A1). Therefore, the (A1) is not restrictive in concrete problems. In particular, our Theorem 2.3 implies [30, Theorem 2.8].

2.2. **Functional settings.** In this subsection, we introduce some spaces and operators stated as follows.

Let $L^p(\mathcal{O})$ and $H^s(\mathcal{O})$ be the usual Sobolev spaces. We set $L^2 = (L^2(\mathcal{O}))^2 \times L^2(\mathcal{O})$, endowed with the following scalar inner product

$$(.,.) = (.,.)_{(L^2(\mathcal{O}))^2} + (.,.)_{L^2(\mathcal{O})},$$

and the norms in $(L^2(\mathcal{O}))^2$, $L^2(\mathcal{O})$ and L^2 are together denoted by the same notation $|.|$, without any confusion. We define a functional space $\mathscr{V}$ integrated the boundary and also the divergence free condition,

$$\mathscr{V} = \{(v, V) \in (C_0^\infty(\mathcal{O}))^2 \times C_0^\infty(\mathcal{O}) : \operatorname{div} v = 0\}.$$

Define $H^1 = (H^1(\mathcal{O}))^2 \times H^1(\mathcal{O})$, where $H^1(\mathcal{O})$ is the usual Sobolev space. Let $\mathcal{H}$ be the closure of $\mathscr{V}$ with respect to the L^2-norm. The norm in $\mathcal{H}$ is still denoted by $|\cdot|$. Moreover, we let $\mathcal{V}$ be the closure of $\mathscr{V}$ with respect to the H^1-norm, possessing the equivalent norm in $\mathcal{V}$ is denoted by $\|\cdot\| = |\nabla \cdot|$. In addition, $\mathcal{V}'$ denotes the dual space of $\mathcal{V}$. Then we have $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$.

For $\mathbf{V} = (v, V)$, we define the operators

$$\begin{gathered}
A_1 v = -(\nu + \kappa)\Delta v, \quad A_2 V = -\gamma \Delta V, \\
B_1(v, v) = v.\nabla v, \quad B_2(v, V) = v.\nabla V, \\
A\mathbf{V} = (A_1 v, A_2 V), \quad B(v, \mathbf{V}) = (B_1(v, v), B_2(v, V)), \\
Lu = (-2\kappa \nabla \times V, -2\kappa \nabla \times v + 4\kappa V), \quad F = (f, g), \quad f \in (L^2(\mathcal{O}))^2, \quad g \in L^2(\mathcal{O}).
\end{gathered}$$

It is obvious that the operator A is a positive self-adjoint unbounded operator and then A^{-1} is also self-adjoint but compact operator in $\mathcal{H}$, and we can utilize the elementary spectral theory in a Hilbert space. We infer that there exists a compete orthonormal family of $\mathcal{H}$, $\{\mathbf{e}_j\}_{j=1}^\infty$ of eigenvectors of A. The corresponding spectrum of A is discrete and denoted by $\{\lambda_j\}_{j=1}^\infty$ which are positive, increasing and tend to infinity as $j \to \infty$.

In particular, we also can use the spectrum theory to allow us to define the operator A^s, the power of A. For $s > 0$, the operator A^s is also a strictly positive and self-adjoint unbounded operator in $\mathcal{H}$ with a dense domain $D(A^s) \subset \mathcal{H}$. This allows us to introduce the function spaces

$$D(A^s) = \Big\{\mathbf{V} = \sum_{j=1}^\infty (\mathbf{V}, \mathbf{e}_j)\mathbf{e}_j : \|\mathbf{V}\|^2_{D(A^s)} = \sum_{j=1}^\infty (\mathbf{V}, \mathbf{e}_j)^2 \lambda_j^{2s} < +\infty\Big\}.$$

This norm $\|\cdot\|_{D(A^s)}$ on $D(A^s)$ is equivalent to the usual norm induced by H^{2s}; see Temam [27] for details. In particular, $D(A^0) = \mathcal{H}$ and $D(A^{1/2}) = \mathcal{V}$. Furthermore, we have

$$\min\{\nu+\kappa,\gamma\}\|\mathbf{V}\|^2 \le (A\mathbf{V},\mathbf{V}) \le \lambda_0^{-1/2}|A\mathbf{V}|\,\|\mathbf{V}\|, \tag{2.4}$$

for all $\mathbf{V}=(v,V)\in D(A)$, where $\lambda_0>0$ satisfies the Poincaré inequality $\lambda_0|\mathbf{V}|^2 \le \|\mathbf{V}\|^2$.

Based on the orthonormal basis $\{\mathbf{e}_j\}_{j=1}^\infty$ of eigenfunctions of A, we define the m-dimensional subspace $\mathcal{V}_m=\operatorname{span}\{\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_m\}\subset\mathcal{V}$ and the canonical orthogonal projection $P_m:\mathcal{V}\mapsto\mathcal{V}_m$ such that for every $\mathbf{V}\in\mathcal{V}$, $\mathbf{V}$ has a unique decomposition: $\mathbf{V}=P_m\mathbf{V}+\mathbf{V}_m$, where

$$P_m\mathbf{V}=\sum_{j=1}^m(\mathbf{V},\mathbf{e}_j)\mathbf{e}_j\in\mathcal{V}_m,\quad \mathbf{V}_m=(I-P_m)\mathbf{V}=\sum_{j=m+1}^\infty(\mathbf{V},\mathbf{e}_j)\mathbf{e}_j\in\mathcal{V}_m^\perp; \tag{2.5}$$

that is, $\mathcal{V}=\mathcal{V}_m\oplus\mathcal{V}_m^\perp$.

According to the above notation, we write (1.1)–(1.3) as the evolution equation

$$d\mathbf{V}+A\mathbf{V}dt+B(v,\mathbf{V})dt+L\mathbf{V}dt=Fdt+dW,\quad \mathbf{V}(0)=\mathbf{V}_0\in\mathcal{H}, \tag{2.6}$$

where $\mathbf{V}=(v,V)$ and $W=(W_1,W_2)$.

3. Existence of $(\mathcal{H},\mathcal{V})$-random attractor for the generated RDS φ

To obtain a priori estimate, we now use the method of Chueshov and Schmalfuß [12] to transform the evolution equation (2.6) to a deterministic partial differential equation with a random parameter without white noise.

A standard model for a spatially correlated noise is the the generalized time derivative of a two-sided *Brownian motion* $\omega=\omega(x,t), x\in\mathbb{R}^2$. Let $\mathcal{H}$ be the separable Hilbert space with norm $|\cdot|$ which is defined in section 2. As usual, we introduce the spatially valued *Brownian motion* MDS $\theta=(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$, where $\Omega=\{\omega\in C_0(\mathbb{R},\mathcal{H}):\omega(0)=0\}$ with compact open topology. This topology is metrizable by the complete metric

$$d(\omega_1,\omega_2)=\sum_{n=1}^\infty\frac{1}{2^n}\frac{d_n(\omega_1,\omega_2)}{1+d_n(\omega_1,\omega_2)},$$

where $d_n(\omega_1,\omega_2)=\max_{|t|\le n}|\omega_1-\omega_2|$ for ω_1 and ω_2 in Ω. $\mathcal{F}=\mathcal{B}(C_0(\mathbb{R},\mathcal{H}))$ is the Borel-σ-algebra induced by the compact open topology of Ω. Suppose the Wiener process ω has covariance operator Q. Let $\mathbb{P}$ be the Wiener measure with respect to Q. The Wiener shift is defined by

$$\theta_s\omega(t)=\omega(t+s)-\omega(s),\quad \omega\in\Omega,\ t,s\in\mathbb{R}.$$

Then the measure $\mathbb{P}$ is ergodic and invariant with respect to the shift θ. Then θ is an ergodic MDS.

The associated probability space defines a canonical Wiener process W. We also note that such a Wiener process W generates a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$,

$$\mathcal{F}_t\equiv\{W(\tau)|\tau\le t\}\subset\mathcal{F}.$$

We introduce the following stochastic partial differential equation on $\mathcal{O}$,

$$d\mathbf{Z}+A\mathbf{Z}dt=dW. \tag{3.1}$$

Because A is a positive and self-adjoint operator, there exists a mild solution to this stochastic equation with the form

$$\mathbf{Z}(t) = \mathbf{Z}(0) + \int_0^t e^{-(t-\tau)A} dW, \quad t > 0,$$

which is called an Ornstein-Uhlenbeck process; see [15]. For the Ornstein-Uhlenbeck process we have the regularity hypothesis; see also [12].

Lemma 3.1. *Suppose that the covariance operator Q of the Wiener process ω has a finite trace; i.e., Q satisfies*

$$\operatorname{tr}_{\mathcal{H}}(QA^{2s-1+\delta}) = \operatorname{tr}_{\mathcal{H}}(A^{s-1/2+\delta/2} Q A^{s-1/2+\delta/2}) < +\infty, \tag{3.2}$$

for some $s \geq 0$ and some (arbitrary small) $\delta > 0$, where $\operatorname{tr}_{\mathcal{H}}$ denotes the trace of the covariance. Then an $\mathcal{F}_0$-measurable Gaussian variable $\mathbf{Z} = (z, Z) \in D(A^s)$ exists, and the process $(t, \omega) \to \mathbf{Z}(\theta_t \omega)$ is a continuous stationary solution to the stochastic equation (3.1)*. Furthermore, the random variable $\|\mathbf{Z}(\omega)\|^2_{D(A^s)}$ is tempered and the expectation*

$$\mathbb{E}\|\mathbf{Z}\|^2_{D(A^s)} = \frac{1}{2} \operatorname{tr}_{\mathcal{H}}(A^{s-\frac{1}{2}} Q A^{s-\frac{1}{2}}) < +\infty.$$

Introducing a new variable $\mathbf{U} = \mathbf{V} - \mathbf{Z}(\theta_t \omega)$, we can rewrite (2.6) as the following evolution equation with a random parameter ω,

$$\begin{gathered} \frac{d\mathbf{U}}{dt} + A\mathbf{U} + B(u + z(\theta_t\omega), \mathbf{U} + \mathbf{Z}(\theta_t\omega)) + L(\mathbf{U} + \mathbf{Z}(\theta_t\omega)) = F(x) \\ \mathbf{U}(0) = \mathbf{U}_0 \in \mathcal{H}, \end{gathered} \tag{3.3}$$

or the following functional form

$$\begin{aligned} &(\frac{d\mathbf{U}}{dt}, \phi) + (A\mathbf{U}, \phi) + (B(u + z(\theta_t\omega), \mathbf{U} + \mathbf{Z}(\theta_t\omega)), \phi) + (L(\mathbf{U} + \mathbf{Z}(\theta_t\omega)), \phi) \\ &= (F(x), \phi), \phi \in \mathcal{V} \end{aligned}$$

where $\mathbf{U} = (u, U)$, $u = v - z(\theta_t\omega)$, $U = V - Z(\theta_t\omega)$.

From the analysis above, we obtain the existence of a weak solution for the problem (3.3)–(3.4) by the standard Galerkin approximation, see, e.g., [7].

Lemma 3.2. *Let $F = (f, g) \in \mathcal{H}$ and $\mathbf{U}_0 = (u_0, U_0) \in \mathcal{H}$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$, the initial problem* (3.3)–(3.4) *possesses a unique solution $\mathbf{U}(t, \omega, \mathbf{U}_0(\omega))$, where*

$$\mathbf{U} = (u, U) \in L^\infty(0, \infty; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \cap C([0, \infty; \mathcal{H}).$$

Furthermore, the mapping $\mathbf{U}_0 \mapsto \mathbf{U}(t, \omega, \mathbf{U}_0(\omega))$ from $\mathcal{H}$ to $\mathcal{H}$ is continuous for all $t \geq 0$.

By a standard argument on the measurability we can show that the solution generates a continuous RDS ψ in the space $\mathcal{H}$ given by $\psi(t, \omega)\mathbf{U}_0(\omega) = \mathbf{U}(t, \omega, \mathbf{U}_0(\omega))$. Put $\mathbf{V}(t, \omega, \mathbf{V}_0(\omega)) = \mathbf{U}(t, \omega, \mathbf{V}_0(\omega) - \mathbf{Z}(\omega)) + \mathbf{Z}(\theta_t\omega)$. Then $\mathbf{V}(t, \omega, \mathbf{V}_0(\omega))$ or briefly $\mathbf{V}(t)$ is a solution to (2.6) with initial value $\mathbf{V}_0(\omega) \in \mathcal{H}$. Given

$$\varphi(t, \omega)\mathbf{V}_0(\omega) = \mathbf{V}(t, \omega, \mathbf{V}_0(\omega)) = \mathbf{U}(t, \omega, \mathbf{V}_0(\omega) - \mathbf{Z}(\omega)) + \mathbf{Z}(\theta_t\omega), \quad \omega \in \Omega,$$

then φ is also a continuous RDS on $\mathcal{H}$ for the original equation (2.6), i.e., system (1.1)–(1.3).

The main conclusion of this study reads as follows.

Theorem 3.3. *We suppose that* (3.2) *holds. Set*

$$M(\omega) = C(\|Z(\omega)\|_{H^2}^2 + \|z(\omega)\|_{H^1}^2) - \frac{1}{2}\lambda_0\varsigma, \tag{3.4}$$

where C is a positive constant depending only on the physical coefficients of the fluid model, $\varsigma = \min\{\nu,\gamma\}$ and λ_0 is the same as in (2.4). *Assume that the mathematical expectation $\mathbb{E}M < 0$. Then the RDS φ generated by* (1.1)–(1.3) *admits a unique $(\mathcal{H},\mathcal{V})$-random attractor $\mathcal{A}_{\mathcal{V}}$. In addition, $\mathcal{A}_{\mathcal{V}} = \mathcal{A}_{\mathcal{H}}$, where $\mathcal{A}_{\mathcal{H}}$ is the $(\mathcal{H},\mathcal{H})$-random attractor.*

4. Proofs of main results

First we list some basic facts. By using the Young's inequality and combination with the divergence free condition, the operators A, L and B possess the following relationships:

$$-(A\mathbf{U},\mathbf{U}) - (L\mathbf{U},\mathbf{U}) \le -\min\{\nu,\gamma\}\|\mathbf{U}\|^2, \quad \forall \mathbf{U} = (u,U) \in \mathcal{V}, \tag{4.1}$$

$$\begin{aligned} -(L\mathbf{U},A\mathbf{U}) &= -2\kappa(\nu+\kappa)(\nabla\times U,\Delta u) - 2\kappa\gamma(\nabla\times u,\Delta U) - 4\kappa\gamma\|U\|^2 \\ &\le \frac{1}{2}|A\mathbf{U}|^2 + 2\kappa^2\|\mathbf{U}\|^2, \quad \forall \mathbf{U} = (u,U) \in D(A), \end{aligned} \tag{4.2}$$

$$\begin{gathered} B(u,\mathbf{U},\mathbf{V}) = -B(u,\mathbf{V},\mathbf{U}), \quad B(u,\mathbf{U},\mathbf{U}) = 0, \\ \forall (u,\mathbf{U},\mathbf{V}) \in (H^1(\mathcal{O}))^2 \times \mathcal{V} \times \mathcal{V}. \end{gathered} \tag{4.3}$$

We recall the Agmon's inequality; see [27],

$$\|u\|_{L^\infty} \le c|u|^{1/2}|A_1u|^{1/2}, \quad \forall\, u \in D(A_1). \tag{4.4}$$

For the projector P_m defined in Section 2, one can easily show that

$$|A_1P_mu|^2 \le \lambda_{m+1}|A_1^{1/2}P_mu|^2, \quad \forall \mathbf{U} = (u,U) \in D(A),$$

and hence by the classic Brezis-Gallouet's inequality; see [4, 5], we have

$$\begin{aligned} \|P_mu\|_{L^\infty} &\le c|A_1^{1/2}P_mu|\Big(1 + \log\frac{|A_1P_mu|^2}{\lambda_1|A_1^{1/2}P_mu|^2}\Big)^{1/2} \\ &\le c\|u\|\Big(1 + \log\frac{\lambda_{m+1}}{\lambda_1}\Big)^{1/2}, \quad \forall u \in D(A_1), \end{aligned} \tag{4.5}$$

where the letter c in (4.4) and (4.5) is a deterministic positive constant and λ_1 is the first eigenvalue of the stokes operator A.

Lemma 4.1. *There exist positive constants C and c depending only on the physical coefficients of this model and the domain $\mathcal{O}$ such that*

$$\frac{d}{dt}|\mathbf{U}|^2 + \varsigma\|\mathbf{U}\|^2 \le M(\theta_t\omega)|\mathbf{U}|^2 + G(\theta_t\omega), \tag{4.6}$$

$$\frac{d}{dt}((\nu+\kappa)\|u\|^2 + \gamma\|U\|^2) \le ((\nu+\kappa)\|u\|^2 + \gamma\|U\|^2)g(t,\omega) + h(t,\omega) \tag{4.7}$$

where $\varsigma = \min\{\nu,\gamma\}$ and

$$\begin{gathered} M(\omega) = C(\|z(\omega)\|^2 + \|Z(\omega)\|_{H^2}^2) - \frac{1}{2}\lambda_0\varsigma, \\ G(\omega) = c(\|z(\omega)\|^4 + |z(\omega)|^2\ \|Z(\omega)\|_{H^2}^2 + |z(\omega)|^2 + |Z(\omega)|^2 + |F|^2), \\ g(t,\omega) = c(\|u\|^2 + \|U\|^2)(|u|^2 + |z|^2), \end{gathered}$$

$$
\begin{gathered}
h(t,\omega)=4\kappa^2\|\mathbf{U}\|^2+c|u|^2(\|z\|^4+\|Z\|^4)+H(\omega),\\
H(\omega)=c(\|z\|^2_{H^2}+|z|^2\|z\|^4+|z|^2\|Z\|^4+\|z\|^2+\|Z\|^2+|Z|^2+|F|^2).
\end{gathered}
$$

Proof We multiply (3.3) by $\mathbf{U}$ and then integrate over $\mathcal{O}$, along with (4.1) and (4.3), to yield

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\mathbf{U}|^2\\
&=-(A\mathbf{U},\mathbf{U})-(L\mathbf{U},\mathbf{U})-(B(u+z,\mathbf{U}+\mathbf{Z}),\mathbf{U})-(L\mathbf{Z},\mathbf{U})+(F,\mathbf{U})\\
&\le -\varsigma\|\mathbf{U}\|^2-(B(u+z,\mathbf{U}+\mathbf{Z}),\mathbf{U})-(L\mathbf{Z},\mathbf{U})+(F,\mathbf{U})\\
&=-\varsigma\|\mathbf{U}\|^2-(B(u+z,\mathbf{Z}),\mathbf{U})-(L\mathbf{Z},\mathbf{U})+(F,\mathbf{U})\\
&=-\varsigma\|\mathbf{U}\|^2-(B_1(u+z,u+z),u)-(B_2(u+z,Z),U)-(L\mathbf{Z},\mathbf{U})+(F,\mathbf{U}),
\end{aligned}
\tag{4.8}
$$

where

$$
\begin{aligned}
&-(L\mathbf{Z},\mathbf{U})+(F,\mathbf{U})\\
&=2\kappa(\nabla\times Z,u)+2\kappa(\nabla\times z,U)-4\kappa(Z,U)+(F,\mathbf{U})\\
&=2\kappa(\nabla\times u,Z)+2\kappa(\nabla\times U,z)-4\kappa(Z,U)+(F,\mathbf{U})\\
&\le\frac{\varsigma}{8}\|\mathbf{U}\|^2+c(|z|^2+|Z|^2+|F|^2).
\end{aligned}
\tag{4.9}
$$

By the Hölder's inequality and Gagliardo-Nirenberg's inequality, we have

$$
\begin{aligned}
(B_1(u+z,u+z),u)&=(B_1(u,z),u)+(B_1(z,z),u)\\
&\le c\|u\|_{L^4}\ \|z\|\ \|u\|_{L^4}+c\|z\|_{L^4}\|z\|\ \|u\|_{L^4}\\
&\le c|u|\ \|z\|\ \|u\|+c\|z\|^2\ \|u\|\\
&\le\frac{\varsigma}{16}\|u\|^2+\frac{C}{2}|u|^2\ \|z\|^2+c\|z\|^4,
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
(B_2(u+z,Z),U)&\le c|u+z|\ \|\nabla Z\|_{L^4}\|U\|_{L^4}\\
&\le c|u|\ \|Z\|_{H^2}\|U\|+c|z|\ \|Z\|_{H^2}\|U\|\\
&\le\frac{\varsigma}{16}\|U\|^2+\frac{C}{2}|u|^2\ \|Z\|^2_{H^2}+c|z|^2\ \|Z\|^2_{H^2}.
\end{aligned}
\tag{4.11}
$$

Then from (4.8)–(4.11) it follows that

$$
\begin{aligned}
&\frac{d}{dt}|\mathbf{U}|^2+\frac{3}{2}\varsigma\|\mathbf{U}\|^2\\
&\le C(\|z\|^2+\|Z\|^2_{H^2})|\mathbf{U}|^2+c(\|z\|^4+|z|^2\ \|Z\|^2_{H^2}+|z|^2+|Z|^2+|F|^2),
\end{aligned}
\tag{4.12}
$$

where c and C is the positive constants independent of t,u,U,z,Z. Further, the Poincaré's inequality implies

$$
\frac{d}{dt}|\mathbf{U}|^2+\varsigma\|\mathbf{U}\|^2\le M(\theta_t\omega)|\mathbf{U}|^2+G(\theta_t\omega),
\tag{4.13}
$$

which shows that (4.6) holds.

Multiplying (3.3) by $A\mathbf{U}$ and then integrating over $\mathcal{O}$, along with (4.2), gives

$$
\begin{aligned}
&\frac{d}{dt}((\nu+\kappa)\|u\|^2+\gamma\|U\|^2)\\
&=-2|A\mathbf{U}|^2-2(L\mathbf{U},A\mathbf{U})-2(B(u+z,\mathbf{U}+\mathbf{Z}),A\mathbf{U})-2(L\mathbf{Z},A\mathbf{U})+2(F,A\mathbf{U})\\
&\le -|A\mathbf{U}|^2+4\kappa^2\|\mathbf{U}\|^2-2(B(u+z,\mathbf{U}+\mathbf{Z}),A\mathbf{U})-2(L\mathbf{Z},A\mathbf{U})+2(F,A\mathbf{U}),
\end{aligned}
\tag{4.14}
$$

where

$$\begin{aligned}
&-(L\mathbf{Z}, A\mathbf{U}) + (F, A\mathbf{U})\\
&= 2\kappa(\nu+\kappa)(\nabla\times Z, \Delta u) + 2\kappa\gamma(\nabla\times z, \Delta U) + 4\kappa\gamma(Z, \Delta U) + (F, A\mathbf{U})\\
&\le \frac{1}{16}|A\mathbf{U}|^2 + c(\|z\|^2 + \|Z\|^2 + |Z|^2 + |F|^2).
\end{aligned} \tag{4.15}$$

Consider that

$$(B(u+z, \mathbf{U}+\mathbf{Z}), A\mathbf{U}) = (B_1(u+z, u+z), A_1u) + (B_2(u+z, U+Z), A_2U). \tag{4.16}$$

Then by (4.4) we have

$$\begin{aligned}
&(B_1(u+z, u+z), A_1u)\\
&\le \|u+z\|_{L^\infty}\|u+z\|\,|A_1u|\\
&\le c|u+z|^{1/2}|A_1(u+z)|^{1/2}\|u+z\|\,|A_1u|\\
&\le \frac{1}{8}|A_1u|^2 + c|u+z|\,|A_1(u+z)|\,\|u+z\|^2\\
&\le \frac{1}{8}|A_1u|^2 + c|u+z|\,\|u+z\|^2\,|A_1u| + c|u+z|\,\|u+z\|^2\ |A_1z|\\
&\le \frac{1}{8}|A_1u|^2 + \frac{1}{8}|A_1u|^2 + c|u+z|^2\|u+z\|^4 + c|A_1z|^2\\
&\le \frac{1}{4}|A_1u|^2 + c(\|u\|^4|u|^2 + \|u\|^4|z|^2 + |u|^2\|z\|^4 + |z|^2\|z\|^4 + |A_1z|^2),
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
&(B_2(u+z, U+Z), A_2U)\\
&\le \|u+z\|_{L^\infty}\|U+Z\|\ |A_2U|\\
&\le c|u+z|^{1/2}|A_1(u+z)|^{1/2}\ \|U+Z\|\ |A_2U|\\
&\le \frac{1}{8}|A_2U|^2 + c|u+z|\ |A_1(u+z)|\ \|U+Z\|^2\\
&\le \frac{1}{8}|A_2U|^2 + c|u+z|\ \|U+Z\|^2|A_1u| + c|u+z|\ \|U+Z\|^2|A_1z|\\
&\le \frac{1}{8}|A_2U|^2 + \frac{1}{8}|A_1u|^2 + c|u+z|^2\|U+Z\|^4 + c|A_1z|^2\\
&\le \frac{1}{8}|A_2U|^2 + \frac{1}{8}|A_1u|^2 + c|u|^2\|U\|^4 + c|u|^2\|Z\|^4 + c|z|^2\|U\|^4\\
&\quad + c|z|^2\|Z\|^4 + c|A_1z|^2.
\end{aligned} \tag{4.18}$$

Therefore, from (4.14)–(4.18) it follows that

$$\begin{aligned}
&\frac{d}{dt}((\nu+\kappa)\|u\|^2 + \gamma\|U\|^2)\\
&\le c(\|u\|^4 + \|U\|^4)(|u|^2 + |z|^2) + 4\kappa^2\|\mathbf{U}\|^2 + c|u|^2(\|z\|^4 + \|Z\|^4)\\
&\quad + c(|A_1z|^2 + |z|^2\|z\|^4 + |z|^2\|Z\|^4 + \|z\|^2 + \|Z\|^2 + |Z|^2 + |F|^2)\\
&\le c((\nu+\kappa)\|u\|^2 + \gamma\|U\|^2)(\|u\|^2 + \|U\|^2)(|u|^2 + |z|^2) + c|u|^2(\|z\|^4 + \|Z\|^4)\\
&\quad + 4\kappa^2\|\mathbf{U}\|^2 + c(\|z\|_{H^2}^2 + |z|^2\|z\|^4 + |z|^2\|Z\|^4 + \|z\|^2 + \|Z\|^2 + |Z|^2 + |F|^2),
\end{aligned}$$

which proves (4.7).

Lemma 4.2. *Assume that the expectation $\mathbb{E}M < 0$, where M is in* (3.4). *Let $D \in \mathcal{D}$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exist tempered random radius $R_1(\omega)$ and* *constant $T = T(D,\omega) > 0$ such that for all $t \ge T$, the solution $\mathbf{U}(t,\omega,\mathbf{V}_0(\omega)-\mathbf{Z}(\omega))$ of the initial problem (3.3) with $\mathbf{V}_0 \in D$ satisfies that for every $l \in [t,t+1]$,*

$$|\mathbf{U}(l,\theta_{-t-1}\omega,\mathbf{V}_0(\theta_{-t-1}\omega)-\mathbf{Z}(\theta_{-t-1}\omega))| \le R_1(\omega), \tag{4.19}$$

where $\mathcal{D}$ is the collection of all tempered random subsets of $\mathcal{H}$.

Proof. Applying the Gronwall's lemma (see [27]) to (4.6), we find that for every $l \ge 0$,

$$|\mathbf{U}(l,\omega,\mathbf{V}_0(\omega)-\mathbf{Z}(\omega))|^2 \le e^{\int_0^l M(\theta_\tau\omega)d\tau}|\mathbf{U}_0(\omega)|^2 + c\int_0^l G(\theta_s\omega)e^{\int_s^l M(\theta_\tau\omega)d\tau}ds,$$

in which we replace ω with $\theta_{-t-1}\omega$ and get that for $l \in [t,t+1]$,

$$\begin{aligned}
&|\mathbf{U}(l,\theta_{-t-1}\omega,\mathbf{V}_0(\theta_{-t-1}\omega)-\mathbf{Z}(\theta_{-t-1}\omega))|^2\\
&\le e^{\int_0^l M(\theta_{\tau-t-1}\omega)d\tau}|\mathbf{U}_0(\theta_{-t-1}\omega)|^2 + \int_0^l G(\theta_{s-t-1}\omega)e^{\int_s^l M(\theta_{\tau-t-1}\omega)d\tau}ds\\
&= e^{\int_{-t-1}^{l-t-1} M(\theta_\tau\omega)d\tau}|\mathbf{U}_0(\theta_{-t-1}\omega)|^2 + \int_{-t-1}^{l-t-1} G(\theta_s\omega)e^{\int_s^{l-t-1} M(\theta_\tau\omega)d\tau}ds \\
&\le 2e^{\int_{-t-1}^{l-t-1} M(\theta_\tau\omega)d\tau}(|\mathbf{V}_0(\theta_{-t-1}\omega)|^2 + |\mathbf{Z}(\theta_{-t-1}\omega)|^2)\\
&\quad + \int_{-t-1}^{0} G(\theta_s\omega)e^{\int_s^{l-t-1} M(\theta_\tau\omega)d\tau}ds.
\end{aligned} \tag{4.20}$$

By noting that for $l \in [t,t+1]$, $l-t-1 \in [-1,0]$, we deduce that for $s \in [-t-1,0]$,

$$\begin{aligned}
e^{\int_s^{l-t-1} M(\theta_\tau\omega)d\tau} &= e^{-\frac12\lambda_0\varsigma(l-t-1)+\frac12\lambda_0\varsigma s}e^{\int_s^{l-t-1} C(\|Z(\theta_\tau\omega)\|_{H^2}^2+\|z(\theta_\tau\omega)\|^2)d\tau}\\
&\le e^{\frac12\lambda_0\varsigma+\frac12\lambda_0\varsigma s}e^{\int_s^0 C(\|Z(\theta_\tau\omega)\|_{H^2}^2+\|z(\theta_\tau\omega)\|^2)d\tau}\\
&\le e^{\frac12\lambda_0\varsigma}e^{\int_s^0 M(\theta_\tau\omega)d\tau}.
\end{aligned} \tag{4.21}$$

Then by (4.20) and (4.21) we obtain that for every $l \in [t,t+1]$,

$$\begin{aligned}
&|\mathbf{U}(l,\theta_{-t-1}\omega,\mathbf{V}_0(\theta_{-t-1}\omega)-\mathbf{Z}(\theta_{-t-1}\omega))|^2\\
&\le K\Big(e^{\int_{-t-1}^0 M(\theta_s\omega)ds}|\mathbf{V}_0(\theta_{-t-1}\omega)|^2 + |\mathbf{Z}(\theta_{-t-1}\omega)|^2)\\
&\quad + \int_{-t-1}^0 G(\theta_s\omega)e^{\int_s^0 M(\theta_\tau\omega)d\tau}ds\Big),
\end{aligned}$$

where $K = 2e^{\frac12\lambda_0\varsigma}$. By the Birkhoff's ergodic theorem and along with our assumption that $\mathbb{E}M(\omega) < 0$, it yields that

$$\lim_{t\to\pm\infty}\frac{1}{t+1}\int_{-t-1}^0 M(\theta_\tau\omega)d\tau = \mathbb{E}M(\omega) = \hat{\mu} < 0,$$

which implies

$$\int_{-t-1}^0 M(\theta_\tau\omega)d\tau \approx \hat{\mu}(t+1), \quad \text{as } t \to +\infty.$$

Since $G(\theta_s)$ is sub-exponential growth,

$$\int_{-\infty}^0 G(\theta_s\omega)e^{\int_s^0 M(\theta_\tau\omega)d\tau}ds < +\infty.$$

Consider that $\|\mathbf{Z}(\omega)\|_{D(A)}$ is tempered. Then $|\mathbf{Z}(\omega)|$ is tempered, whence by the initial data $\mathbf{V}_0(\omega) \in D(\omega)$, there exists constant $T = T(D,\omega) > 0$ such that for all $t \geq T$ with $l \in [t, t+1]$,

$$\begin{aligned}
&|\mathbf{U}(l, \theta_{-t-1}\omega, \mathbf{V}_0(\theta_{-t-1}\omega) - \mathbf{Z}(\theta_{-t-1}\omega))|^2 \\
&\leq R_1(\omega)^2 := K(1 + \int_{-\infty}^0 G(\theta_s\omega) e^{\int_s^0 M(\theta_\tau\omega)d\tau} ds).
\end{aligned}$$

We can use the method in [6, Lemma 4.6] to show that the random variable $R_1(\omega)$ is tempered. Indeed, for an arbitrary $\lambda > 0$, and let $t < 0$. We then have

$$\begin{aligned}
&e^{-2\lambda|t|} R_1(\theta_t\omega)^2 \\
&= Ke^{\lambda 2t} + Ke^{\lambda t} \int_{-\infty}^0 G(\theta_{s+t}\omega) e^{\lambda t + \int_s^0 M(\theta_{\tau+t}\omega)d\tau} ds. \\
&= Ke^{\lambda 2t} + Ke^{\lambda t} \int_{-\infty}^0 G(\theta_{s+t}\omega) e^{\lambda t + \int_{s+t}^0 (M(\theta_\tau\omega) - \hat{\mu})d\tau - \int_t^0 (M(\theta_\tau\omega) - \hat{\mu})d\tau - \hat{\mu}s} ds.
\end{aligned} \tag{4.22}$$

Let $0 < \varepsilon < \frac{1}{4}\min\{-\hat{\mu}, \lambda\}$. Then using again the sub-exponential growth of $G(\theta_\tau)$, there exists $t_1 = t_1(\varepsilon, \omega) < 0$ such that for all $t < t_1$,

$$G(\theta_{s+t}) \leq e^{-\varepsilon(s+t)}. \tag{4.23}$$

On the other hand, there exists $t_2 = t_2(\varepsilon, \omega) < 0$ such that for all $t < t_2$,

$$\int_{s+t}^0 (M(\theta_\tau\omega) - \hat{\mu})d\tau \leq -\varepsilon(t+s); \quad -\int_t^0 (M(\theta_\tau\omega - \hat{\mu}) \leq -\varepsilon(t+s). \tag{4.24}$$

Put $t_0 = \min\{t_1, t_2\}$. Then it follows from (4.22)–(4.24) that for all $t < t_0$,

$$\begin{aligned}
e^{-2\lambda|t|} R_1(\theta_t\omega)^2 &\leq Ke^{\lambda 2t} + Ke^{\lambda t} \int_{-\infty}^0 e^{\lambda t - 3\varepsilon(t+s) - \hat{\mu}s} ds \\
&\leq Ke^{\lambda 2t} + Ke^{\lambda t} \int_{-\infty}^0 e^{\varepsilon(t+s)} ds \to 0
\end{aligned}$$

as $t \to -\infty$. Similarly, we can prove the convergence for $t \to +\infty$. □

Lemma 4.3. *Assume that the expectation $\mathbb{E}M < 0$, where M is in (3.4). Let $D \in \mathcal{D}$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exist random radius $R_2(\omega)$ and constant $T = T(D,\omega) > 0$ such that for all $t \geq T$, the solution $\mathbf{U}(t, \omega, \mathbf{V}_0(\omega) - \mathbf{Z}(\omega))$ of the initial problem (3.3) with $\mathbf{V}_0 \in D$ satisfies that for every $l \in [t, t+1]$,*

$$\|\mathbf{U}(l, \theta_{-t-1}\omega, \mathbf{V}_0(\theta_{-t-1}\omega) - \mathbf{Z}(\theta_{-t-1}\omega))\| \leq R_2(\omega),$$

where $\mathcal{D}$ is the collection of tempered random subsets of $\mathcal{H}$.

Proof. Integrating (4.6) from t to $t+1$ yields

$$\begin{aligned}
&\int_t^{t+1} \|\mathbf{U}(s, \omega, \mathbf{V}_0(\omega) - \mathbf{Z}(\omega))\|^2 ds \\
&\leq c\int_t^{t+1} M(\theta_s\omega) |\mathbf{U}(s, \omega, \mathbf{V}_0(\omega) - \mathbf{Z}(\omega))|^2 ds \\
&\quad + c\int_t^{t+1} G(\theta_s\omega) ds + c|\mathbf{U}(t, \omega, \mathbf{V}_0(\omega) - \mathbf{Z}(\omega))|^2.
\end{aligned}$$

Then by Lemma 4.2 we know that there exists a $T = T(D,\omega) > 0$ such that for every $t \ge T$,

$$\begin{aligned}
&\int_t^{t+1} \|\mathbf{U}(s,\theta_{-t-1}\omega,\mathbf{V}_0(\theta_{-t-1}\omega)-\mathbf{Z}(\theta_{-t-1}\omega))\|^2 ds \\
&\le cR_1(\omega)^2\int_t^{t+1} M(\theta_{s-t-1}\omega)ds + c\int_t^{t+1} G(\theta_{s-t-1}\omega)ds + cR_1(\omega)^2 \\
&= c(R_1(\omega)^2\int_{-1}^0 M(\theta_s\omega)ds + \int_{-1}^0 G(\theta_s\omega)ds + R_1(\omega)^2) := C_1(\omega).
\end{aligned} \tag{4.25}$$

Using the classic Gronwall's lemma to (4.7) on the interval $[s,l]$ with $t \le s \le l \le t+1$, we obtain

$$\begin{aligned}
\|\mathbf{U}(l,\omega,\mathbf{V}_0(\omega)-\mathbf{Z}(\omega))\|^2 &\le ce^{\int_t^{t+1} g(\tau,\omega)d\tau}(\|\mathbf{U}(s,\omega,\mathbf{V}_0(\omega)-\mathbf{Z}(\omega))\|^2 \\
&\quad + \int_t^{t+1} h(s,\omega)ds),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\|\mathbf{U}(l,\theta_{-t-1}\omega,\mathbf{V}_0(\theta_{-t-1}\omega)-\mathbf{Z}(\theta_{-t-1}\omega))\| \\
&\le ce^{\int_t^{t+1} g(\tau,\theta_{-t-1}\omega)d\tau}(\int_t^{t+1} \|\mathbf{U}(s,\theta_{-t-1}\omega,\mathbf{V}_0(\theta_{-t-1}\omega)-\mathbf{Z}(\theta_{-t-1}\omega))\|^2 ds \\
&\quad + \int_t^{t+1} h(s,\theta_{-t-1}\omega)ds).
\end{aligned} \tag{4.26}$$

Note that by Lemma 4.1 and association with (4.25), there exists $T = T(D,\omega) > 0$ such that for all $t \ge T$,

$$\begin{aligned}
&\int_t^{t+1} g(\tau,\theta_{-t-1}\omega)d\tau \\
&= c\int_t^{t+1} (\|u(s)\|^2 + \|U(s)\|^2)(|u(s)|^2 + |z(\theta_{s-t-1}\omega)|^2)ds \\
&\le c(R_1(\omega)^2 + \max_{-1\le t\le 0}\{|z(\theta_t\omega)|^2\}\int_t^{t+1} (\|u(s)\|^2 + \|U(s)\|^2)ds \\
&\le c(R_1(\omega)^2 + \max_{-1\le t\le 0}\{|z(\theta_t\omega)|^2\})C_1(\omega) := C_2(\omega),
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
&\int_t^{t+1} h(s,\theta_{-t-1}\omega)ds \\
&= c\int_t^{t+1} \|\mathbf{U}(s)\|^2 ds + c\int_t^{t+1} |u(s)|^2(\|z(\theta_{s-t-1}\omega)\|^4 + \|Z(\theta_{s-t-1}\omega)\|^4)ds \\
&\quad + \int_t^{t+1} H(\theta_{s-t-1}\omega)ds \\
&\le c\max_{-1\le t\le 0}\{\|z(\theta_t\omega)\|^4 + \|Z(\theta_t\omega)\|^4\}(C_1(\omega) + R_1(\omega)^2) + \int_{-1}^0 H(\theta_s\omega)ds \\
&:= C_3(\omega).
\end{aligned} \tag{4.28}$$

Then by (4.26)–(4.28) it gives that for all $t \ge T$ and $l \in [t,t+1]$,

$$\|\mathbf{U}(l,\theta_{-t-1}\omega,\mathbf{V}_0(\theta_{-t}\omega)-\mathbf{Z}(\theta_{-t}\omega))\|^2 \le ce^{C_2(\omega)}(C_1(\omega)+C_3(\omega)) := R_2(\omega)^2.$$

This completes the proof. □

Lemma 4.4. *Assume that the expectation $\mathbb{E}M < 0$, where M is in (3.4). Let $D \in \mathcal{D}$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $\varepsilon > 0$, there are $N = N(\varepsilon, \omega), K = K(\omega)$, and $T = T(\varepsilon, D, \omega) > 0$ such that for all $t \geq T$ and $m \geq N$, the solution $\mathbf{U}(t, \omega, \mathbf{V}_0(\omega) - \mathbf{Z}(\omega))$ of the initial problem (3.3) with $\mathbf{V}_0 \in D$ satisfies that*

$$\|P_m \mathbf{U}(t, \theta_{-t}\omega, \mathbf{V}_0(\theta_{-t}\omega) - \mathbf{Z}(\theta_{-t}\omega))\| \leq K,$$
$$\|(I - P_m)\mathbf{U}(t, \theta_{-t}\omega, \mathbf{V}_0(\theta_{-t}\omega) - \mathbf{Z}(\theta_{-t}\omega))\| \leq \varepsilon,$$

where $\mathcal{D}$ is the collection of tempered random subsets of $\mathcal{H}$.

Proof. Multiplying (3.3) by $A\mathbf{U}_m$ and then integrating over $\mathcal{O}$, we have

$$\begin{aligned} &\frac{d}{dt}((\nu+\kappa)\|u_m\|^2 + \gamma\|U_m\|^2) + 2|A\mathbf{U}_m|^2 \\ &= -2(L\mathbf{U}, A\mathbf{U}_m) - 2(B(u+z, \mathbf{U}+\mathbf{Z}), A\mathbf{U}_m) - 2(L\mathbf{Z}, A\mathbf{U}_m) + 2(F, A\mathbf{U}_m), \end{aligned} \tag{4.29}$$

where

$$\begin{aligned} &-(L\mathbf{U}, A\mathbf{U}_m) \\ &= -2\kappa(\nu+\kappa)(\nabla \times U, \Delta u_m) - 2\kappa\gamma(\nabla \times u, \Delta U_m) - 4\kappa\gamma\|U_m\|^2 \\ &\leq \frac{1}{2}|A\mathbf{U}_m|^2 + 2\kappa^2\|\mathbf{U}\|^2, \end{aligned} \tag{4.30}$$

$$\begin{aligned} &-(L\mathbf{Z}, A\mathbf{U}_m) \\ &= -2\kappa(\nu+\kappa)(\nabla \times Z, \Delta u_m) - 2\kappa\gamma(\nabla \times z, \Delta U_m) + 4\kappa\gamma(Z, \Delta U_m) \\ &\leq \frac{1}{64}|A\mathbf{U}_m|^2 + c(\|z\|^2 + \|Z\|^2 + |Z|^2), \end{aligned} \tag{4.31}$$

$$2(F, A\mathbf{U}_m) \leq \frac{1}{32}|A\mathbf{U}_m|^2 + c|F|^2. \tag{4.32}$$

Then from (4.29)–(4.32), we obtain

$$\begin{aligned} &\frac{d}{dt}((\nu+\kappa)\|u_m\|^2 + \gamma\|U_m\|^2) + |A\mathbf{U}_m|^2 \\ &\leq -2(B(u+z, \mathbf{U}+\mathbf{Z}), A\mathbf{U}_m) + \frac{1}{16}|A\mathbf{U}_m|^2 \\ &\quad + c\|\mathbf{U}\|^2 + c(\|z\|^2 + \|Z\|^2 + |Z|^2 + |F|^2). \end{aligned} \tag{4.33}$$

Likewise, we have

$$|A\mathbf{U}_m|^2 \geq \lambda_{m+1}|A^{1/2}\mathbf{U}_m|^2 = \lambda_{m+1}((\nu+\kappa)\|u_m\|^2 + \gamma\|U_m\|^2). \tag{4.34}$$

It remains to estimate the first term on the right hand side of (4.33). To this end, we rewrite

$$\begin{aligned} &(B(u+z, \mathbf{U}+\mathbf{Z}), A\mathbf{U}_m) \\ &= (B_1(u+z, u+z), A_1u_m) + (B_2(u+z, U+Z), A_2U_m) = I_1 + I_2, \end{aligned} \tag{4.35}$$

where

$$I_1 = (B_1(u+z, u+z), A_1u_m), \quad I_2 = (B_2(u+z, U+Z), A_2U_m).$$

To estimate I_1, we rewrite it as

$$\begin{aligned} I_1 &= (B_1(u, u), A_1u_m) + (B_1(u, z(\theta_t\omega)), A_1u_m) \\ &\quad + (B_1(z(\theta_t\omega), z(\theta_t\omega)), A_1u_m) + (B_1(z(\theta_t\omega), v), A_1u_m), \end{aligned} \tag{4.36}$$

where by (4.4) and (4.5), we calculate that

$$
\begin{aligned}
&|(B_1(u,u),A_1u_m)|\\
&\le |(B_1(P_mu,u),A_1u_m)|+|(B_1(u_m,u),A_1u_m)|\\
&\le \|P_mu\|_{L^\infty}\|u\|\ |A_1u_m|+\|u_m\|_{L^\infty}\|u\|\ |A_1u_m|\\
&\le c(1+\log\frac{\lambda_{m+1}}{\lambda_1})^{1/2}\|u\|^2\ |A_1u_m|+c\|u_m\|^{1/2}|A_1u_m|^{3/2}\|u\|\\
&\le \frac{1}{32}|A_1u_m|^2+c(1+\log\frac{\lambda_{m+1}}{\lambda_1})\|u\|^4+c|u|^2\|u\|^4,
\end{aligned}
\tag{4.37}
$$

$$
\begin{aligned}
&|(B_1(u,z(\theta_t\omega)),A_1u_m)|\\
&\le |(B_1(P_mu,z(\theta_t\omega)),A_1u_m)|+|(B_1(u_m,z(\theta_t\omega)),A_1u_m)|\\
&\le \|P_mu\|_{L^\infty}\|z(\theta_t\omega)\|\ |A_1u_m|+\|u_m\|_{L^\infty}\|z(\theta_t\omega)\|\ |A_1u_m|\\
&\le c(1+\log\frac{\lambda_{m+1}}{\lambda_1})^{1/2}\|u\|\ \|z(\theta_t\omega)\|\ |A_1u_m|+c|u_m|^{1/2}|A_1u_m|^{3/2}\|z(\theta_t\omega)\|^2\\
&\le \frac{1}{32}|A_1u_m|^2+c(1+\log\frac{\lambda_{m+1}}{\lambda_1})\|u\|^2\|z(\theta_t\omega)\|^2+c|u|^2\|z(\theta_t\omega)\|^4,
\end{aligned}
\tag{4.38}
$$

$$
|(B_1(z(\theta_t\omega),z(\theta_t\omega)),A_1u_m)|\le\frac{1}{32}|A_1u_m|^2+c\|z(\theta_t\omega)\|_{H^2}^2\|z(\theta_t\omega)\|^2, \tag{4.39}
$$

$$
|(B_1(z(\theta_t\omega),u),A_1u_m)|\le\frac{1}{32}|A_1u_m|^2+c\|z(\theta_t\omega)\|_{H^2}^2\|u\|^2. \tag{4.40}
$$

Then it follows from (4.35)–(4.40) that

$$
\begin{aligned}
I_1\le{}&\frac{1}{8}|A_1u_m|^2+c(1+\log\frac{\lambda_{m+1}}{\lambda_1})(\|u\|^2\|z(\theta_t\omega)\|^2+\|u\|^4)\\
&+c(|u|^2\|u\|^4+\|z(\theta_t\omega)\|_{H^2}^2\|u\|^2+|u|^2\|z(\theta_t\omega)\|^4\\
&+\|z(\theta_t\omega)\|_{H^2}^2\|z(\theta_t\omega)\|^2).
\end{aligned}
\tag{4.41}
$$

Then we estimate I_2 in (4.13), by writing it as

$$
\begin{aligned}
I_2={}&(B_2(u,Z(\theta_t\omega)),A_2U_m)+(B_2(u,U),A_2U_m)\\
&+(B_2(z(\theta_t\omega),Z(\theta_t\omega)),A_2U_m)+(B_2(z(\theta_t\omega),U),A_2U_m),
\end{aligned}
\tag{4.42}
$$

where

$$
\begin{aligned}
&|(B_2(u,U),A_2U_m)|\\
&\le |(B_2(P_mu,U),A_2U_m)|+|(B_2(u_m,U),A_2U_m)|\\
&\le \|P_mu\|_{L^\infty}\|U\|\ |A_2U_m|+\|u_m\|_{L^\infty}\|U\|\ |A_2U_m|\\
&\le c\|u\|(1+\log\frac{\lambda_{m+1}}{\lambda_1})^{1/2}\|U\|\ |A_2U_m|+c|u|^{1/2}|A_1u_m|^{1/2}\|U\|\ |A_2U_m|\\
&\le \frac{1}{32}|A_2U_m|^2+c(1+\log\frac{\lambda_{m+1}}{\lambda_1})\|u\|^2\|U\|^2+c|u|\ |A_1u_m|\ \|U\|^2\\
&\le \frac{1}{32}|A_2U_m|^2+c(1+\log\frac{\lambda_{m+1}}{\lambda_1})\|u\|^2\|U\|^2+\frac{1}{32}|A_1u_m|^2+c|u|^2\|U\|^4,
\end{aligned}
\tag{4.43}
$$

$$
\begin{aligned}
|(B_2(u,Z(\theta_t\omega)),A_2U_m)|\le{}&\frac{1}{32}|A_2U_m|^2+c(1+\log\frac{\lambda_{m+1}}{\lambda_1})\|u\|^2\|Z(\theta_t\omega)\|^2\\
&+\frac{1}{32}|A_1u_m|^2+c|u|^2\|Z(\theta_t\omega)\|^4,
\end{aligned}
\tag{4.44}
$$

$$|(B_2(z(\theta_t\omega), Z(\theta_t\omega)), A_2U_m)| \le \frac{1}{32}|A_2U_m|^2 + c\|z(\theta_t\omega)\|_{H^2}^2\|Z(\theta_t\omega)\|^2, \tag{4.45}$$

$$|(B_2(z(\theta_t\omega), U), A_2U_m)| \le \frac{1}{32}|A_2U_m|^2 + c\|z(\theta_t\omega)\|_{H^2}^2\|U\|^2. \tag{4.46}$$

Then it follows from (4.42)–(4.46) that

$$\begin{aligned} I_2 \le{}& \frac{1}{8}|A_2U_m|^2 + \frac{1}{8}|A_1u_m|^2 + c(1+\log\frac{\lambda_{m+1}}{\lambda_1})\big(\|u\|^2\|U\|^2 \\ &+ \|u\|^2\|Z(\theta_t\omega)\|^2\big) + c(|u|^2\|U\|^4 + |u|^2\|Z(\theta_t\omega)\|^4 + c\|z(\theta_t\omega)\|_{H^2}^2\|U\|^2 \\ &+ \|z(\theta_t\omega)\|_{H^2}^2\|Z(\theta_t\omega)\|^2). \end{aligned} \tag{4.47}$$

Then we incorporate (4.34), (4.35), (4.41) and (4.47) into (4.33) to give

$$\begin{aligned} \frac{d}{dt}((\nu+\kappa)\|u_m\|^2 + \gamma\|U_m\|^2) \le{}& -\lambda_{m+1}((\nu+\kappa)\|u_m\|^2 + \gamma\|U_m\|^2) \\ &+ (1+\log\frac{\lambda_{m+1}}{\lambda_1})P(t,\omega) + Q(t,\omega), \end{aligned} \tag{4.48}$$

where

$$\begin{aligned} P(t,\omega) ={}& c(\|u\|^4 + \|u\|^2\|z(\theta_t\omega)\|^2 + \|u\|^2\|U\|^2 + \|u\|^2\|Z(\theta_t\omega)\|^2), \\ Q(t,\omega) ={}& c(\|\mathbf{U}\|^2 + |u|^2\|u\|^4 + |u|^2\|z(\theta_t\omega)\|^4 + |u|^2\|U\|^4 + \|u\|^2\|Z(\theta_t\omega)\|^4 \\ &+ \|z(\theta_t\omega)\|_{H^2}^2\|U\|^2 + \|z(\theta_t\omega)\|_{H^2}^2\|Z(\theta_t\omega)\|^2 + \|z(\theta_t\omega)\|_{H^2}^2\|z(\theta_t\omega)\|^2 \\ &+ \|z(\theta_t\omega)\|_{H^2}^2\|u\|^2 + |Z(\theta_t\omega)|^2 + \|z(\theta_t\omega)\|^2 + \|Z(\theta_t\omega)\|^2 + |F|^2). \end{aligned}$$

Multiplying (4.48) by $e^{\lambda_{m+1}t}$ and then integrating over the interval $[t, t+1]$, we infer that

$$\begin{aligned} &\|\mathbf{U}_m(t+1,\omega,\mathbf{V}_0(\omega)-\mathbf{Z}(\omega))\|^2 \\ &\le c(1+\log\frac{\lambda_{m+1}}{\lambda_1})e^{-\lambda_{m+1}(t+1)}\int_t^{t+1} e^{\lambda_{m+1}s}P(s,\omega)ds \\ &\quad + e^{-\lambda_{m+1}(t+1)}\int_t^{t+1} e^{\lambda_{m+1}s}Q(s,\omega)ds \\ &\quad + e^{-\lambda_{m+1}(t+1)}\int_t^{t+1} e^{\lambda_{m+1}s}\|\mathbf{U}_m(s,\omega,\mathbf{V}_0(\omega)-\mathbf{Z}(\omega))\|^2ds. \end{aligned} \tag{4.49}$$

According to Lemma 4.3, there exists $T = T(D,\omega) > 0$ such that for all $t \ge T$,

$$\begin{aligned} &\int_t^{t+1} e^{\lambda_{m+1}s}P(s,\theta_{-t-1}\omega)ds \\ &\le c\int_t^{t+1} e^{\lambda_{m+1}s}\Big(R_2(\omega)^4 + R_2(\omega)^2\|z(\theta_{s-t-1}\omega)\|^2 \\ &\quad + R_2(\omega)^4 + R_2(\omega)^2\|Z(\theta_{s-t-1}\omega)\|^2\Big)ds \\ &\le \frac{c}{\lambda_{m+1}}e^{\lambda_{m+1}(t+1)}(2R_2(\omega)^4 + R_2(\omega)^2\sup_{-1\le s\le 0}\|z(\theta_s\omega)\|^2 \\ &\quad + R_2(\omega)^2\sup_{-1\le s\le 0}\|Z(\theta_s\omega)\|^2) \\ &:= \frac{\hat{R}(\omega)}{\lambda_{m+1}}e^{\lambda_{m+1}(t+1)}, \end{aligned} \tag{4.50}$$

where
$$\hat{R}(\omega)=c(2R_2(\omega)^4+R_2(\omega)^2\sup_{-1\le s\le 0}\|z(\theta_s\omega)\|^2+R_2(\omega)^2\sup_{-1\le s\le 0}\|Z(\theta_s\omega)\|^2)$$
is independent of λ_{m+1}. By a similar calculation as (4.50), we find that there exists a random variable $\hat{\hat{R}}(\omega)$ such that for all $t\ge T$,
$$\int_t^{t+1} e^{\lambda_{m+1}s}Q(t,\theta_{-t-1}\omega)ds\le \frac{\hat{\hat{R}}(\omega)}{\lambda_{m+1}}e^{\lambda_{m+1}(t+1)}, \tag{4.51}$$
and
$$\int_t^{t+1} e^{\lambda_{m+1}s}\|\mathbf{U}_m(s,\theta_{-t-1}\omega)\|^2ds\le \frac{R_2(\omega)^2}{\lambda_{m+1}}e^{\lambda_{m+1}(t+1)}, \tag{4.52}$$
where $R_2(\omega)$ is in Lemma 4.3. Then (4.49), together with (4.50)–(4.52), implies that for all $t\ge T$,
$$\begin{aligned}&\|\mathbf{U}_m(t+1,\theta_{-t-1}\omega,\mathbf{V}_0(\theta_{-t-1}\omega)-\mathbf{Z}(\theta_{-t-1}\omega))\|^2\\ &\le \frac{1}{\lambda_{m+1}}(1+\log\frac{\lambda_{m+1}}{\lambda_1})\hat{R}(\omega)+\frac{1}{\lambda_{m+1}}(\hat{\hat{R}}(\omega)+R_2(\omega)^2)\to 0,\end{aligned}$$
as $m\to+\infty$. As a consequence, for every $\varepsilon>0$ and $D\in\mathcal{D}$, there exists an integer N and positive constant $K=K(\omega)$ such that for all $t\ge T$ and $m\ge N$,
$$\begin{gathered}\|(I-P_m)\mathbf{U}(t,\theta_{-t}\omega,\mathbf{V}_0(\theta_{-t}\omega)-\mathbf{Z}(\theta_{-t}\omega))\|\le\varepsilon,\\ \|P_m\mathbf{U}(t,\theta_{-t}\omega,\mathbf{V}_0(\theta_{-t}\omega)-\mathbf{Z}(\theta_{-t}\omega))\|\le K.\end{gathered}$$
This concludes the proof. □

Lemma 4.5. *Assume that the expectation $\mathbb{E}M<0$, where M is in* (3.4)*. Then the RDS φ generated by the solution of stochastic micropolar fluid flows* (1.1) *is omega-limit compact in $\mathcal{V}$; i.e., for every $\varepsilon>0$ and an arbitrary $D\in\mathcal{D}$, there is an $T=T(\varepsilon,D,\omega)>0$ such that for $\mathbb{P}$-a.e. $\omega\in\Omega$,*
$$k\Big(\cup_{t\ge T}\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)\Big)\le\varepsilon,$$
where $\mathcal{D}$ is the collection of tempered random subsets of $\mathcal{H}$.

Proof. By Lemma 4.4, there exist constants $\hat{K}(\omega)$ and $T=T(\varepsilon,D,\omega)$ and $N_1\in\mathbb{N}$ such that for all $t\ge T$ and $m\ge N_1$, there hold $\|P_m\mathbf{U}(t,\theta_{-t}\omega,\mathbf{V}_0(\theta_{-t}\omega)-\mathbf{Z}(\theta_{-t}\omega))\|\le\hat{K}(\omega)$ and
$$\|(I-P_m)\mathbf{U}(t,\theta_{-t}\omega,\mathbf{V}_0(\theta_{-t}\omega)-\mathbf{Z}(\theta_{-t}\omega))\|\le\frac{\varepsilon}{2}. \tag{4.53}$$
Note that $\|P_m\mathbf{Z}(\omega)\|\le\|\mathbf{Z}(\omega)\|$, and
$$\|(I-P_m)\mathbf{Z}(\omega)\|\le\frac{1}{\lambda_{m+1}}\|\mathbf{Z}(\omega)\|_{H^2}\to 0$$
as $m\to\infty$; and then there exists $N_2\in\mathbb{N}$ such that for every $m\ge N_2$,
$$\|P_m\mathbf{Z}(\omega)\|\le\|\mathbf{Z}(\omega)\|,\quad \|(I-P_m)\mathbf{Z}(\omega)\|\le\frac{\varepsilon}{2}. \tag{4.54}$$
Put $N=\max\{N_1,N_2\}$, by (4.53) and (4.54) we find that there exist $K(\omega)=\hat{K}(\omega)+\|\mathbf{Z}(\omega)\|^2$ and $T=T(\varepsilon,D,\omega)>0$ such that for all $t\ge T$,
$$\|P_N\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)\|\le K(\omega), \tag{4.55}$$
$$\|(I-P_N)\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)\|\le\varepsilon. \tag{4.56}$$

That is to say the RDS φ satisfies the flattening conditions in $\mathcal{V}$; see [21]. By utilizing the additive property of Kuratowski measure of non-compactness; see, [22, Lemma 2.5 (ii)], it follows from (4.55) and (4.56) that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\begin{aligned} k\Big(\cup_{t\geq T} \varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)\Big) &\leq k\Big(P_N\Big(\cup_{t\geq T} \varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)\Big)\Big) \\ &\quad + k\Big((I-P_N)\Big(\cup_{t\geq T} \varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)\Big)\Big) \\ &\leq 0 + k(B_{\mathcal{V}}(0,\varepsilon)) = 2\varepsilon, \end{aligned}$$

where $B_{\mathcal{V}}(0,\varepsilon)$ is the ε- neighborhood at centre 0 in $\mathcal{V}$. This completes the proof. □

Proof of Theorem 3.3. According to Lemma 4.5, by the embedding relation $\mathcal{V} \hookrightarrow \mathcal{H}$, we can show that φ is omega-limit compact in $\mathcal{H}$. But by Lemma 4.2, φ possesses a tempered random absorbing set $\hat{D} \in \mathcal{D}$ in $\mathcal{H}$. Thus by the first conclusion in Theorem 2.3, we know that φ admits an $(\mathcal{H},\mathcal{H})$-random attractor $\mathcal{A}_{\mathcal{H}}$. From Lemma 4.5, φ is $(\mathcal{H},\mathcal{V})$-omega-limit compact. Then the second conclusion in Theorem 2.3 implies that the existence of $(\mathcal{H},\mathcal{V})$-random attractor $\mathcal{A}_{\mathcal{V}}$. Furthermore, $\mathcal{A}_{\mathcal{V}} = \mathcal{A}_{\mathcal{H}}$. □

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