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# WELL-POSEDNESS AND EXPONENTIAL STABILITY FOR A LINEAR DAMPED TIMOSHENKO SYSTEM WITH SECOND SOUND AND INTERNAL DISTRIBUTED DELAY

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Abstract. In this article we consider one-dimensional linear thermoelastic system of Timoshenko type with linear frictional damping and a distributed delay acting on the displacement equation. The heat flux of the system is governed by Cattaneo's law. Under suitable assumption on the weight of the delay and that of frictional damping, we establish the well-posedness result and prove that the system is exponentially stable regardless of the speeds of wave propagation.

## 1. Introduction

It is well-known that the model using classic Fourier's law of heat conduction (which states that the heat flux is proportional to the gradient of temperature) predicts the physical paradox of infinite speed of heat propagation. In other words, any thermal disturbance at one point has an instantaneous effect elsewhere in the body. To overcome this physical paradox but still keeping the essentials of heat conduction process, many theories have merged. One of which is the advent of the second sound effects observed experimentally in materials at low temperature. This theory suggests replacing the classic Fourier's law $\beta q + \theta_x = 0$ by a modified law of heat conduction called Cattaneo's law $\tau q_t + \beta q + \theta_x = 0$. Consequently, heat is transported by a wave propagation process instead of the usual diffusion thereby eliminating the physical paradox of infinite speed of heat propagation. We refer the reader to [5, 6, 7, 11, 15, 29, 35] and the references therein, for more discussion on Cattaneo's law and thermoelasticity with second sound.

In this article we are concerned with the following thermoelastic system of Timoshenko type with a linear frictional damping and an internal distributed delay acting on the transverse displacement, where the heat flux is given by Cattaneo's

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law:

$$\begin{aligned}
\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\varphi_t(x,t-s)ds = 0 \quad \text{in } (0,1)\times(0,\infty),\\
\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0 \quad \text{in } (0,1)\times(0,\infty),\\
\rho_3\theta_t + q_x + \delta\psi_{tx} = 0 \quad \text{in } (0,1)\times(0,\infty),\\
\tau q_t + \beta q + \theta_x = 0 \quad \text{in } (0,1)\times(0,\infty),\\
\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \theta(x,0) = \theta_0(x) \quad \text{in } (0,1),\\
\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad q(x,0) = q_0(x) \quad \text{in } (0,1),\\
\varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = \theta(0,t) = \theta(1,t) = 0 \quad \text{in } (0,\infty),\\
\varphi_t(x,-t) = f_0(x,t) \quad \text{in } (0,1)\times(0,\tau_2).
\end{aligned} \tag{1.1}$$

Here $\varphi = \varphi(x,t)$ is the transverse displacement of the beam, $\psi = \psi(x,t)$ is the rotation angle, $\theta = \theta(x,t)$ is the difference temperature, $q = q(x,t)$ is the heat flux, $\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0$ are initial data, and $f_0$ is history function. The coefficients, $\rho_1, \rho_2, b, \kappa, \delta, \beta, \mu_1$ are positive constants, and $\mu_2\colon [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function, where $\tau_1$ and $\tau_2$ are two real numbers satisfying $0 \le \tau_1 < \tau_2$. The parameter $\tau > 0$ is the relaxation time describing the time lag in the response of the heat flux to a gradient in the temperature. The purpose of this paper is to study the well-posedness and the asymptotic behavior of the solution of (1.1) regardless of the speeds of wave propagation.

Delay effects arise in many applications and practical problems (see for instance [3, 31]) and it has attracted lots of attentions from researchers in diverse fields of human endeavor such as mathematics, engineering, science, and economics. It has been established that voluntary introduction of delay can benefit the control (see [1]). On the other hand, it may not only destabilize a system which is asymptotically stable in the absence of delay but may also lead to ill-posedness (see [8, 30] and the references therein). Therefore, the issue of well-posedness and the stability result of systems with delay are of practical and theoretical importance.

Nicaise and Pignotti [24] considered wave equation with linear frictional damping and internal distributed delay

$$u_{tt} - \Delta u + \mu_1 u_t + a(x)\int_{\tau_1}^{\tau_2} \mu_2(s)u_t(t-s)ds = 0$$

in $\Omega\times(0,\infty)$, with initial and mixed Dirichlet-Neumann boundary conditions and $a$ is a function chosen in an appropriate space. They established exponential stability of the solution under the assumption that

$$\|a\|_\infty \int_{\tau_1}^{\tau_2} \mu_2(s)ds < \mu_1. \tag{1.2}$$

The authors also obtained the same result when the distributed delay acted on the part of the boundary. Recently, Mustafa and Kafini [21] considered a thermoelastic system with internal distributed delay

$$\begin{gathered}
au_{tt} - du_{xx} + \beta\theta_x = 0\\
b\theta_t - \mu_1\theta_{xx} - \int_{\tau_1}^{\tau_2} \mu_2(s)\theta_{xx}(t-s)ds + \beta u_{xt} = 0
\end{gathered}$$

in $(0,L)\times(0,\infty)$, and proved that the damping effect via heat conduction is strong enough to exponentially stabilize the system provided (1.2) (with $a=1$) holds. See [22] for similar result concerning boundary distributed delay. Interested reader is referred to [2, 12, 13, 17, 19, 20, 23, 25, 26, 28, 32, 33], for more results concerning other types of delay (constant or time-varying delay).

In the absence of delay ($\mu_2 \equiv 0$), Messaoudi et al [16] considered (1.1) for both linear and nonlinear case and proved that the system is exponentially stable without any restriction on the coefficients. Whereas, in the absence of both the frictional damping ($\mu_1 = 0$) and delay, Fernández Sare and Racke [9] proved that the system is no longer exponentially stable even in the presence of viscoelastic damping term of the form $\int_0^\infty g(s)\varphi_{xx}(t-s)ds$ in the second equation of (1.1). The results of [9] were generalized by Guesmia et al [10] to the case where $g$ does not converge exponentially to zero. On the other hand, if the infinite memory is considered in the first equation, then it was proved in [10] that the uniform stability (exponential, polynomial or others depending on the growth of $g$ at infinity) holds without any restriction on the parameters. Very recently, Santos et al [34] improved the result of [9] (for $g=0$) by introducing a new stability number

$$\chi = \Big(\tau - \frac{\rho_1}{\kappa\rho_3}\Big)\Big(\rho_2 - \frac{b\rho_1}{\kappa}\Big) - \frac{\tau\rho_1\delta^2}{\kappa\rho_3}$$

and proved that the corresponding semigroup associated to the system is exponentially stable if and only if $\chi = 0$, otherwise there is a lack of exponential stability. In addition to the absence of frictional damping and delay, if $\tau = 0$ (classical thermoelasticity) then Rivera and Racke [18] proved that (1.1) is exponentially stable if the propagation speeds are equal i.e. $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$, otherwise a weaker rate of decay is obtained for strong solutions.

In this present work we consider (1.1) and prove the well-posedness and establish exponential stability results regardless of the speeds of wave propagation. Our work extends the stability results in [16] to Timoshenko systems with distributed delay acting on the displacement equation.

The rest of this article is organized as follows. In section 2, we introduce some transformations and state the assumption needed in our work. In section 3, we use the semigroup method to prove the well-posedness of our problem. In the last section, we state and prove our stability result. We use $c$ throughout this paper to denote a generic positive constant.

## 2. Preliminaries

As in [24], we introduce the new variable

$$z(x,\rho,s,t) = \varphi_t(x,t-\rho s) \quad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty). \tag{2.1}$$

It is straight forward to check that $z$ satisfies

$$sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0 \quad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Consequently, problem (1.1) is equivalent to

$$\begin{gathered}
\rho_1\varphi_{tt} - \kappa(\varphi_x+\psi)_x + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)ds = 0 \quad \text{in } (0,1)\times(0,\infty),\\
\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x+\psi) + \delta\theta_x = 0 \quad \text{in } (0,1)\times(0,\infty),\\
\rho_3\theta_t + q_x + \delta\psi_{tx} = 0 \quad \text{in } (0,1)\times(0,\infty),
\end{gathered}$$

$$
\begin{gathered}
\tau q_t+\beta q+\theta_x=0 \quad \text{in } (0,1)\times(0,\infty),\\
sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0 \quad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty),\\
z(x,0,s,t)=\varphi_t \quad \text{in } (0,1)\times(\tau_1,\tau_2)\times(0,\infty)\\
\varphi(x,0)=\varphi_0(x),\quad \varphi_t(x,0)=\varphi_1(x),\quad \theta(x,0)=\theta_0(x) \quad \text{in } (0,1),\\
\psi(x,0)=\psi_0(x),\quad \psi_t(x,0)=\psi_1(x),\quad q(x,0)=q_0(x) \quad \text{in } (0,1),\\
\varphi(0,t)=\varphi(1,t)=\psi_x(0,t)=\psi_x(1,t)=\theta(0,t)=\theta(1,t)=0 \quad \text{in } (0,\infty),\\
z(x,\rho,s,0)=f_0(x,\rho s) \quad \text{in } (0,1)\times(0,1)\times(0,\tau_2).
\end{gathered} \tag{2.2}
$$

Concerning the weight of the delay, we assume that

$$
\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds<\mu_1 \tag{2.3}
$$

and establish the well-posedness as well as the exponential stability results of the energy $E$, defined by

$$
\begin{aligned}
E(t)=&\frac{1}{2}\int_0^1\Big[\rho_1\varphi_t^2+\rho_2\psi_t^2+b\psi_x^2+\kappa(\varphi_x+\psi)^2+\rho_3\theta^2+\tau q^2\Big]\,dx\\
&+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,\rho,s,t)\,ds\,d\rho\,dx.
\end{aligned} \tag{2.4}
$$

Meanwhile, using $(2.2)_2$, $(2.2)_4$, and the boundary conditions, we conclude that

$$
\frac{d^2}{dt^2}\int_0^1\psi(x,t)+\frac{\kappa}{\rho_2}\int_0^1\psi(x,t)=0 \quad\text{and}\quad \frac{d}{dt}\int_0^1 q(x,t)+\frac{\beta}{\tau}\int_0^1 q(x,t)=0. \tag{2.5}
$$

So, by solving (2.5) and using the initial data of $\psi$ and $q$, we obtain

$$
\int_0^1\psi(x,t)\,dx=\Big(\int_0^1\psi_0(x)\,dx\Big)\cos\sqrt{\frac{\kappa}{\rho_2}}t+\sqrt{\frac{\rho_2}{\kappa}}\Big(\int_0^1\psi_1(x)\,dx\Big)\sin\sqrt{\frac{\kappa}{\rho_2}}t
$$

and

$$
\int_0^1 q(x,t)\,dx=\Big(\int_0^1 q_0(x)\,dx\Big)\exp(-\frac{\beta}{\tau}t).
$$

Consequently, if we let

$$
\begin{gathered}
\overline{\psi}(x,t)=\psi(x,t)-\Big(\int_0^1\psi_0(x)\,dx\Big)\cos\sqrt{\frac{\kappa}{\rho_2}}t-\sqrt{\frac{\rho_2}{\kappa}}\Big(\int_0^1\psi_1(x)\,dx\Big)\sin\sqrt{\frac{\kappa}{\rho_2}}t,\\
\overline{q}(x,t)=q(x,t)-\Big(\int_0^1 q_0(x)\,dx\Big)\exp(-\frac{\beta}{\tau}t).
\end{gathered}
$$

Then it follows that

$$
\int_0^1\overline{\psi}(x,t)\,dx=0 \quad\text{and}\quad \int_0^1\overline{q}(x,t)\,dx=0,\quad \forall t\ge 0.
$$

Therefore, the use of Poincaré's inequality for $\overline{\psi}$ is justified. In addition, simple substitution shows that $(\varphi,\overline{\psi},\theta,\overline{q},z)$ satisfies system (2.2) with initial data for $\overline{\psi}$ and $\overline{q}$ given as

$$
\begin{gathered}
\overline{\psi}_0(x)=\psi_0(x)-\int_0^1\psi_0(x)\,dx,\quad \overline{\psi}_1(x)=\psi_1(x)-\int_0^1\psi_1(x)\,dx,\\
\overline{q}_0(x)=q_0(x)-\int_0^1 q_0(x)\,dx
\end{gathered}
$$

instead of $\psi_0(x), \psi_1(x)$ for $\psi$, and $q_0$ for $q$, respectively. Henceforth, we work with $\overline{\psi}$ and $\overline{q}$ instead of $\psi$ and $q$ but write $\psi$ and $q$ for simplicity of notation.

## 3. Well-posedness of the problem

In this section, we prove the existence and uniqueness of solutions for (2.2) using semigroup theory. Introducing the vector function $\Phi = (\varphi, u, \psi, v, \theta, q, z)^T$, where $u = \varphi_t$ and $v = \psi_t$, system (2.2) can be written as

$$\begin{gathered}\Phi'(t) + \mathcal{A}\Phi(t) = 0, \quad t > 0, \\ \Phi(0) = \Phi_0 = \big(\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0, f_0\big)^T,\end{gathered} \tag{3.1}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\Phi = \begin{pmatrix} -u \\ -\frac{\kappa}{\rho_1}(\varphi_x + \psi)_x + \frac{\mu_1}{\rho_1}u + \frac{1}{\rho_1}\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s)\, ds \\ -v \\ -\frac{b}{\rho_2}\psi_{xx} + \frac{\kappa}{\rho_2}(\varphi_x + \psi) + \frac{\delta}{\rho_2}\theta_x \\ \frac{1}{\rho_3}q_x + \frac{\delta}{\rho_3}v_x \\ \frac{\beta}{\tau}q + \frac{1}{\tau}\theta_x \\ \frac{1}{s}z_\rho(x,\rho,s) \end{pmatrix}.$$

We consider the following spaces

$$\begin{gathered} L^2_\star(0,1) = \{w \in L^2(0,1) : \int_0^1 w(s)\, ds = 0\}, \quad H^1_\star(0,1) = H^1(0,1) \cap L^2_\star(0,1), \\ H^2_\star(0,1) = \{w \in H^2(0,1) : w_x(0) = w_x(1) = 0\}. \end{gathered}$$

Let

$$\begin{aligned} \mathcal{H} := {} & H^1_0(0,1) \times L^2(0,1) \times H^1_\star(0,1) \times L^2_\star(0,1) \times L^2(0,1) \\ & \times L^2_\star(0,1) \times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \end{aligned}$$

be the Hilbert space equipped with the inner product

$$\begin{aligned} (\Phi, \tilde{\Phi})_{\mathcal{H}} = {} & \kappa \int_0^1 (\varphi_x + \psi)(\tilde{\varphi}_x + \tilde{\psi})\, dx + \rho_1 \int_0^1 u\tilde{u} dx + b \int_0^1 \psi_x \tilde{\psi}_x\, dx \\ & + \rho_2 \int_0^1 v\tilde{v}\, dx + \rho_3 \int_0^1 \theta\tilde{\theta}\, dx + \tau \int_0^1 q\tilde{q}\, dx \\ & + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z(x,\rho,s)\tilde{z}(x,\rho,s)\, ds\, d\rho\, dx. \end{aligned}$$

The domain of $\mathcal{A}$ is

$$\begin{aligned} D(\mathcal{A}) = \Big\{ & \Phi \in \mathcal{H} : \varphi \in H^2(0,1) \cap H^1_0(0,1),\ \psi \in H^2_\star(0,1) \cap H^1_\star(0,1), \\ & u, \theta \in H^1_0(0,1),\ v, q \in H^1_\star(0,1),\ z(x,0,s) = u, \\ & z, z_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \Big\} \end{aligned}$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

We have the following existence and uniqueness result.

**Theorem 3.1.** *Let $\Phi_0 \in \mathcal{H}$, then there exists a unique solution $\Phi \in C(\mathbb{R}^+, \mathcal{H})$ of problem (3.1). Moreover, if $\Phi_0 \in D(\mathcal{A})$, then $\Phi \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

*Proof.* The result follows from Lumer-Phillips theorem provided we prove that $\mathcal{A}$ is a maximal monotone operator. In what follows, we prove that $\mathcal{A}$ is monotone. For any $\Phi \in D(\mathcal{A})$, and using the inner product, we obtain

$$\begin{aligned}
&(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} \\
&= \beta \int_0^1 q^2 dx + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s)\, ds\, dx \\
&+ (\mu_1 - \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds) \int_0^1 u^2 dx + \int_0^1 u \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s)\, ds\, dx.
\end{aligned} \tag{3.2}$$

Using Young's inequality, the last term in (3.2), we have

$$\begin{aligned}
&- \int_0^1 u \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s)\, ds\, dx \\
&\le \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_0^1 u^2\, dx + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s)\, ds\, dx.
\end{aligned} \tag{3.3}$$

Substituting (3.3) in (3.2) yields

$$(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} \ge \beta \int_0^1 q^2\, dx + \Big(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\Big) \int_0^1 u^2\, dx.$$

By (2.3), it follows that $(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} \ge 0$, which implies that $\mathcal{A}$ is monotone. Next, we prove that the operator $I + \mathcal{A}$ is surjective. Given $G = (g_1, g_2, g_3, g_4, g_5, g_6, g_7)^T \in \mathcal{H}$, we prove that there exists $\Phi \in D(\mathcal{A})$ satisfying

$$\Phi + \mathcal{A}\Phi = G; \tag{3.4}$$

that is,

$$\begin{gathered}
-u + \varphi = g_1 \in H_0^1(0,1) \\
-\kappa(\varphi_x + \psi)_x + (\rho_1 + \mu_1) u + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s) ds = \rho_1 g_2 \in L^2(0,1) \\
-v + \psi = g_3 \in H_\star^1(0,1) \\
-b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x + \rho_2 v = \rho_2 g_4 \in L_\star^2(0,1) \\
q_x + \delta v_x + \rho_3 \theta = \rho_3 g_5 \in L^2(0,1) \\
(\beta + \tau) q + \theta_x = \tau g_6 \in L_\star^2(0,1) \\
z_\rho(x,\rho,s) + s z(x,\rho,s) = s g_7(x,\rho,s) \in L^2((0,1) \times (0,1) \times (\tau_1, \tau_2)).
\end{gathered} \tag{3.5}$$

We note that the last equation in (3.5) with $z(x,0,s) = u$, has a unique solution

$$z(x,\rho,s) = e^{-s\rho} u + s e^{-s\rho} \int_0^\rho e^{s\tau} g_7(x,\tau,s)\, d\tau. \tag{3.6}$$

From the sixth equation in (3.5), we define

$$\theta = \tau \int_0^x g_6\, dx - (\beta + \tau) \int_0^x q\, dx, \tag{3.7}$$

then $\theta(0) = \theta(1) = 0$. Inserting $u = \varphi - g_1$, $v = \psi - g_3$, and (3.7) in $(3.5)_2$, $(3.5)_4$, and $(3.5)_5$, we obtain

$$\begin{gathered}
-\kappa(\varphi_x + \psi)_x + \mu\varphi = h_1 \in L^2(0,1) \\
-b\psi_{xx} + \kappa(\varphi_x + \psi) + \rho_2\psi - (\beta+\tau)\delta q = h_2 \in L^2_\star(0,1) \\
-q_x + (\beta+\tau)\rho_3 \int_0^x q(y)dy - \delta\psi_x = h_3 \in L^2(0,1),
\end{gathered} \tag{3.8}$$

where

$$\begin{gathered}
\mu = \mu_1 + \rho_1 + \int_{\tau_1}^{\tau_2} \mu_2(s)e^{-s} ds \\
h_1 = \mu g_1 + \rho_1 g_2 - \int_{\tau_1}^{\tau_2} s\mu_2(s)e^{-s} \int_0^1 e^{s\tau} g_5(x,\tau,s) d\tau ds \\
h_2 = \rho_2(g_3 + g_4) - \tau\delta g_6 \\
h_3 = -\delta g_{3x} - \rho_3\Big(g_5 - \tau\int_0^x g_6(y)dy\Big).
\end{gathered} \tag{3.9}$$

To solve (3.8) we consider

$$B\big((\varphi,\psi,q),(\tilde\varphi,\tilde\psi,\tilde q)\big) = F\big(\tilde\varphi,\tilde\psi,\tilde q\big), \tag{3.10}$$

where $B : \big[H^1_0(0,1) \times H^1_\star(0,1) \times L^2_\star(0,1)\big]^2 \to \mathbb{R}$ is the bilinear form

$$\begin{aligned}
B\big((\varphi,\psi,q),(\tilde\varphi,\tilde\psi,\tilde q)\big) &= \kappa\int_0^1 (\varphi_x+\psi)(\tilde\varphi_x+\tilde\psi)\,dx + (\beta+\tau)\int_0^1 q\tilde q\,dx \\
&\quad + b\int_0^1 \psi_x\tilde\psi_x\,dx + \rho_2\int_0^1 \psi\tilde\psi\,dx - \delta(\beta+\tau)\int_0^1 q\tilde\psi\,dx \\
&\quad + \mu\int_0^1 \varphi\tilde\varphi\,dx + \delta(\beta+\tau)\int_0^1 \psi\tilde q\,dx \\
&\quad + \rho_3(\beta+\tau)^2\int_0^1 \Big(\int_0^x q(y)dy\int_0^x \tilde q(y)dy\Big)\,dx
\end{aligned}$$

and $F : \big[H^1_0(0,1) \times H^1_\star(0,1) \times L^2_\star(0,1)\big] \to \mathbb{R}$ is the linear form

$$F(\tilde\varphi,\tilde\psi,\tilde q) = \int_0^1 h_1\tilde\varphi\,dx + \int_0^1 h_2\tilde\psi\,dx + \int_0^1 h_3\int_0^x \tilde q(y)\,dy\,dx.$$

Now, for $V = H^1_0(0,1) \times H^1_\star(0,1) \times L^2_\star(0,1)$ equipped with the norm

$$\|(\varphi,\psi,q)\|_V = \|(\varphi_x+\psi)\|_2^2 + \|\varphi\|_2^2 + \|\psi_x\|_2^2 + \|q\|_2^2,$$

one can easily see that $B$ and $F$ are bounded. Furthermore, using integration by parts, we obtain

$$\begin{aligned}
&B\big((\varphi,\psi,q),(\varphi,\psi,q)\big) \\
&= \kappa\int_0^1 (\varphi_x+\psi)^2\,dx + (\beta+\tau)\int_0^1 q^2\,dx + b\int_0^1 \psi_x^2\,dx \\
&\quad + \rho_2\int_0^1 \psi^2\,dx + \mu\int_0^1 \varphi^2\,dx + \rho_3(\beta+\tau)^2\int_0^1 \Big(\int_0^x q(y)\,dy\Big)^2\,dx \\
&\ge c\|(\varphi,\psi,q)\|_V^2.
\end{aligned}$$

Thus $B$ is coercive. Consequently, by Lax-Milgram Lemma, system (3.8) has a unique solution

$$\varphi \in H_0^1(0,1), \quad \psi \in H_\star^1(0,1), \quad q \in L_\star^2(0,1).$$

Substituting $\varphi$, $\psi$, and $q$ in $(3.5)_1$, $(3.5)_3$, and $(3.5)_6$, respectively, we obtain

$$u \in H_0^1(0,1), \quad v \in H_\star^1(0,1), \quad \theta \in H_0^1(0,1).$$

Similarly, inserting $u$ in (3.6) and bearing in mind $(3.6)_7$, we obtain

$$z, z_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)).$$

Now, if $(\tilde{\varphi},\tilde{q}) \equiv (0,0) \in H_0^1(0,1)\times L_\star^2(0,1)$, then (3.10) reduces to

$$\begin{aligned} &\kappa\int_0^1(\varphi_x+\psi)\tilde{\psi}\,dx + b\int_0^1\psi_x\tilde{\psi}_x\,dx + \rho_2\int_0^1\psi\tilde{\psi}\,dx - \delta(\beta+\tau)\int_0^1 q\tilde{\psi}\,dx \\ &= \int_0^1 h_2\tilde{\psi}\,dx, \quad \forall \tilde{\psi}\in H_\star^1(0,1), \end{aligned} \tag{3.11}$$

which implies

$$-b\psi_{xx} = -(\kappa+\rho_2)\psi - \kappa\varphi_x + (\beta+\tau)\delta q + h_2 \in L^2(0,1). \tag{3.12}$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$\psi \in H^2(0,1)\cap H_\star^1(0,1).$$

Moreover, (3.11) is also true for any $\phi \in C^1([0,1]) \subset H_\star^1(0,1)$. Hence, we have

$$b\int_0^1\psi_x\phi_x dx + \int_0^1\Big(\kappa(\varphi_x+\psi)+\rho_2\psi - \delta(\beta+\tau)q - h_2\Big)\phi dx = 0$$

for all $\phi\in C^1([0,1])$. Thus, using integration by parts and bearing in mind (3.12), we obtain

$$\psi_x(1)\phi(1) - \psi_x(0)\phi(0) = 0, \quad \forall\phi\in C^1([0,1]).$$

Therefore, $\psi_x(0)=\psi_x(1)=0$. Consequently, we obtain

$$\psi \in H_\star^2(0,1)\cap H_\star^1(0,1).$$

Similarly, we obtain

$$\begin{gathered} -\kappa\varphi_{xx} = -\mu\varphi - \kappa\psi_x + h_1 \ \in L^2(0,1) \\ -q_x = \delta\psi_x - (\beta+\tau)\rho_3\int_0^x q(y)\,dy + h_3 \ \in L^2(0,1); \end{gathered}$$

thus, we have

$$\varphi \in H^2(0,1)\cap H_0^1(0,1), \quad q \in H_\star^1(0,1).$$

Finally, the application of the regularity theory for the linear elliptic equations guarantees the existence of unique $\Phi \in D(\mathcal{A})$ such that (3.4) is satisfied. Consequently, $\mathcal{A}$ is a maximal operator. Hence, the result of Theorem 3.1 follows from Lumer-Phillips theorem (see [14, 27]). $\square$

## 4. Exponential Stability

In this section, we state and prove our stability result for the energy of the solution of system (2.2), using the multiplier technique. To achieve our goal, we need the following lemmas.

**Lemma 4.1.** *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (2.2) and assume (2.3) holds. Then the energy functional, defined by (2.4) satisfies*

$$E'(t) \leq -m_0 \int_0^1 \varphi_t^2 dx - \beta \int_0^1 q^2 dx \leq 0, \quad \forall t \geq 0, \tag{4.1}$$

*for some positive constant $m_0$.*

*Proof.* Multiplying $(2.2)_1$, $(2.2)_2$, $(2.2)_3$, and $(2.2)_4$ by $\varphi_t$, $\psi_t$, $\theta$, and $q$, respectively, and integrating over $(0,1)$, using integration by parts and the boundary conditions, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \Big[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + \kappa(\varphi_x+\psi)^2 + \rho_3\theta^2 + \tau q^2\Big]\, dx \\ &= -\mu_1 \int_0^1 \varphi_t^2\, dx - \beta\int_0^1 q^2 dx - \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\, ds\, dx. \end{aligned} \tag{4.2}$$

Multiplying $(2.2)_3$ by $|\mu_2(s)|z$, integrating the product over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, and recall that $z(x,0,s,t) = \varphi_t$, yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t)\, ds\, d\rho\, dx \\ &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds\, dx + \frac{1}{2}\int_0^1 \varphi_t^2 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\, dx. \end{aligned} \tag{4.3}$$

A combination of (4.2) and (4.3) gives

$$\begin{aligned} E'(t) &= -\Big(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds\Big) \int_0^1 \varphi_t^2 dx - \beta \int_0^1 q^2 dx \\ &\quad - \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\, ds\, dx - \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds\, dx. \end{aligned} \tag{4.4}$$

Meanwhile, using Young's inequality, we have

$$\begin{aligned} &-\int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\, ds\, dx \\ &\leq \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 \varphi_t^2 dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds\, dx. \end{aligned} \tag{4.5}$$

Simple substitution of (4.5) into (4.4) and using (2.3) give (4.1), which concludes the proof. □

**Lemma 4.2.** *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (2.2). Then the functional*

$$F_1(t) := \rho_2 \int_0^1 \psi\psi_t\, dx$$

*satisfies, the estimate*

$$F_1'(t) \le -\frac{b}{2}\int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + c\int_0^1 (\varphi_x+\psi)^2 dx + c\int_0^1 \theta^2 dx. \tag{4.6}$$

*Proof.* A simple differentiation of $F_1$, using $(2.2)_2$, gives

$$F_1'(t) = -b\int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + \delta\int_0^1 \theta\psi_x dx - \kappa\int_0^1 \psi(\varphi_x+\psi) dx.$$

Using Young's and Poincaré inequalities, estimate (4.6) is established. □

**Lemma 4.3.** *Let $(\varphi,\psi,\theta,q,z)$ be the solution of* (2.2). *Then the functional*

$$F_2(t) := -\frac{\rho_2\rho_3}{\delta}\int_0^1 \theta \int_0^x \psi_t(y)\,dy\,dx$$

*satisfies, for any $\varepsilon_1 > 0$, the estimate*

$$\begin{aligned} F_2'(t) \le &-\frac{\rho_2}{2}\int_0^1 \psi_t^2 dx + \varepsilon_1 \int_0^1 \psi_x^2 dx + c\int_0^1 q^2 dx \\ &+ c\big(1+\frac{1}{\varepsilon_1}\big)\int_0^1 \theta^2 dx + c\int_0^1 (\varphi_x+\psi)^2 dx. \end{aligned} \tag{4.7}$$

*Proof.* By differentiating $F_2$, then exploiting the second and the third equations in (2.2), and integrating by parts, we obtain

$$\begin{aligned} F_2'(t) = &-\rho_2\int_0^1 \psi_t^2 dx - \frac{\rho_2}{\delta}\int_0^1 q\psi_t dx - \frac{b\rho_3}{\delta}\int_0^1 \theta\psi_x dx \\ &+ \rho_3\int_0^1 \theta^2 dx + \frac{\rho_3\kappa}{\delta}\int_0^1 \theta\Big(\varphi + \int_0^x \psi(y)dy\Big)\,dx. \end{aligned}$$

Using Poincaré and Young's inequalities with $\varepsilon_1 > 0$, we obtain estimate (4.7). □

**Lemma 4.4.** *Let $(\varphi,\psi,\theta,q,z)$ be the solution of* (2.2). *Then the functional*

$$F_3(t) := \rho_1\int_0^1 \varphi_t\left(\varphi + \int_0^x \psi(y)dy\right)\,dx$$

*satisfies, for any $\varepsilon_2 > 0$, the estimate*

$$\begin{aligned} F_3'(t) \le &-\frac{\kappa}{2}\int_0^1 (\varphi_x+\psi)^2 dx + \varepsilon_2\int_0^1 \psi_t^2 dx \\ &+ c\big(1+\frac{1}{\varepsilon_2}\big)\int_0^1 \varphi_t^2 dx + c\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\,ds\,dx. \end{aligned} \tag{4.8}$$

*Proof.* Taking the derivative of $F_3$, using $(2.2)_2$ and integration by parts, we obtain

$$\begin{aligned} F_3'(t) = {}&\rho_1\int_0^1 \varphi_t\int_0^x \psi_t(y)\,dy\,dx - \int_0^1 \Big(\varphi + \int_0^x \psi(y)dy\Big)\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\,ds\,dx \\ &- \kappa\int_0^1 (\varphi_x+\psi)^2 dx + \rho_1\int_0^1 \varphi_t^2 dx - \mu_1\int_0^1 \varphi_t\Big(\varphi + \int_0^x \psi(y)dy\Big)\,dx. \end{aligned} \tag{4.9}$$

Now, we estimate the terms in the right hand side of (4.9) using Young's, Poincaré, and Cauchy-Schwarz inequalities

$$\rho_1\int_0^1 \varphi_t\int_0^x \psi_t(y)dy)\,dy \le \varepsilon_2\int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon_2}\int_0^1 \varphi_t^2 dx, \tag{4.10}$$

where $\varepsilon_2 > 0$, and

$$
\begin{aligned}
&-\int_0^1 \Big(\varphi + \int_0^x \psi(y)dy\Big) \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\, ds\, dx \\
&\le \frac{\kappa}{4}\int_0^1 \Big(\varphi + \int_0^x \psi(y)dy\Big)^2 dx + \frac{1}{\kappa}\int_0^1 \Big(\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) ds\Big)^2 dx \\
&\le \frac{\kappa}{4}\int_0^1 (\varphi_x + \psi)^2 dx + \frac{1}{\kappa} \underbrace{\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds}_{<\ \mu_1} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds\, dx.
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
-\mu_1 \int_0^1 \varphi_t \Big(\varphi + \int_0^x \psi(y)dy\Big)\, dx &\le \frac{\kappa}{4}\int_0^1 \Big(\varphi + \int_0^x \psi(y)dy\Big)^2 dx + \frac{\mu_1^2}{\kappa}\int_0^1 \varphi_t^2 dx \\
&\le \frac{\kappa}{4}\int_0^1 (\varphi_x + \psi)^2 dx + c\int_0^1 \varphi_t^2 dx,
\end{aligned}
\tag{4.12}
$$

Estimate (4.8) follows by substituting (4.10)–(4.12) into (4.9). ☐

**Lemma 4.5.** *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (2.2). Then the functional*

$$
F_4(t) := \tau\rho_3 \int_0^1 \theta \int_0^x q(y)\, dy\, dx
$$

*satisfies, for any $\varepsilon_2 > 0$, the estimate*

$$
F_4'(t) \le -\frac{\rho_3}{2}\int_0^1 \theta^2 dx + \varepsilon_2 \int_0^1 \psi_t^2 dx + c\big(1 + \frac{1}{\varepsilon_2}\big)\int_0^1 q^2 dx. \tag{4.13}
$$

*Proof.* Taking the derivative of $F_4$, using the third and the fourth equations in (2.2) and integration by parts, we obtain

$$
F_4'(t) = -\rho_3 \int_0^1 \theta^2 dx + \tau\beta\int_0^1 q^2 dx + \tau\delta\int_0^1 q\psi_t dx - \beta\rho_3 \int_0^1 \theta \int_0^x q(y)\, dy\, dx. \tag{4.14}
$$

We now use Cauchy-Schwarz and Young's inequalities with $\varepsilon_2 > 0$ on (4.14) to obtain (4.13). ☐

**Lemma 4.6.** *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (2.2). Then the functional*

$$
F_5(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x,\rho,s,t)\, ds\, d\rho\, dx
$$

*satisfies, for some positive constant $m_1$, the following estimate*

$$
\begin{aligned}
F_5'(t) \le & -m_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t) ds d\rho dx \\
& - m_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds\, dx + \mu_1 \int_0^1 \varphi_t^2 dx.
\end{aligned}
\tag{4.15}
$$

*Proof.* Differentiating $F_5$, and using the fifth equation in (2.2), we obtain

$$
\begin{aligned}
F_5'(t) &= -2\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)| z(x,\rho,s,t) z_\rho(x,\rho,s,t)\, ds\, d\rho\, dx \\
&= -\frac{d}{d\rho}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)| z^2(x,\rho,s,t)\, ds\, d\rho\, dx
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x,\rho,s,t)\, ds\, d\rho\, dx \\
&= - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| [e^{-s} z^2(x,1,s,t) - z^2(x,0,s,t)]\, ds\, dx \\
&\quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x,\rho,s,t)\, ds\, d\rho\, dx.
\end{aligned}
$$

Using the fact that $z(x,0,s,t) = \varphi_t$ and $e^{-s} \le e^{-s\rho} \le 1$, for all $\rho \in [0,1]$, we obtain

$$
\begin{aligned}
F_5'(t) \le & - \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s} |\mu_2(s)| z^2(x,1,s,t)\, ds\, dx + \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 \varphi_t^2 dx \\
& - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s} |\mu_2(s)| z^2(x,\rho,s,t)\, ds\, d\rho\, dx.
\end{aligned}
$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s} \le -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$. Finally, setting $m_1 = e^{-\tau_2}$ and recalling (2.3), we obtain (4.15). ☐

Next, we define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy functional $E$.

**Lemma 4.7.** *For $N$ sufficiently large, the functional defined by*

$$
\mathcal{L}(t) := NE(t) + F_1(t) + 4F_2(t) + N_1(F_3(t) + F_4(t)) + N_2 F_5(t), \tag{4.16}
$$

*where $N_1$ and $N_2$ are positive real numbers to be chosen appropriately later, satisfies*

$$
c_1 E(t) \le \mathcal{L}(t) \le c_2 E(t), \quad \forall t \ge 0, \tag{4.17}
$$

*for two positive constants $c_1$ and $c_2$.*

*Proof.* Let $\mathscr{L}(t) = F_1(t) + 4F_2(t) + N_1(F_3(t) + F_4(t)) + N_2 F_5(t)$

$$
\begin{aligned}
|\mathscr{L}(t)| \le & \rho_2 \int_0^1 |\psi \psi_t| dx + \rho_1 N_1 \int_0^1 \Big| \varphi_t \Big( \varphi + \int_0^x \psi(y) dy \Big) \Big| dx \\
& + \frac{4\rho_2\rho_3}{\delta} \int_0^1 \Big| \theta \int_0^x \psi_t(y) dy \Big| dx + \tau \rho_3 N_1 \int_0^1 \Big| \theta \int_0^x q(y) dy \Big| dx \\
& + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s) e^{-s\rho}| z^2(x,\rho,s,t)\, ds\, d\rho\, dx.
\end{aligned}
$$

Exploiting Young's, Poincaré, Cauchy-Schwarz inequalities, (2.4), and the fact that $e^{-s\rho} \le 1$ for all $\rho \in [0,1]$, we obtain

$$
\begin{aligned}
|\mathscr{L}(t)| &\le c \int_0^1 \Big( \varphi_t^2 + \psi_t^2 + \psi_x^2 + (\varphi_x + \psi)^2 + \theta^2 + q^2 \Big) dx \\
&\quad + c \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x,\rho,s,t)\, ds\, d\rho\, dx \\
&\le cE(t).
\end{aligned}
$$

Consequently, $|\mathcal{L}(t) - NE(t)| \le cE(t)$, which yields

$$
(N - c)E(t) \le \mathcal{L}(t) \le (N + c)E(t).
$$

Choosing $N$ large enough, we obtain estimate (4.17). ☐

Now, we are ready to state and prove the main result of this section.

**Theorem 4.8.** *Let $(\varphi, \psi, \theta, q, z)$ be the solution of* (2.2)*. Then the energy functional* (2.4) *satisfies,*

$$E(t) \leq k_0 e^{-k_1 t}, \quad \forall t \geq 0, \tag{4.18}$$

*where $k_0$ and $k_1$ are positive constants.*

*Proof.* By differentiating (4.16) and recalling (4.1), (4.6), (4.7), (4.8), (4.13), and (4.15), and letting $\varepsilon_1 = \frac{b}{16}$, we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\big[m_0 N - cN_1\big(1 + \frac{1}{\varepsilon_2}\big) - \mu_1 N_2\big] \int_0^1 \varphi_t^2 dx - \big[\rho_2 - 2\varepsilon_2 N_1\big] \int_0^1 \psi_t^2 dx \\
& - [\frac{\rho_3}{2} N_1 - c] \int_0^1 \theta^2 dx - m_1 N_2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x, \rho, s, t)\, ds\, d\rho\, dx \\
& - [\frac{\kappa}{2} N_1 - c] \int_0^1 (\varphi_x + \psi)^2 dx - \big[\beta N - c - cN_1\big(1 + \frac{1}{\varepsilon_2}\big)\big] \int_0^1 q^2 dx \\
& - \frac{b}{4} \int_0^1 \psi_x^2 dx - [m_1 N_2 - cN_1] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t)\, ds\, dx.
\end{aligned}$$

At this point, we set $\varepsilon_2 = \rho_2/(4N_1)$, then we choose $N_1$ large enough so that

$$\gamma_0 := \frac{\rho_3}{2} N_1 - c > 0 \quad \text{and} \quad \gamma_1 := \frac{\kappa}{2} N_1 - c > 0.$$

Once $N_1$ is fixed, we then choose $N_2$ large enough so that $m_1 N_2 - cN_1 > 0$.

Finally, we choose $N$ large enough such that (4.17) remains valid and

$$\gamma_2 := m_0 N - cN_1\Big(1 + \frac{1}{\varepsilon_2}\Big) - \mu_1 N_2 > 0, \quad \gamma_3 := \beta N - c - cN_1\Big(1 + \frac{1}{\varepsilon_2}\Big) > 0.$$

Thus, by letting $\gamma_4 := m_1 N_2$, we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\int_0^1 \Big(\gamma_2 \varphi_t^2 + \frac{\rho_2}{2} \psi_t^2 + \frac{b}{4} \psi_x^2 + \gamma_1 (\varphi_x + \psi)^2 + \gamma_0 \theta^2 dx + \gamma_3 q^2\Big) dx \\
& - \gamma_4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x, \rho, s, t)\, ds\, d\rho\, dx.
\end{aligned}$$

By (2.4), we obtain

$$\mathcal{L}'(t) \leq -\alpha_0 E(t), \quad \forall t \geq 0, \tag{4.19}$$

for some $\alpha_0 > 0$. A combination of (4.17) and (4.19) gives

$$\mathcal{L}'(t) \leq -k_1 \mathcal{L}(t), \quad \forall t \geq 0, \tag{4.20}$$

where $k_1 = \alpha_0 / c_2$. A simple integration of (4.20) over $(0, t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-k_1 t}, \quad \forall t \geq 0. \tag{4.21}$$

Finally, by combining (4.17) and (4.21) we obtain (4.18) with $k_0 = \frac{c_2 E(0)}{c_1}$, which completes the proof. □

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