Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 259, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR FIRST-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH TWO-POINT AND INTEGRAL BOUNDARY CONDITIONS

MISIR J. MARDANOV, YAGUB A. SHARIFOV, HABIB H. MOLAEI

Abstract. In this article, we study the existence of solutions to boundary-value problems for ordinary differential equations with two-point and integral boundary conditions. Existence and uniqueness results are obtained by using well known fixed point theorems. Some illustrative examples are also discussed.

1. Introduction

Many of the physical systems can be described by the differential equations with integral boundary conditions. Integral boundary conditions are encountered in various applications, such as population dynamics, blood flow models, chemical engineering and cellular systems. Moreover, boundary-value problems with integral conditions constitute an interesting and important class of problems. They include two, three, multi and nonlocal boundary-value problems as special cases. For boundary-value problems with nonlocal boundary conditions and comments on their importance, we refer the reader to [1, 2, 3, 4, 5, 6] and the references therein.

In this article, we study existence and uniqueness of the solutions of nonlinear differential equations of the type

$$\dot{x}(t) = f(t, x(t)), \quad \text{for } t \in [0, T], \tag{1.1}$$

with two-point and integral boundary conditions

$$Ax(0) + \int_0^T m(s)x(s)ds + Bx(T) = \int_0^T g(s, x(s))ds, \tag{1.2}$$

where $A, B \in R^{n\times n}$ are given matrices, $\det\big(A + \int_0^T m(s)ds + B\big) \neq 0$; $f, g : [0, T] \times R^n \to R^n$, are given functions. By $C([0, T]\mathbb{R}^n)$ we denote the Banach space of all continuous functions from $[0, T]$ into R^n with the norm

$$\|x\| = \max\{|x(t)| : t \in [0, T]\},$$

where $|\cdot|$ is the norm in space R^n.

2000 *Mathematics Subject Classification.* 34A37, 34G60, 34G20.
Key words and phrases. Nonlocal boundary conditions; contraction principle; existence and uniqueness; fixed point theorem.

Submitted November 1, 2014. Published December 11, 2014.

We prove some new existence and uniqueness results by using a variety of fixed point theorems. In Theorem 3.1 we prove an existence and uniqueness result by using Banach's contraction principle. In Theorem 3.2 we prove the existence of a solution by using Schaefer's fixed point theorem, while in Theorem 3.3 we prove the existence of a solution via Leray-Schauder nonlinear alternative.

It is worth mention that the methods used in this paper are standard. Our impact is implementation of these methods to the solution of the problem (1.1), (1.2).

2. Preliminaries

We define a solution of the problem (1.1)-(1.2) as follows:

Definition. A function $x \in C([0,T]\mathbb{R}^n)$ is said to be a solution of problem (1.1)-(1.2) if $\dot{x}(t) = f(t,x(t))$, for each $t \in [0,T]$, and the boundary conditions (1.2) are satisfied.

Lemma 2.1. *Let $y, g \in C([0,T]\mathbb{R}^n)$. Then the unique solution of the boundary-value problem for the differential equation*

$$\dot{x}(t) = y(t), t \in [0,T] \tag{2.1}$$

with boundary condition

$$Ax(0) + \int_0^T m(s)x(s)ds + Bx(T) = \int_0^T g(s)ds \tag{2.2}$$

is given by

$$x(t) = C + \int_0^T K(t,\tau)y(\tau)d\tau, \tag{2.3}$$

where

$$K(t,\tau) = \begin{cases} \Gamma^{-1}\big(A + \int_0^t m(\tau)d\tau\big) & 0 \le \tau \le t \\ -\Gamma^{-1}(\int_t^T m(\tau)d\tau + B), & t \le \tau \le T \end{cases},$$

$$C = \Gamma^{-1}\int_0^T g(s)ds,$$

$$\Gamma = (A + \int_0^T m(t)dt + B).$$

Proof. If $x = x(\cdot)$ is a solution of the differential equation (2.1), then for $t \in (0,T)$,

$$x(t) = x(0) + \int_0^t y(\tau)d\tau, \tag{2.4}$$

where $x(0)$ is an arbitrary constant vector. In order to determine $x(0)$ we require that the function in equality (2.1) should satisfy condition (2.2), i.e.,

$$\Gamma x(0) = \int_0^T g(t)dt - \int_0^T m(t)\int_0^t y(\tau)d\tau dt - B\int_0^T y(t)\,dt.$$

Since $\det\Gamma \ne 0$, we have

$$x(0) = C + \Gamma^{-1}\int_0^T \int_t^T m(\tau)d\tau y(t)dt - \Gamma^{-1}B\int_0^T y(t)dt. \tag{2.5}$$

Now in (2.4) we take into account the value $x(0)$ determined from the equality (2.5) and obtain

$$x(t) = C + \int_0^T K(t,\tau)y(\tau)d\tau.$$

Thus we have proved that one can write the boundary-value problem (2.1), (2.2) as the integral equation (2.3). One can immediately verify that a solution to the integral equation (2.3) also satisfies the boundary-value problem (2.1), (2.2). □

Lemma 2.2. *Assume that $f, g \in C([0,T] \times R^n\mathbb{R}^n)$. Then the function $x(t)$ is a solution of the boundary-value problem* (1.1)-(1.2) *if and only if $x(t)$ is a solution of the integral equation*

$$x(t) = \int_0^T K(t,s)f(s,x(s))ds + \Gamma^{-1}\int_0^T g(s,x(s))ds. \tag{2.6}$$

Proof. Let $x(t)$ be a solution of the boundary-value problem (1.1), (1.2). Then in the same way as in Lemma 2.1, we can prove that it is also a solution of the integral equation (2.6). By direct verification we can show that the solution of the integral equation (2.6) also satisfies equation (1.1) and nonlocal boundary condition (1.2). Lemma 2.2 is proved. □

3. Main results

Define the operator $P : C([0,T]\mathbb{R}^n) \to P([0,T]\mathbb{R}^n)$ as

$$Px(t) = \Gamma^{-1}\int_0^T g(t,x(t))dt + \int_0^T K(t,\tau)f(\tau,x(\tau))d\tau. \tag{3.1}$$

Obviously, the problem (1.1), (1.2) is equivalent to the fixed point problem $x = Px$. In consequence, problem (1.1), (1.2) has a solution if and only if the operator P has a fixed point.

Our first result is based on the Banach fixed point theorem. It uses the assumptions:

(H1) There exists a continuous function $N(t) > 0$ such that

$$|f(t,x) - f(t,y)| \le N(t)|x-y|,$$

for each $t \in [0,T]$ and all $x, y \in R^n$;

(H2) There exists a continuous function $M(t) > 0$ such that

$$|g(t,x) - g(t,y)| \le M(t)|x-y|,$$

for each $t \in [0,T]$ and all $x, y \in R^n$.

Theorem 3.1. *Assume* (H1), (H2) *hold, and*

$$L = T[SN + M]\|\Gamma^{-1}\| < 1\,. \tag{3.2}$$

Then the boundary-value problem (1.1)-(1.2) *has a unique solution on* $[0,T]$*, where*

$$N = \max_{[0,T]} N(t), \quad M = \max_{[0,T]} M(t),$$

$$S = \max\{\|(A + \int_0^t m(\tau)d\tau)\|, \|(\int_t^T m(\tau)d\tau + B)\|\}.$$

Proof. Setting $\max_{[0,T]}|f(t,0)| = M_f$, $\max_{[0,T]}|g(t,0)| = M_g$ and choosing

$$r \geq [1 - \|\Gamma^{-1}\|T(SN+M)]^{-1}\|\Gamma^{-1}\|(M_f + M_g),$$

we show that $PB_r \subset B_r$, where

$$B_r = \{x \in C([0,T]\mathbb{R}^n) : \|x\| \leq r\}.$$

For $x \in B_r$, we have

$$\begin{aligned}
\|(Px)(t)\| &\leq \max_{[0,T]}\Big[\int_0^T |K(t,s)||f(s,x(s))|ds\Big] + \Big[\|\Gamma^{-1}\|\int_0^T |g(s,x(s))|ds\Big]\\
&\leq \max_{[0,T]}\Big[\int_0^T |K(t,s)|(|f(s,x(s)) - f(s,0)| + |f(s,0)|)ds\Big]\\
&\quad + \|\Gamma^{-1}\|\int_0^T (|g(s,x(s)) - g(s,0)| + |g(s,0)|)ds\\
&\leq \|\Gamma^{-1}\|S(Nr + M_f)T + \|\Gamma^{-1}\|(Mr + M_g)T \leq r.
\end{aligned}$$

Now, for any $u, v \in B_r$ we have

$$\begin{aligned}
&|(Pu)(t) - (Pv)(t)|\\
&\leq \|\Gamma^{-1}\|\int_0^T |g(t,u(t)) - g(t,v(t))|dt + \int_0^T |K(t,\tau)||f(\tau,u(\tau)) - f(\tau,v(\tau))|d\tau\\
&\leq \|\Gamma^{-1}\|\int_0^T M(t)|u(t) - v(t)|dt + \|\Gamma^{-1}\|S\int_0^T N(t)|u(t) - v(t)dt|dt\\
&\leq \|\Gamma^{-1}\|[M + NS]T\|u - v\|,
\end{aligned}$$

or

$$\|Pu - Pv\| \leq L\|u - v\|. \tag{3.3}$$

From condition (3.2) it follows that $\|Pu - Pv\| < \|u - v\|$. Therefore, P is a contraction in B_r. Therefore, in view of the contraction principle the operator P defined by (3.1) has a unique fixed point in $C([0,T]\mathbb{R}^n)$. Consequently, the integral equation (2.6) (or the boundary-value problem (1.1), (1.2)) has a unique solution. □

The second result is based on Schaefer's fixed point theorem. It uses the assumptions:

- (H3) The function $f : [0,T] \times R^n \to R^n$ is continuous;
- (H4) There exists a constant $N_1 > 0$ such that $|f(t,x)| \leq N_1$ for each $t \in [0,T]$ and all $x \in R^n$;
- (H5) The function $g : [0,T] \times R^n \to R^n$ is continuous;
- (H6) There exists a constant $N_2 > 0$ such that $|g(t,x)| \leq N_2$ for each $t \in [0,T]$ and all $x \in R^n$.

Theorem 3.2. *Assume* (H3)–(H6) *Then the boundary-value problem* (1.1)-(1.2) *has at least one solution on* $[0,T]$.

Proof. We divide the proof into several main steps in which we show that under the assumptions of the theorem, the operator P has a fixed point.

Step 1. The operator P under the assumptions of the theorem is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C([0,T]\mathbb{R}^n)$. Then for any $t \in (0,T)$,

$$|P(x_n)(t) - P(x)(t)|$$

$$
\begin{aligned}
&\leq \|\Gamma^{-1}\| \int_0^T |g(t,x_n(t)) - g(t,x(t))|dt + \int_0^T |K(t,\tau)||f(\tau,x_n(\tau)) - f(\tau,x(\tau))|d\tau \\
&\leq TM\|\Gamma^{-1}\| \max_{[0,T]} |g(t,x_n(t)) - g(t,x(t))| \\
&\quad + TNS\|\Gamma^{-1}\| \max_{[0,T]} |f(t,x_n(t)) - f(t,x(t))|.
\end{aligned}
$$

Since f and g are continuous functions, we have

$$\|P(x_n)(t) - P(x)(t)\| \to 0, \quad \text{as } n \to \infty.$$

Step 2. The operator P maps bounded sets into $C([0,T]\mathbb{R}^n)$. Indeed, it is sufficient to show that for any $\eta > 0$, there exists a positive constant l such that for each $x \in B_\eta = \{x \in C([0,T]\mathbb{R}^n) : \|x\| \leq \eta\}$, we have $\|P(x)\| \leq l$. By (H4) and (H6) we have for each $t \in [0,T]$,

$$|P(x)(t)| \leq \int_0^T |K(t,s)||f(s,x(s))|ds + \|\Gamma^{-1}\| \int_0^T |g(s,x(s))|ds.$$

Hence,

$$|P(x)(t)| \leq \|\Gamma^{-1}\|STN_1 + \|\Gamma^{-1}\|TN_2.$$

Thus,

$$\|P(x)(t)\| \leq \|\Gamma^{-1}\|STN_1 + \|\Gamma^{-1}\|TN_2 = l.$$

Step 3. The operator P maps bounded sets into equicontinuous sets of $C([0,T]\mathbb{R}^n)$. Let $t_1, t_2 \in (0,T]$, $t_1 < t_2$, B_η be a bounded set of $C([0,T]\mathbb{R}^n)$ as in Step 2, and let $x \in B_\eta$. Then

$$
\begin{aligned}
&|P(x)(t_2) - P(x)(t_1)| \\
&= |\int_0^T [K(t_2,s) - K(t_1,s)]f(s,x(s))ds| \\
&\leq |\Gamma^{-1} \int_{t_1}^{t_2} [A + \int_0^t m(\tau)d\tau]f(t,x(t))dt + \Gamma^{-1} \int_{t_1}^{t_2} [B + \int_t^T m(\tau)d\tau]f(t,x(t))dt| \\
&\leq 2S\|\Gamma^{-1}\| \int_{t_1}^{t_2} |f(t,x(t))|dt.
\end{aligned}
$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $P : C([0,T]\mathbb{R}^n) \to C([0,T]\mathbb{R}^n)$ is completely continuous.

Step 4. A priori bounds. Now it remains to show that the set

$$\Delta = \{x \in C([0,T]\mathbb{R}^n) : x = \lambda P(x), \text{ for some } 0 < \lambda < 1\}$$

is bounded.

Let $x = \lambda(Px)$ for some $0 < \lambda < 1$. Thus, for each $t \in [0,T]$ we have

$$x(t) = \lambda\Big[\int_0^T K(t,s)f(s,x(s))ds + \Gamma^{-1} \int_0^T g(s,x(s))ds\Big].$$

This implies by (H4) and (H6) (as in Step 2) that for each $t \in [0,T]$,

$$|P(x)(t)| \leq \|\Gamma^{-1}\|(SN_1 + N_2)T.$$

Thus, for every $t \in [0,T]$ we have

$$\|x\| \leq \|\Gamma^{-1}\|(SN_1 + N_2)T.$$

This shows that the set Δ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that P has a fixed point which is a solution of the problem (1.1)-(1.2). $\square$

In the following theorem we give an existence result for the problem (1.1)-(1.2) by means of an application of the Leray-Schauder type nonlinear alternative, where the conditions (H4) and (H6) are weakened.

- (H7) There exist $\theta_f \in L_1([0,T], R^+)$ and continuous and nondecreasing $\psi_f : [0,\infty) \to [0,\infty)$ such that
$$|f(t,x)| \le \theta_f(t)\psi_f(|x|),$$
for each $t \in [0,T]$ and all $x \in R$;
- (H8) There exist $\theta_g \in L^1([0,T], R^+)$ and continuous and nondecreasing $\psi_g : [0,\infty) \to [0,\infty)$ such that
$$|g(t,x)| \le \theta_g(t)\psi_g(|x|),$$
for each $t \in [0,T]$ and all $x \in R$.
- (H9) There exists a number $K > 0$ such that
$$\frac{K}{\|\Gamma^{-1}\|S\psi_f(K)\|\theta_f\|_{L_1} + \psi_g(K)\|\Gamma^{-1}\|\|\theta\|_{L_1}} > 1.$$

Theorem 3.3. *Assume that* (H3), (H5), (H7)–(H9) *hold. Then the boundary-value problem* (1.1)-(1.2) *has at least one solution on* $[0,T]$.

Proof. Consider the operator P defined above. It can be easily shown that P is continuous and completely continuous. For $\lambda \in [0,1]$ let x be such that for each $t \in [0,T]$ we have $x(t) = \lambda(Px)(t)$. Then from (H7) and (H8), for each $t \in [0,T]$ we have
$$\begin{aligned}
|x(t)| &\le \int_0^t |K(t,s)|\theta_f(s)\psi(|x(s)|)ds + \|\Gamma^{-1}\| \int_0^T \theta_g(s)\psi_g(|x(s)|)ds \\
&\le \|\Gamma^{-1}\|S\psi_f(\|x\|) \int_0^T \theta_f(s)ds + \psi_g(\|x\|)\|\Gamma^{-1}\| \int_0^T \theta_g(s)ds.
\end{aligned}$$
Thus,
$$\frac{\|x\|}{\|\Gamma^{-1}\|S\psi_f(\|x\|)\|\theta_f\|_{L_1} + \psi_g(\|x\|)\|\Gamma^{-1}\|\|\theta_g\|_{L_1}} \le 1.$$
Then, in view of (H9), there exists K such that $\|x\| \ne K$. Let us set
$$U = \{x \in C([0,T], R) : \|x\| < K\}.$$

Note that the operator $P : \overline{U} \to C([0,T], \mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda P(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [7], we deduce that P has a fixed point x in $\overline{U}$ which is a solution of the problem (1.1)-(1.2). This completes the proof. $\square$

4. Examples

In this section, we give examples to illustrate the usefulness of our main results.

Example 4.1. Let us consider the following nonlocal boundary-value problem for the system of differential equations

$$\begin{aligned} \dot{x}_1(t) &= 0.1\sin x_2, \quad t\in[0,1],\\ \dot{x}_2(t) &= \frac{|x_1|}{(9+e^t)(1+|x_1|)}, \end{aligned} \tag{4.1}$$

with

$$x_1(0)=1, \quad x_2(1)=\int_0^1 2tx_1(t)dt. \tag{4.2}$$

We can rewrite the boundary conditions (4.2) in the equivalent form:

$$\begin{aligned} &\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}x_1(0)\\x_2(0)\end{pmatrix}-\int_0^1\begin{pmatrix}0&0\\2t&0\end{pmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}dt+\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}x_1(1)\\x_2(1)\end{pmatrix}\\ &=\begin{pmatrix}1\\0\end{pmatrix}. \end{aligned}$$

Obviously,

$$\Gamma=\begin{pmatrix}1&0\\0&0\end{pmatrix}-\int_0^1\begin{pmatrix}0&0\\2t&0\end{pmatrix}dt+\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\-1&1\end{pmatrix},$$

and the matrix Γ is invertible.

Evidently, $\Gamma^{-1}=\begin{pmatrix}1&0\\1&1\end{pmatrix}$, and $\|\Gamma^{-1}\|=2$.

Hence, the conditions (H1)–(H2) hold with $N=0.1$, $M=0$, $S=2$. We can easily see that the condition (3.2) is satisfied. Indeed,

$$L=\|\Gamma^{-1}\|SNT=4\times 0.1=0.4<1. \tag{4.3}$$

Then, by Theorem 3.1 the boundary-value problem (4.1)-(4.2) has a unique solution on $[0,1]$.

Example 4.2. On $[0,1]$ we consider the boundary-value problem

$$\begin{aligned} \dot{x}_1 &= \sin x_2,\\ \dot{x}_2 &= \cos x_1,\\ \dot{x}_3 &= \frac{1}{3}\Big(\frac{1}{1+x_1^2}+\frac{1}{1+x_2^2}+\frac{1}{1+x_3^2}\Big). \end{aligned} \tag{4.4}$$

with

$$x_1(0)=0, \quad x_2(0)+\int_0^1 tx_3(t)dt=2, x_3(1)=1. \tag{4.5}$$

Obviously, the matrix

$$\Gamma=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}+\int_0^1\begin{pmatrix}0&0&0\\0&0&t\\0&0&0\end{pmatrix}dt+\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0.5\\0&0&1\end{pmatrix}$$

is invertible, and the function

$$\begin{pmatrix}f_1\\f_2\\f_3\end{pmatrix}=\begin{pmatrix}\sin x_1\\ \cos x_2\\ \frac{1}{3}\big(\frac{1}{1+x_1^2}+\frac{1}{1+x_2^2}+\frac{1}{1+x_3^2}\big)\end{pmatrix}$$

is continuous and bounded. Hence, by Theorem 3.2 the boundary-value problem (4.4), (4.5) has at least one solution on $[0, 1]$.

Conclusion. The boundary condition considered in this article is general enough to cover a wide class of boundary-value problems. To illustrate this point, we consider the following three cases:

(1) If $g \equiv 0$, $A = E$ (E is a unit matrix), $m(t) \equiv \theta$ and $B = \theta$, (θ is a zero matrix), then we get the Cauchy problem.
(2) If $g \equiv 0$, $m(t) \equiv \theta$, we get a two-point boundary-value problem.
(3) If $g \equiv 0$, $A = \theta$, $B = \theta$, we get a boundary problem with integral condition.

Also note that the methods of this article may be used for more general case of (1.2),

$$\sum_{i=0}^{n} A_i x(t_i) + \int_0^T m(t)x(t)dt = C,$$

where $0 = t_0 < t_1 < \cdots < t_n = T$, $\det\left(\sum_{i=0}^{n} A_i + \int_0^T m(t)dt\right) \neq 0$.

References

[1] A. Ashyralyev, Y. A. Sharifov; *Existence and uniqueness of solutions for nonlinear impulsive differential equations with two-point and integral boundary conditions*, Advances in Difference Equations 2013:173, 2013.
[2] A. Belarbi, M. Benchohra; *Existence results for nonlinear boundary-value problems with integral boundary conditions*, Electronic Journal of Differential Equations, vol. 2005, no. 6, p. 10, 2005.
[3] A. Belarbi, M. Benchohra, A. Quahab; *Multiple positive solutions for nonlinear boundary-value problems with integral boundary conditions*, Archivum Mathematicum, vol. 44, no. 1, pp. 1-7, 2008.
[4] M. Benchohra, J. J. Nieto, A. Quahab; *Second-order boundary-value problem with integral boundary conditions.* Boundary Value Problems, vol. 2011, Article ID 260309, 9 pages, 2011.
[5] M. Benchohra, S. Hamani, J. J. Nieto; *The method of upper and lower solutions for the second order differential inclusions with integral boundary conditions*, The Rocky Mountain Journal of Mathematics, vol. 40, No. 1, pp. 13-26, 2010.
[6] S. A. Brykalov; *A second order nonlinear problems with two-point and integral boundary conditions*, Georgian Mathematical Journal, vol. 1, pp.243-249, 1994.
[7] A. Granas, J. Dugundji; *Fixed Point Theory, Springer Monographs in Mathematics*, Springer, New York, NY, USA, 2003.

Misir J. Mardanov
Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, Baku, Azerbaijan
E-mail address: misirmardanov@yahoo.com

Yagub A. Sharifov
Baku State University, Institute of Control Systems, National Academy of Sciences of Azerbaijan, Baku, Azerbaijan
E-mail address: sharifov22@rambler.ru

Habib H. Molaei
Department of Mathematics, Technical and Vocational University, Urmia, Iran
E-mail address: habibmolaei@gmail.com