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EXISTENCE OF SOLUTIONS FOR CROSS CRITICAL EXPONENTIAL N-LAPLACIAN SYSTEMS

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Abstract. In this article we consider cross critical exponential N-Laplacian systems. Using an energy estimate on a bounded set and the Ekeland variational principle, we prove the existence of a nontrivial weak solution, for a parameter large enough.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and $N \geq 2$. Firstly we consider the problem

$$\begin{gathered}
-\Delta_N u = au|u|^{N-2} + bu|u|^{\frac{N-4}{2}}|v|^{N/2} + du(N|u|^{N-2} \\
+ \frac{\alpha_0 N}{N-1}|u|^{\frac{N^2-2N+2}{N-1}})|v|^N \exp\{\alpha_0|u|^{\frac{N}{N-1}} + \beta_0|v|^{\frac{N}{N-1}}\} \quad \text{in } \Omega, \\
-\Delta_N v = bv|v|^{\frac{N-4}{2}}|u|^{N/2} + cv|v|^{N-2} + dv(N|v|^{N-2} \\
+ \frac{\beta_0 N}{N-1}|v|^{\frac{N^2-2N+2}{N-1}})|u|^N \exp\{\alpha_0|u|^{\frac{N}{N-1}} + \beta_0|v|^{\frac{N}{N-1}}\} \quad \text{in } \Omega, \\
u = 0, \quad v = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}$$

where $a, b, c, d, \alpha_0, \beta_0$ are real constants and $\alpha_0, \beta_0 > 0$. For similar problem, to our knowledge, de Figueiredo, do O and Ruf [3] firstly discussed the coupled system of exponential type in $\mathbb{R}^2$

$$\begin{gathered}
-\Delta u = g(v) \quad \text{in } \Omega, \\
-\Delta v = f(u) \quad \text{in } \Omega, \\
u = 0, \quad v = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.2}$$

where $f(u), g(v)$ behave like $\exp\{\alpha|u|^2\}$ and $\exp\{\alpha|v|^2\}$ respectively for some $\alpha > 0$ at infinity. They obtained the existence of the positive solution by a linking theorem in Hilbert space. Recently, Lam and Lu [5] extended this existence result of problem (1.2) on the condition that the nonlinear terms satisfy a weak Ambrosetti-Rabinowitz condition. Furthermore, the author [9] proved a similar result for a class of cross critical exponential system even if these critical nonlinear terms without Ambrosetti-Rabinowitz condition. For further and recent researches on exponential

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system, we refer to [4, 7, 8] and the references therein. Our main propose of this article is to study a class nonuniform critical exponential terms similar to (1.1), which weaken the critical assumptions used in [9], and further elaborate the idea of [9] that proper energy estimate guarantees the nontrivial weak solutions for some critical growth systems.

In the last section, we will extend this existence result to a wider class of nonlinear terms with cross critical growth. More exactly, we study the problem

$$\begin{gathered} -\Delta_N u = a|u|^{N-2}u + bu|u|^{N/2-2}|v|^{N/2} + df(x,u,v) \quad \text{in } \Omega, \\ -\Delta_N v = bv|v|^{N/2-2}|u|^{N/2} + c|v|^{N-2}v + dg(x,u,v) \quad \text{in } \Omega, \\ u = 0, \quad v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where a, b, c, d are constants and $f(x,u,v), g(x,u,v)$ with critical growth at $\alpha_0, \beta_0 > 0$ respectively. Here we say $f(x,u,v)$ and $g(x,u,v)$ have critical growth at α_0, β_0 respectively, if there exist positive constants α_0, β_0 such that: For any $v \neq 0$,

$$\lim_{u\to\infty} \frac{|f(x,u,v)|}{\exp\{\alpha|u|^{\frac{N}{N-1}}\}} = 0,\ \forall \alpha > \alpha_0 \quad \text{and} \quad \lim_{u\to\infty} \frac{|f(x,u,v)|}{\exp\{\alpha|u|^{\frac{N}{N-1}}\}} = +\infty,\ \forall \alpha < \alpha_0; \tag{1.4}$$

and for any $u \neq 0$,

$$\lim_{v\to\infty} \frac{|g(x,u,v)|}{\exp\{\beta|v|^{\frac{N}{N-1}}\}} = 0,\ \forall \beta > \beta_0 \quad \text{and} \quad \lim_{v\to\infty} \frac{|g(x,u,v)|}{\exp\{\beta|v|^{\frac{N}{N-1}}\}} = +\infty,\ \forall \beta < \beta_0. \tag{1.5}$$

Since the system is not variational in general, we assume that there exists the primitive $F(x,u,v)$ such that

$$F_u(x,u,v) = f(x,u,v), \quad F_v(x,u,v) = g(x,u,v).$$

We weaken some of the critical exponential assumptions used in [9], as follows:

- (F1) $f(x,t,s), g(x,t,s) : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying $f(x,t,0) = f(x,0,s) = g(x,t,0) = g(x,0,s) = 0$;
- (F2) $F(x,s,t) > 0$, for $t, s \in \mathbb{R}^+$ and a.e. $x \in \Omega$.

We note that the above assumptions have been simplified. From the exponential growth condition, the explicit exponential nonlinear term

$$F(x,u,v) = h(x,u,v)\exp\{k(x,u,v)u^{N/(N-1)}\}\exp\{l(x,u,v)v^{N/(N-1)}\}$$

satisfies the Ambrosetti-Rabinowitz condition, where $\lim_{u\to\infty} k(x,u,v) = \alpha_0$, $\lim_{v\to\infty} l(x,u,v) = \beta_0$ and $h(x,u,v) \geq 0$. It is obvious that

$$\begin{aligned} f(x,u,v) &= h_u(x,u,v)\exp\{k(x,u,v)u^{N/(N-1)}\}\exp\{l(x,u,v)v^{N/(N-1)}\} \\ &\quad + h(x,u,v)\Big(\frac{N}{N-1}k(x,u,v)u^{\frac{1}{N-1}} + k_u(x,u,v)u^{\frac{N}{N-1}}\Big) \\ &\quad \times \exp\{k(x,u,v)u^{N/(N-1)}\}\exp\{l(x,u,v)v^{N/(N-1)}\}, \end{aligned}$$

and

$$\begin{aligned} g(x,u,v) &= h_v(x,u,v)\exp\{k(x,u,v)u^{N/(N-1)}\}\exp\{l(x,u,v)v^{N/(N-1)}\} \\ &\quad + h(x,u,v)\Big(\frac{N}{N-1}k(x,u,v)v^{\frac{1}{N-1}} + k_v(x,u,v)v^{\frac{N}{N-1}}\Big) \\ &\quad \times \exp\{k(x,u,v)u^{N/(N-1)}\}\exp\{l(x,u,v)v^{N/(N-1)}\}, \end{aligned}$$

Since $h_u(x,u,v), h_v(x,u,v), k_u(x,u,v), k_v(x,u,v)$ and $h(x,u,v) \geq 0$, there exist constants $C, M > 0$ such that for all $|u|, |v| \geq C$,

$$0 < F(x,u,v) \leq M(f(x,u,v) + g(x,u,v)) \quad \text{for a.e. } x \in \Omega;$$

i. e. the Ambrosetti-Rabinowitz condition is satisfied. On the other hand, without the assumption $\limsup_{t\to 0} \frac{F(x,t,s)}{|t|^N+|s|^N} = 0$, we could not have mountain pass geometry. A typical example is given as follows:

$$F(x,u,v) = \sqrt{|u||v|}\exp\{\alpha_0 e^{|u|^{-3}}|u|^{N/(N-1)}\}\exp\{\beta_0 e^{|v|^{-3}}|v|^{N/(N-1)}\}.$$

Here are the main results of this article for problem (1.1).

Theorem 1.1. *Under the assumptions $a, c < \lambda_1$, there exists a positive constant Λ^* such that* (1.1) *has at least one solution for all $d > \Lambda^*$, where λ_1 as in* (2.2) *and Λ^* depends on $a, b, c, \alpha_0, \beta_0$, the dimension N and the domain Ω.*

The following theorem extends partially the existence result of nontrivial weak solution presented in [9].

Theorem 1.2. *If $a, c < \lambda_1$ and the assumption* (F1)-(F2) *are satisfied, there exists a positive constant Θ^* such that* (1.3) *has at least one solution for all $d > \Theta^*$, where λ_1 as in* (2.2) *and Θ^* depends on $a, b, c, \alpha_0, \beta_0$, the dimension N and the domain Ω.*

This article is organized as follows. Section 2 contains the preliminaries. Section 3 shows two important estimate results. Section 4 shows the proof of Theorem 1.1. Section 5 provides a simple proof of Theorem 1.2.

2. Preliminaries

Throughout this paper, we define

$$\|u\|_N = \Big(\int_\Omega |\nabla u|^N\Big)^{1/N}, \quad |u|_N = \Big(\int_\Omega |u|^N\Big)^{1/N},$$

and

$$\begin{aligned} I(u,v) = \frac{1}{N}\int_\Omega |\nabla u|^N + \frac{1}{N}\int_\Omega |\nabla v|^N - \frac{a}{N}\int_\Omega |u|^N - \frac{c}{N}\int_\Omega |v|^N \\ - \frac{2b}{N}\int_\Omega |u|^{N/2}|v|^{N/2} - d\int_\Omega |u|^N|v|^N \exp\{\alpha_0|u|^{\frac{N}{N-1}}\}\exp\{\beta_0|v|^{\frac{N}{N-1}}\}. \end{aligned} \tag{2.1}$$

It is well known that

$$\lambda_1 = \min_{u\in W_0^{1,N}(\Omega)\setminus\{0\}} \frac{\|u\|_N^N}{|u|_N^N} > 0, \tag{2.2}$$

The space X designates the product space $W_0^{1,N}(\Omega)\times W_0^{1,N}(\Omega)$ equipped by the norm $\|(u,v)\|_X = \|u\|_N + \|v\|_N$. It is well known that the maximal growth of $u \in W_0^{1,N}(\Omega)$ is of exponential type, see references [6] and [9]. More precisely, we have the following uniform bound estimate (see also [2]):

Trudinger-Moser inequality. Let $u \in W_0^{1,N}(\Omega)$, then $\exp\{|u|^{\frac{N}{N-1}}\} \in L^\theta(\Omega)$ for all $1 \le \theta < \infty$. That is to say that for any given $\theta > 0$, any $u \in W_0^{1,N}(\Omega)$ holds $\exp\{\theta|u|^{\frac{N}{N-1}}\} \in L^1(\Omega)$. Moreover, there exists a constant $C = C(N,\alpha) > 0$ such that

$$\sup_{\|u\|_N \le 1} \int_\Omega \exp(\alpha|u|^{\frac{N}{N-1}}) \le C|\Omega|, \quad \text{if } 0 \le \alpha \le \alpha_N, \tag{2.3}$$

where $|\Omega|$ is the N dimension Lebesgue measure of Ω, $\alpha_N = N\omega_N^{\frac{1}{N-1}}$ and ω_N is the $N-1$ dimension Hausdorff measure of the unit sphere in $\mathbb{R}^N$. Furthermore, if $\alpha > \alpha_N$, then $C = +\infty$. Here and throughout this paper, we often denote various constants by same C. The reader can recognize them easily. Thanks to Trudinger-Moser inequality, we know the functional $I(u,v)$ is well defined. Using a standard argument, we also deduce that the functional $I(u,v)$ is of class C^1 and

$$\begin{aligned}
&\langle I'(u,v),(\varphi,\phi)\rangle \\
&= \int_\Omega |\nabla u|^{N-2}\nabla u\nabla\varphi + \int_\Omega |\nabla v|^{N-2}\nabla v\nabla\phi - a\int_\Omega |u|^{N-2}u\varphi - c\int_\Omega |v|^{N-2}v\phi \\
&\quad - b\int_\Omega u\varphi|u|^{N/2-2}|v|^{N/2} - b\int_\Omega v\phi|v|^{N/2-2}|u|^{N/2} \\
&\quad - d\int_\Omega u\varphi(N|u|^{N-2} + \frac{\alpha_0 N}{N-1}|u|^{\frac{N^2-2N+2}{N-1}})|v|^N \exp\{\alpha_0|u|^{\frac{N}{N-1}} + \beta_0|v|^{\frac{N}{N-1}}\} \\
&\quad - d\int_\Omega v\phi(N|v|^{N-2} + \frac{\beta_0 N}{N-1}|v|^{\frac{N^2-2N+2}{N-1}})|u|^N \exp\{\alpha_0|u|^{\frac{N}{N-1}} + \beta_0|v|^{\frac{N}{N-1}}\},
\end{aligned} \tag{2.4}$$

for any $\varphi,\phi \in W_0^{1,N}(\Omega)$. Obviously, the critical points of $I(u,v)$ are precisely the weak solutions for problem (1.1). By the critical assumptions (1.4), (1.5) and (F1), the functional

$$\begin{aligned}
J(u,v) &= \frac{1}{N}\int_\Omega |\nabla u|^N + \frac{1}{N}\int_\Omega |\nabla v|^N - \frac{1}{N}\int_\Omega a|u|^N - \frac{1}{N}\int_\Omega c|v|^N \\
&\quad - \frac{2}{N}\int_\Omega b|u|^{N/2}|v|^{N/2} - d\int_\Omega F(x,u,v),
\end{aligned}$$

is well defined and of class C^1 such that the critical points of $J(u,v)$ are precisely the weak solutions for problem (1.3); i.e.,

$$\begin{aligned}
\langle J'(u,v),(\varphi,\phi)\rangle &= \int_\Omega |\nabla u|^{N-2}\nabla u\nabla\varphi + \int_\Omega |\nabla v|^{N-2}\nabla v\nabla\phi - a\int_\Omega |u|^{N-2}u\varphi \\
&\quad - c\int_\Omega |v|^{N-2}v\phi - b\int_\Omega u\varphi|u|^{N/2-2}|v|^{N/2} \\
&\quad - b\int_\Omega v\phi|v|^{N/2-2}|u|^{N/2} - d\int_\Omega f(x,u,v)\varphi - d\int_\Omega g(x,u,v)\phi.
\end{aligned} \tag{2.5}$$

3. Energy estimates

Lemma 3.1. *If $\|u\|_N^{\frac{N}{N-1}} < \frac{\alpha_N}{\alpha_0}$ and $\|v\|_N^{\frac{N}{N-1}} < \frac{\alpha_N}{\beta_0}$, there exists $q > 1$ such that*

$$\int_\Omega (N|u|^{N-1} + \frac{\alpha_0 N}{N-1}|u|^{\frac{N^2-N+1}{N-1}})^q |v|^{qN} \exp\{q\alpha_0|u|^{\frac{N}{N-1}} + q\beta_0|v|^{\frac{N}{N-1}}\} \le C$$

and

$$\int_\Omega (N|v|^{N-1}+\frac{\beta_0 N}{N-1}|v|^{\frac{N^2-N+1}{N-1}})^q|u|^{qN}\exp\{q\alpha_0|u|^{\frac{N}{N-1}}+q\beta_0|v|^{\frac{N}{N-1}}\}\le C.$$

Proof. By contradiction. Then for any $\varepsilon_1,\varepsilon_2>0$ and any $q>1$, we estimate that

$$\begin{aligned}
&\int_\Omega (N|u|^{N-1}+\frac{\alpha_0 N}{N-1}|u|^{\frac{N^2-N+1}{N-1}})^q|v|^{qN}\exp\{q\alpha_0|u|^{\frac{N}{N-1}}+q\beta_0|v|^{\frac{N}{N-1}}\}\\
&\le C\int_\Omega \exp\{q(\alpha_0+\varepsilon_1)|u|^{\frac{N}{N-1}}\}\exp\{q(\beta_0+\varepsilon_2)|v|^{\frac{N}{N-1}}\}\\
&= C\int_\Omega \exp\{q(\alpha_0+\varepsilon_1)\|u\|_N^{\frac{N}{N-1}}(\frac{|u|}{\|u\|_N})^{\frac{N}{N-1}}\}\exp\{q(\beta_0+\varepsilon_2)\|v\|_N^{\frac{N}{N-1}}(\frac{|v|}{\|v\|_N})^{\frac{N}{N-1}}\},
\end{aligned}$$

tends to infinite. Then by Trudinger-Moser inequality (2.3), we get that $q(\alpha_0+\varepsilon_1)\|u\|_N^{\frac{N}{N-1}}>\alpha_N$ or $q(\beta_0+\varepsilon_2)\|v\|_N^{\frac{N}{N-1}}>\alpha_N$. Since $q>1$ and $\varepsilon_1,\varepsilon_2>0$ are arbitrary, we have

$$\|u\|_N^{\frac{N}{N-1}}\ge\frac{\alpha_N}{\alpha_0}\quad\text{or}\quad\|v\|_N^{\frac{N}{N-1}}\ge\frac{\alpha_N}{\beta_0},$$

which contradicts our assumptions. Applying similar argument to $\int_\Omega (N|v|^{N-1}+\frac{\beta_0 N}{N-1}|v|^{\frac{N^2-N+1}{N-1}})^q|u|^{qN}\exp\{q\alpha_0|u|^{\frac{N}{N-1}}+q\beta_0|v|^{\frac{N}{N-1}}\}$, we deduce the conclusion. $\square$

We denote the Moser functions as follows

$$\overline{M_n}(x):=\omega_N^{-1/N}\begin{cases}(\log n)^{\frac{N-1}{N}}, & |x|\le 1/n;\\ \frac{\log(1/|x|)}{(\log n)^{1/N}}, & 1/n\le|x|\le 1;\\ 0, & |x|\ge 1;\end{cases}$$

where $2\le n\in\mathbb{N}^+$ and ω_N as in (2.3), i.e. $N^{N-1}\omega_N=\alpha_N^{N-1}$. Let r be the inner radius of Ω and $x_0\in\Omega$ such that $B_r(x_0)\subset\Omega$. Then the functions

$$M_n(x):=\overline{M_n}(\frac{x-x_0}{r})$$

satisfy $\|M_n\|_N=1$, $|M_n|_N^N=O(1/\log n)$ and $\operatorname{supp}M_n\subset B_r(x_0)$. We define a close convex ball as

$$\overline{B}_{\alpha_0,\beta_0}:=\{(u,v)\in X|\|(u,v)\|_X^{\frac{N}{N-1}}\le\min(\frac{\alpha_N}{\alpha_0},\frac{\alpha_N}{\beta_0})\}.$$

Now, we give an estimate from below for the functional $I(u,v)$ on the ball in $\overline{B}_{\alpha_0,\beta_0}$.

Lemma 3.2. *There exist a constant Λ^* such that for all $d>\Lambda^*$,*

$$\inf_{(u,v)\in\overline{B}_{\alpha_0,\beta_0}} I(u,v)=c_0<0,\tag{3.1}$$

where Λ^* depends on a,b,c,α_0,β_0, the dimension N and the domain Ω.

Proof. Without loss generality, we assume that $\alpha_0\ge\beta_0$. Here we take $u_n=\frac12(\frac{\alpha_N}{\alpha_0})^{\frac{N-1}{N}}M_n$ and

$$v_n=\frac12(\frac{\alpha_N}{\alpha_0})^{\frac{N-1}{N}}M_n\le\frac12(\frac{\alpha_N}{\beta_0})^{\frac{N-1}{N}}M_n.$$

Then $\|u_n\|_N=\|v_n\|_N=\frac{1}{2}(\frac{\alpha_N}{\alpha_0})^{\frac{N-1}{N}}$ (i.e. $(u_n,v_n)\in\overline{B}_{\alpha_0,\beta_0}$). Form the definition of $M_n(x)$, we have

$$
\begin{aligned}
&\frac{a}{N}\int_\Omega |u_n|^N+\frac{c}{N}\int_\Omega |v_n|^N+\frac{2b}{N}\int_\Omega |u_n|^{N/2}|v_n|^{N/2}\\
&=\frac{a+2b+c}{2^N N\omega_N}(\frac{\alpha_N}{\alpha_0})^{N-1}\int_{B(x_0,\frac{r}{n})}(\log n)^{N-1}\\
&\quad+\frac{a+2b+c}{2^N N\omega_N}(\frac{\alpha_N}{\alpha_0})^{N-1}\int_{B(x_0,r)\setminus B(x_0,\frac{r}{n})}\frac{(\log\frac{r}{|x-x_0|})^N}{\log n}\\
&=\frac{(a+2b+c)r^N}{2^N N^2 n^N}(\frac{\alpha_N}{\alpha_0})^{N-1}(\log n)^{N-1}+\frac{a+2b+c}{2^N N\log n}(\frac{\alpha_N}{\alpha_0})^{N-1}\int_{\frac{r}{n}}^r(\log\frac{r}{l})^N l^{N-1}dl\\
&=O(1/\log n),
\end{aligned}\tag{3.2}
$$

and

$$
\begin{aligned}
&\int_\Omega |u_n|^N|v_n|^N\exp\{\alpha_0|u_n|^{\frac{N}{N-1}}\}\exp\{\beta_0|v_n|^{\frac{N}{N-1}}\}\\
&=\frac{\alpha_N^{2(N-1)}}{4^N\omega_N^2\alpha_0^{2(N-1)}}\int_{B(x_0,\frac{r}{n})}(\log n)^{2(N-1)}\exp\{\frac{N}{2^{\frac{N}{N-1}}}\log n+\frac{N\beta_0}{2^{\frac{N}{N-1}}\alpha_0}\log n\}\\
&\quad+\frac{\alpha_N^{2(N-1)}}{4^N\omega_N^2\alpha_0^{2(N-1)}}\int_{B(x_0,r)\setminus B(x_0,\frac{r}{n})}\frac{(\log\frac{r}{|x-x_0|})^{2N}}{(\log n)^2}\\
&\quad\times\exp\{(\frac{N}{2^{\frac{N}{N-1}}}+\frac{N\beta_0}{2^{\frac{N}{N-1}}\alpha_0})\frac{(\log\frac{r}{|x-x_0|})^{\frac{N}{N-1}}}{(\log n)^{\frac{1}{N-1}}}\}\\
&\geq\frac{\omega_N r^N}{4^N n^N}\frac{N^{2N-3}}{\alpha_0^{2(N-1)}}(\log n)^{2(N-1)}n^{\frac{N}{2^{\frac{N}{N-1}}}+\frac{N\beta_0}{2^{\frac{N}{N-1}}\alpha_0}}+\frac{\alpha_N^{2(N-1)}}{4^N\omega_N\alpha_0^{2(N-1)}(\log n)^2}\\
&\quad\times\int_{\frac{r}{n}}^r(\log\frac{r}{l})^{2N}\exp\{(\frac{N}{2^{\frac{N}{N-1}}}+\frac{N\beta_0}{2^{\frac{N}{N-1}}\alpha_0})\frac{(\log\frac{r}{l})^{\frac{N}{N-1}}}{(\log n)^{\frac{1}{N-1}}}\}l^{N-1}dl.
\end{aligned}\tag{3.3}
$$

Obviously, for fixed n, we deduce that expression (3.2) is bounded and expression (3.3) is larger than a positive constant. Noticing the definitions of u_n,v_n, we obtain that there exists a positive constant Λ^* such that for all $d>\Lambda^*$ holds $I(u_n,v_n)<0$, which implies that

$$
\inf_{(u,v)\in\overline{B}_{\alpha_0,\beta_0}} I(u,v)=c_0<0.
$$

□

4. Proof of Theorem 1.1

Since $\overline{B}_{\alpha_0,\beta_0}$ is a Banach space with the norm given by the norm of X, the functional $I(u,v)$ is of class C^1 and bounded below on $\overline{B}_{\alpha_0,\beta_0}$. In fact, if $\|u\|_N^{\frac{N}{N-1}}$ equals to $\min(\frac{\alpha_N}{\alpha_0},\frac{\alpha_N}{\beta_0})$, then $\|v\|_N=0$. Hence that

$$
\int_\Omega |u|^N|v|^N\exp\{\alpha_0|u|^{\frac{N}{N-1}}\}\exp\{\beta_0|v|^{\frac{N}{N-1}}\}=0.\tag{4.1}
$$

And same result holds for $\|v\|_N^{\frac{N}{N-1}} = \min(\frac{\alpha_N}{\alpha_0}, \frac{\alpha_N}{\beta_0})$. By a similar argument of Lemma 3.1, we conclude that

$$\int_\Omega |u|^N |v|^N \exp\{\alpha_0 |u|^{\frac{N}{N-1}}\} \exp\{\beta_0 |v|^{\frac{N}{N-1}}\} \le C \tag{4.2}$$

for $\|u\|_N^{\frac{N}{N-1}}, \|v\|_N^{\frac{N}{N-1}} < \min(\frac{\alpha_N}{\alpha_0}, \frac{\alpha_N}{\beta_0})$. That is to say that the functional $I(u,v)$ is bounded below on $\overline{B}_{\alpha_0,\beta_0}$.

Thanks to Ekeland's variational principle [1, Corollary A.2], there exists some minimizing sequence $\{(u_n, v_n)\} \subset \overline{B}_{\alpha_0,\beta_0}$ such that

$$I(u_n, v_n) \to \inf_{(u,v)\in \overline{B}_{\alpha_0,\beta_0}} I(u,v) = c_0 < 0, \tag{4.3}$$

and

$$I'(u_n, v_n) \to 0 \quad \text{in } X^*, \text{ as as } n \to \infty. \tag{4.4}$$

From (2.4) and (4.4), taking $(\varphi, \phi) = (u_n, 0)$ and $(\varphi, \phi) = (0, v_n)$ respectively, we have

$$\begin{aligned} &\int_\Omega |\nabla u_n|^N - a\int_\Omega |u_n|^N - b\int_\Omega |u_n|^{N/2}|v_n|^{N/2} \\ &- d\int_\Omega (N|u_n|^N + \frac{\alpha_0 N}{N-1}|u_n|^{\frac{N^2}{N-1}})|v_n|^N \exp\{\alpha_0|u_n|^{\frac{N}{N-1}} + \beta_0|v_n|^{\frac{N}{N-1}}\} \to 0, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &\int_\Omega |\nabla v_n|^N - c\int_\Omega |v_n|^N - b\int_\Omega |u_n|^{N/2}|v_n|^{N/2} \\ &- d\int_\Omega (N|v_n|^N + \frac{\beta_0 N}{N-1}|v_n|^{\frac{N^2}{N-1}})|u_n|^N \exp\{\alpha_0|u_n|^{\frac{N}{N-1}} + \beta_0|v_n|^{\frac{N}{N-1}}\} \to 0. \end{aligned} \tag{4.6}$$

Since u_n, v_n are uniform bounded in $W_0^{1,N}(\Omega)$, by Lemma 6.1 in the Appendix, we conclude that

$$u_n \rightharpoonup u_0, \quad v_n \rightharpoonup v_0 \quad \text{in } W_0^{1,N}(\Omega),$$

and (u_0, v_0) is a weak solution for problem (1.1).

Now, we prove that this weak solution is nontrivial.

Proposition 4.1. *The above weak solution* (u_0, v_0) *is a nontrivial solution for problem* (1.1).

Proof. By the assumptions $a, c < \lambda_1$ and $d > 0$, we have that $u_0 = 0$ if and only if $v_0 = 0$. The condition $d > 0$ guarantees this problem is nontrivial. In fact, if $u_0 = 0$, then v_0 is a solution of the equation

$$\begin{gathered} -\Delta_p v = c|v|^{N-2}v < \lambda_1 |v|^{N-2} v \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Obviously, we have $v_0 = 0$.

Now we suppose that $u_0 = v_0 = 0$. Then by $u_n, v_n \rightharpoonup 0$ weak convergence in $W_0^{1,N}(\Omega)$, we have

$$\lim_{n\to\infty} \frac{a}{N}\int_\Omega |u_n|^N, \lim_{n\to\infty} \frac{c}{N}\int_\Omega |v_n|^N, \lim_{n\to\infty} \frac{2b}{N}\int_\Omega |u_n|^{N/2}|v_n|^{N/2} = 0.$$

These together with (4.5) and (4.6), from Lemma 3.1, by Hölder inequality, we obtain

$$\|u_n\|_N, \|v_n\|_N \to 0.$$

i.e. $(u_0, v_0) \to (0, 0)$ strongly in X. Obviously,

$$\int_\Omega |u_n|^N |v_n|^N \exp\{\alpha_0 |u_n|^{\frac{N}{N-1}}\} \exp\{\beta_0 |v_n|^{\frac{N}{N-1}}\} \to 0,$$

as $n \to \infty$. Hence that

$$\lim_{n\to\infty} I(u_n, v_n) = 0,$$

which contracts with (4.3). □

Thus the proof of theorem 1.1 is complete.

5. Proof of Theorem 1.2

In this section we show the existence of nontrivial weal solution for more general quasilinear system (i.e. problem (1.3)). As the proofs are similar we will sketch from place to place. Noticing the assumptions (1.4) and (1.5), by similar arguments of Lemma 3.1, we would see that

$$\int_\Omega |f(x, u, v)|^q, \int_\Omega |g(x, u, v)|^q \le C \tag{5.1}$$

for some $q > 1$ and $\|u\|_N^{\frac{N}{N-1}} < \frac{\alpha_N}{\alpha_0}$ and $\|v\|_N^{\frac{N}{N-1}} < \frac{\alpha_N}{\beta_0}$. By assumption (F2), choosing a proper constant $c \neq 0$ such that $(u_n, v_n) = (cM_n(x), cM_n(x)) \in \overline{B}_{\alpha_0,\beta_0}$, we have

$$\int_\Omega F(x, u_n, v_n) \ge C \tag{5.2}$$

for some fixed $n > 1$, which means that

$$\inf_{(u,v)\in \overline{B}_{\alpha_0,\beta_0}} J(u, v) = \widetilde{c_0} < 0$$

for d large enough. Form the assumption (F1), similar to equality (4.1) and inequality (4.2), we have that $J(u, v)$ is bounded below on $\overline{B}_{\alpha_0,\beta_0}$. This combined with $\overline{B}_{\alpha_0,\beta_0}$ is a Banach space with the norm given by the norm of X and the functional $J(u, v)$ is of class C^1, by Ekeland's variational principle [1, Corollary A.2], there exists some minimizing sequence $\{(u_n, v_n)\} \subset \overline{B}_{\alpha_0,\beta_0}$ such that

$$J(u_n, v_n) \to \inf_{(u,v)\in \overline{B}_{\alpha_0,\beta_0}} J(u, v) = \widetilde{c_0} < 0, \tag{5.3}$$

and

$$J'(u_n, v_n) \to 0 \quad \text{in } X^*, \text{ as } n \to \infty. \tag{5.4}$$

From (2.5) and (5.4), taking $(\varphi, \phi) = (u_n, 0)$ and $(\varphi, \phi) = (0, v_n)$ respectively, we have

$$\int_\Omega |\nabla u_n|^N - a \int_\Omega |u_n|^N - b \int_\Omega |u_n|^{N/2} |v_n|^{N/2} - d \int_\Omega f(x, u_n, v_n) u_n \to 0, \tag{5.5}$$

and

$$\int_\Omega |\nabla v_n|^N - c \int_\Omega |v_n|^N - b \int_\Omega |u_n|^{N/2} |v_n|^{N/2} - d \int_\Omega g(x, u_n, v_n) v_n \to 0. \tag{5.6}$$

Since u_n, v_n are uniform bounded in $W_0^{1,N}(\Omega)$, by Lemma 6.1 in the appendix, we conclude that

$$u_n \rightharpoonup u_0, \quad v_n \rightharpoonup v_0 \quad \text{in } W_0^{1,N}(\Omega),$$

and (u_0, v_0) is a weak solution for problem (1.3).

Now, we will prove this weak solution is nontrivial.

Proposition 5.1. *The above weak solution (u_0, v_0) is nontrivial.*

Proof. By the assumptions (F1), $a, c < \lambda_1$ and $d > 0$, using same argument for Proposition 4.1, we can get that $u = 0$ if and only if $v = 0$.

Now we suppose that $u_0 = v_0 = 0$. Then by $u_n, v_n \rightharpoonup 0$ weak convergence in $W_0^{1,N}(\Omega)$, we have

$$\lim_{n\to\infty} \frac{a}{N} \int_\Omega |u_n|^N, \lim_{n\to\infty} \frac{c}{N} \int_\Omega |v_n|^N, \lim_{n\to\infty} \frac{2b}{N} \int_\Omega |u_n|^{N/2} |v_n|^{N/2} = 0.$$

These together with (5.1), (5.5) and (5.6), by Hölder inequality, we obtain

$$\|u_n\|_N, \|v_n\|_N \to 0.$$

i.e. $(u_0, v_0) \to (0, 0)$ strong convergence in X, which means $\int_\Omega F(x, u_n, v_n) \to 0$, as $n \to \infty$. Hence

$$\lim_{n\to\infty} J(u_n, v_n) = 0,$$

which contracts with (5.3). $\square$

Thus the proof of Theorem 1.2 is complete.

6. Appendix

Here we give a brief proof for the existence result of the weak solution for problem (1.3), see also [11], however the non-triviality of this weak solution need to be clarified.

Lemma 6.1. *Suppose the sequences $\{u_n\}, \{v_n\}$ are bounded in $W_0^{1,N}(\Omega)$, and the $\lim_{n\to\infty} J'(u_n, v_n) \to 0$ in X^*, then there exist u_0, v_0 such that $u_n \rightharpoonup u_0$, $v_n \rightharpoonup v_0$ in $W_0^{1,N}(\Omega)$ and $\langle J'(u_0, v_0), (\varphi, \phi)\rangle = 0$ for all $\varphi, \phi \in W_0^{1,N}(\Omega)$.*

Proof. Since $\{u_n\}, \{v_n\}$ are bounded in $W_0^{1,N}(\Omega)$, there exist u_0, v_0 such that

$$u_n \to u_0 \quad \text{and} \quad v_n \to v_0,$$

which implies $u_n \to u_0$ and $v_n \to v_0$ in $L^1(\Omega)$. By assumptions (1.4) and (1.5), using Trudinger-Moser inequality, we have

$$\int_\Omega |f(x, u_n, v_n) u_n| \le C, \quad \int_\Omega |f(x, u_n, v_n) v_n| \le C,$$
$$\int_\Omega |g(x, u_n, v_n) v_n| \le C, \quad \int_\Omega |g(x, u_n, v_n) u_n| \le C.$$

Combining the above results, we find that

$$f(x, u_n, v_n) \to f(x, u_0, v_0), \quad g(x, u_n, v_n) \to g(x, u_0, v_0) \quad \text{in} \quad L^1(\Omega). \tag{6.1}$$

Now, taking test function $(\tau(u_n - u_0), 0)$, the assumption $\lim_{n\to\infty} J'(u_n, v_n) \to 0$ becomes

$$\langle I_2'(u_n, v_n), (\tau(u_n - u_0), 0)\rangle$$

$$
\begin{aligned}
&= \int_\Omega |\nabla u_n|^{N-2}\nabla u_n \nabla\tau(u_n-u_0) + \int_\Omega a u_n \tau(u_n-u_0)|u_n|^{N-2} \\
&\quad + \int_\Omega b u_n \tau(u_n-u_0)|u_n|^{N/2-2}|v_n|^{N/2} + \int_\Omega f(x,u_n,v_n)\tau(u_n-u_0) \to 0,
\end{aligned}
$$

where

$$
\tau(t) = \begin{cases} t, & \text{if } |t| \le 1; \\ t/|t|, & \text{if } |t| > 1. \end{cases}
$$

Hence by (6.1) and $|\tau(u_n-u_0)|_\infty \to 0$, we deduce

$$
\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0)\nabla\tau(u_n-u_0) \to 0,
$$

which implies $\nabla u_n \to \nabla u_0$ a.e. in Ω; see [10, Theorem 1.1]. Since $N \ge 2$, we know

$$
|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u_0|^{N-2}\nabla u_0 \quad \text{in } (L^{N/(N-1)}(\Omega))^N.
$$

Using similar argument, we get the same result for sequence $\{v_n\}$. By these results combined with (6.1) and $J'(u_n,v_n) \to 0$, we obtain that

$$
\langle J'(u_0,v_0),(\varphi,\phi)\rangle = 0
$$

for any $\varphi,\phi \in \mathcal{D}(\Omega)$. By using an argument of density, this identity holds for all $\varphi,\phi \in W_0^{1,N}(\Omega)$. Then the proof is complete. □

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