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EXACT NUMBER OF SOLUTIONS FOR A NEUMANN PROBLEM INVOLVING THE p-LAPLACIAN

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Abstract. We study the exact number of solutions of the quasilinear Neumann boundary-value problem

$$(\varphi_p(u'(t)))' + g(u(t)) = h(t) \quad \text{in } (a,b),$$
$$u'(a) = u'(b) = 0,$$

where $\varphi_p(s) = |s|^{p-2}s$ denotes the one-dimensional p-Laplacian. Under appropriate hypotheses on g and h, we obtain existence, multiplicity, exactness and non existence results. The existence of solutions is proved using the method of upper and lower solutions.

1. Introduction

The p-Laplacian operator appears in the study of non-Newtonian fluids in which the quantity p is a characteristic of the medium. In particular, media with $1 < p < 2$ are called pseudo-plastics. If $p = 2$, they are Newtonian fluids. They also appear in the study of flows through porous media ($p = 3/2$), as well as in glaciology ($1 < p \leq 4/3$). For a general description of diffusion processes, see, e.g., [7]. See also [8] for a study of flows through porous media in one dimension.

Let us consider the nonlinear problem

$$-\Delta_p v(x) = \rho(x)g(v(x)) - h(x),$$

in the annulus $\Omega = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$, $N \geq 3$, with radially symmetric functions ρ and h, subject to zero Neumann boundary conditions. Thus this problem can be reduced to an ODE's problem. Indeed, applying the following two changes of variables

$$s = -\int_r^{r_2} \tau^{-\frac{n-1}{p-1}} d\tau, \ z(s) = v(r(s))$$

and $s = \omega \frac{b-t}{b-a}$, $u(t) = z(s)$ where

$$\omega = -\int_{r_1}^{r_2} \tau^{-\frac{n-1}{p-1}} \, d\tau$$

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we obtain, for a suitable function ρ, the following Neumann boundary value problem

$$\begin{gathered} (\varphi_p(u'(t)))' + g(u(t)) = h(t) \quad \text{in } (a,b), \\ u'(a) = u'(b) = 0. \end{gathered} \tag{1.1}$$

Here $\varphi_p(s) = |s|^{p-2}s$, for $s \neq 0$, with $\varphi_p(0) = 0$ for $p > 1$, denotes the one-dimensional p-Laplacian and the functions g and h satisfy suitable conditions.

Our goal is to obtain the exact number of solutions to problem (1.1) which belongs to a certain set, depending only on the nonlinearity g. For this, we mainly apply the method of lower and upper solutions. This method allows us to establish the existence of at least one solution of the problem considered. We consider two cases, the first one when the lower and the upper solutions are well ordered; i.e., the lower solution is less than the upper one, and the second (less common) case when the lower and the upper solution are reversely ordered. In the last case, even when $p = 2$, the existence of solutions to problem (1.1) is not certain, in general. A typical example is given by the problem

$$u'' + u = \cos t, \quad u'(0) = u'(\pi) = 0,$$

which has no solution even though $\alpha(t) \equiv 1$ and $\beta(t) \equiv -1$ are lower and upper solutions, respectively. In this direction, when $p = 2$, $a = 0$ and $b = 1$, by using Sobolev and Wirtinger inequalities, lower and upper solutions method and fixed-point techniques for completely continuous and increasing operators in ordered Banach spaces, results of exact number of solutions and positive solutions for this problem are established in [9]. One of the main results reads as follows (see [9, Theorem 4.1]): let $g \in C^1(\mathbb{R})$. If $g'(x) < \pi^2/4$, $x \in \mathbb{R}$, g' is a strictly increasing function and $\lim_{x\to\pm\infty} g(x) = +\infty$, then there exists an $M \in \mathbb{R}$ such that

(a) if $h(t) \leq M$, with strict inequality on a set of positive measure, then problem (1.1) has no solution.
(b) if $h(t) \equiv M$, problem (1.1) has exactly one solution.
(c) if $h(t) \geq M$, with strict inequality on a set of positive measure, then problem (1.1) has exactly two solutions.

We point out that to obtain the exact number of solutions in part (c) it is necessary to show that the solutions of problem (1.1) may be ordered. For this, the author of [9] studies a homogeneous problem associated with (1.1), where $g(u(t))$ is replaced by $q(t)u(t)$ with $q \in L^r(0,1)$ for some $r \in [1,+\infty)$; i.e., the problem

$$\begin{gathered} u''(t) + q(t)u(t) = 0 \quad \text{in } (0,1), \\ u'(0) = u'(1) = 0. \end{gathered} \tag{1.2}$$

In that case the linear nature of the underlying equation is used in an essential way.

In this article, since we deal with the nonlinear case, to get accurate results with respect to the number of solutions (multiple solutions) we need to impose a restriction on the range of values of p to the interval $(1,2]$ (such restriction is necessary in view of the counterexample given in Section 5 of [3]) and apply [3, Theorem 4.1]. For this, the following property of the p-Laplacian ($1 < p \leq 2$) is crucial: for all compact interval $[k_1,k_2]$ there exists $K > 0$ (depending on p if $p \neq 2$) such that for all $u, v \in [k_1,k_2]$

$$(\varphi_p(u) - \varphi_p(v))(u - v) \geq K(u - v)^2. \tag{1.3}$$

Note that this property is not verified by the p-Laplacian, with $p > 2$. Although the condition (1.3) restricts the range of values of p, this condition is optimal for generating the approximation of solutions between lower and upper solutions in the reversed order by means of monotone iterative techniques and anti-maximum principles. We point out that this kind of nonlinear elliptic problems has been the object of intensive research in recent years, mostly in the linear case $p = 2$, see for example [10, 11, 12, 13, 14, 15, 16]. Methods used in the cited literature include fixed point theorems in cones and degree arguments. However, there have not been many results in the nonlinear case $p \neq 2$. Further, all these results deal with a single solution, or the least number of solutions. The reason for this is that exact multiplicity results are usually difficult to establish. As mentioned here we use mainly the very important technique of lower and upper solutions. For a survey of this technique, see [5, 6]. We refer the reader to [4] for a recent review on the formidable literature about this method. Our results are inspired by those of [9], for the corresponding second-order Neumann boundary value problem.

To state our main result we impose the following two hypotheses:

- (H1) g belongs to $C^1(\mathbb{R})$ with $g'(x)$ strictly increasing and $\lim_{x\to\pm\infty} g(x) = +\infty$.
- (H2) The function h belongs to $C([a, b])$.

By hypothesis (H1) there exists $\theta \in \mathbb{R}$ such that

$$g(\theta) = \min_{x\in\mathbb{R}} g(x), \quad g'(\theta) = 0.$$

Let $m = g(\theta)$. Then $g(x) \geq m$ for all $x \in \mathbb{R}$. Since $\lim_{x\to\pm\infty} g(x) = +\infty$, there exist constants c_1 and c_2 such that $c_1 < \theta < c_2$ and

$$\widetilde{m} := g(c_1) = g(c_2) > h(t), \quad \text{for all } \ t \in [a, b]. \tag{1.4}$$

We obtain multiple solutions of problem (1.1) belonging to the following set

$$S := \{u \in C^1([a, b]) : m < g(u(t)) < \widetilde{m}, \text{ for all } t \in [a, b]\}.$$

Let $l > 0$, we denote by K_l the constant given in (1.3) for the symmetric interval $[-l, l]$. When $1 < p < 2$ the largest of such constants is $(p - 1)l^{p-2}$ (this is the inverse of the maximum of $\frac{d}{dy}\,\varphi_p^{-1}(y)$ on the interval $[-\varphi_p(l), \varphi_p(l)]$).

The main result in this article reads as follows.

Theorem 1.1. *Assume that the hypotheses* (H1) *and* (H2) *hold. Let c and k be given by* (2.1) *and* (2.2)*, respectively. If*

$$g'(x) < \frac{\pi^2}{4(b-a)^2} \min\{K_c, K_k\}, \; x \in \mathbb{R},$$

then the following holds:

1. *if $h(t) \leq m$, with strict inequality on a subinterval of $[a, b]$, then problem* (1.1) *has no solution.*
2. *if $h(t) \equiv m$, then problem* (1.1) *has exactly one solution.*
3. *if $1 < p \leq 2$ and $h(t) \geq m$, with strict inequality on a subinterval of $[a, b]$, then problem* (1.1) *has exactly two solutions in the set S.*

Remark 1.2. (i) The conclusions (1) and (2) of the theorem hold for all $p > 1$. On the other hand, in (3) we seek solutions of problem (1.1) in the set $\{u : c_1 \leq u(t) \leq c_2, \text{ for all } t \in [a, b]\}$. When $u \in [\theta, c_2]$, we have the a priori bound k over the derivatives of these solutions. So, if the lower and the upper solution are reversely ordered, then the behavior of φ_p outside of a compact interval plays no role. This is precisely the meaning of (1.3).

(ii) Note that when $p = 2$ we can take $K_k = 1$ (independently of k). Also in this case $K_c = 1$ (see Remark 2.6). Thus, we recover [9, Theorem 4.1].

This article is organized as follows. In Section 2, we establish some notation, as well as some basic facts, and we prove the Lemmas 2.5 and 2.7 that will be used in Section 3 to prove our main result, Theorem 1.1. Finally, in Section 4, we give an example to illustrate our results.

2. Preliminaries

We say that u is a solution of (1.1) if $u \in C^1([a,b])$, $|u'|^{p-2}u' \in W^{1,1}((a,b))$, $u'(a) = u'(b) = 0$, and $(\varphi_p(u'(t)))' + g(u(t)) = h(t)$ for almost all $t \in (a,b)$. Here $W^{1,1}((a,b))$ denotes the Banach space of absolutely continuous functions on (a,b). For later use, it is convenient to define $f_+(t,u) := g(u) - h(t)$ and $f_-(t,u) := h(t) - g(u)$. We use the following symbols. Let $I = [a,b]$ and $q \geq 1$. For $u \in L^q(I)$, we write

$$\|u\|_q = \Big(\int_a^b |u(s)|^q ds\Big)^{1/q}$$

and for $u \in C(I)$,

$$\|u\|_\infty = \sup_{t\in I} |u(t)|.$$

Let us first recall the following classical integral inequality.

Lemma 2.1. *Let $u \in C^1(I)$. If $u(a) = 0$ or $u(b) = 0$, then*

$$\frac{\pi}{2(b-a)} \|u\|_2 \leq \|u'\|_2 .$$

We need the following version of the Gronwall's lemma for showing uniqueness of solutions of an initial value problem for the p-Laplacian.

Lemma 2.2 (Gronwall's lemma)**.** *Suppose that $a < b$, and let z, v be nonnegative continuous functions defined on $[a,b]$. Furthermore, suppose that C is a nonnegative constant. If*

$$v(t) \leq C + \int_a^t z(s)v(s)ds, \quad t \in [a,b],$$

then

$$v(t) \leq C\, e^{\int_a^t z(s)ds}, \quad t \in [a,b].$$

Remark 2.3. In particular, if $C = 0$, we have $v \equiv 0$ on $[a,b]$.

Remark 2.4. (i) Note that, for every $R > 0$, we have

$$|f_+(t,u)| \leq h_R(t) \text{ for all } t \in [a,b] \text{ and all } u \text{ with } |u| \leq R,$$

where $h_R(t) := \max_{|s|\leq R} |g(s)| + |h(t)|$.

(ii) Note that we have an a priori estimate over the derivatives of the solutions of problem (1.1). Indeed, let $u(t)$ be a solution of (1.1). Define $\overline{h}(t) := h(t) - m$, then $\varphi_p(|u'(t)|) \leq \|\overline{h}\|_1$ for all $t \in [a,b]$. Therefore, if $u(t)$ is a solution of (1.1), then

$$\|u'\|_\infty \leq \|\overline{h}\|_1^{\frac{1}{p-1}} =: c \quad \text{(only depending on } g,h,p). \tag{2.1}$$

We shall say that $\alpha \in C^1([a,b])$ is a *lower solution* of (1.1) if $\varphi_p \circ \alpha' \in W^{1,1}((a,b))$ and

$$-(\varphi_p(\alpha'(t)))' \le f_+(t,\alpha(t)), \quad \alpha'(a) \ge 0 \ge \alpha'(b).$$

An *upper solution* is defined by reversing inequalities in the previous definition. Let α and $\beta \in C^1([a,b])$ be such that $\beta(t) \le \alpha(t)$ on $[a,b]$. We write

$$[\beta,\alpha] := \{v \in C^1([a,b]) : \beta(t) \le v(t) \le \alpha(t) \text{ on } [a,b]\}.$$

By Remark 2.4, part (i), we can find a continuous function $\widetilde{h}$ such that $|f_+(t,u)| \le \widetilde{h}(t)$ for all $t \in [a,b]$ and all $u \in [\beta(t),\alpha(t)]$. We define

$$k(\alpha,\beta) := \|\widetilde{h}\|_1^{\frac{1}{p-1}}.$$

If further α and β are lower and upper solutions, it is easy to check that for all $t \in [a,b]$,

$$\alpha'(t), \beta'(t) \in [-k(\alpha,\beta), k(\alpha,\beta)].$$

Note that the constant θ is an upper solution while c_1 and c_2 are lower solutions of problem (1.1). Moreover, since g is increasing for $u \in [\theta, c_2]$, we have

$$|f_+(t,u)| \le |g(u)| + |h(t)| \le |g(c_2)| + |h(t)| = |\widetilde{m}| + |h(t)| =: \widetilde{h}(t).$$

Using the previous notation we define

$$k := k(c_2,\theta) = \|\widetilde{h}\|_1^{\frac{1}{p-1}} = (|\widetilde{m}|(b-a) + \|h\|_1)^{\frac{1}{p-1}}. \tag{2.2}$$

The next result is key to study the exact number of solutions of (1.1).

Lemma 2.5. *Let $1 < p \le 2$. Suppose that $g \in C^1(\mathbb{R})$ with $g'(x) < \frac{\pi^2 K_c}{(b-a)^2}$, $x \in \mathbb{R}$, then the solutions of (1.1) do not cross each other, in other words, if u_1, u_2 are different solutions of (1.1), then $u_1(t) \ne u_2(t)$ for every $t \in [a,b]$.*

Proof. Let u_1, u_2 be different solutions of (1.1), then

$$(\varphi_p(u_i'(t)))' + g(u_i(t)) = h(t) \quad \text{in } (a,b), u_i'(a) = u_i'(b) = 0$$

for $i = 1, 2$; and so

$$(\varphi_p(u_1'(t)))' - (\varphi_p(u_2'(t)))' + g(u_1(t)) - g(u_2(t)) = 0, \tag{2.3}$$

for almost all $t \in (a,b)$. Define

$$q(t) = \begin{cases} \frac{g(u_1(t)) - g(u_2(t))}{u_1(t) - u_2(t)}, & u_1(t) \ne u_2(t) \\ g'(u_1(t)), & u_1(t) = u_2(t). \end{cases}$$

Then q is a continuous function satisfying $q(t) < \pi^2 K_c/(b-a)^2$ for all $t \in [a,b]$. Now (2.3) can be rewritten as

$$(\varphi_p(u_1'(t)))' - (\varphi_p(u_2'(t)))' + q(t)(u_1(t) - u_2(t)) = 0. \tag{2.4}$$

Suppose that there is a $\zeta \in (a,b)$, such that $u_1(\zeta) = u_2(\zeta)$. Multiplying (2.4) by $u_1 - u_2$ and integrating by parts over $[a,\zeta]$ and $[\zeta,b]$, we obtain

$$\int_a^\zeta [\varphi_p(u_1'(t)) - \varphi_p(u_2'(t))](u_1'(t) - u_2'(t))dt = \int_a^\zeta q(t)(u_1(t) - u_2(t))^2 dt,$$

$$\int_\zeta^b [\varphi_p(u_1'(t)) - \varphi_p(u_2'(t))](u_1'(t) - u_2'(t))dt = \int_\zeta^b q(t)(u_1(t) - u_2(t))^2 dt.$$

Set $u(t) = u_1(t) - u_2(t)$ for $t \in [a, b]$. From inequality (1.3), Remark 2.4 (ii) and Lemma 2.1, we have

$$\begin{gathered}
\frac{\pi^2 K_c}{4(\zeta - a)^2} \|u\|^2_{L^2(a,\zeta)} \le \int_a^\zeta q(t)(u(t))^2 dt \le \|q\|_\infty \|u\|^2_{L^2(a,\zeta)}, \\
\frac{\pi^2 K_c}{4(b - \zeta)^2} \|u\|^2_{L^2(\zeta,b)} \le \int_\zeta^b q(t)(u(t))^2 dt \le \|q\|_\infty \|u\|^2_{L^2(\zeta,b)}.
\end{gathered}$$

Thus

$$\frac{\pi^2 K_c}{4} \Big[\frac{1}{(\zeta - a)^2} + \frac{1}{(b - \zeta)^2}\Big] \le 2\|q\|_\infty.$$

The term in brackets reaches its minimum at $\zeta = \frac{a+b}{2}$, then

$$\|q\|_\infty \ge \frac{\pi^2 K_c}{(b - a)^2},$$

which is a contradiction.

If $\zeta = a$ or $\zeta = b$, then $u_1 \equiv u_2$. This is immediate by uniqueness if $p = 2$. The proof that $u_1 \equiv u_2$ when $1 < p < 2$ is more complicated. Let, u_1 and u_2 be solutions of (1.1) with $u_1(a) = u_2(a)$. Then

$$\varphi_p(u_1'(t)) = \int_a^t f_+(s, u_1(s))ds, \quad \varphi_p(u_2'(t)) = \int_a^t f_+(s, u_2(s))ds.$$

Hence

$$\begin{aligned}
|\varphi_p(u_1'(t)) - \varphi_p(u_2'(t))| &\le \int_a^t |f_+(s, u_1(s)) - f_+(s, u_2(s))|ds \\
&\le \frac{\pi^2 K_c}{(b - a)^2} \int_a^t |u_1(s) - u_2(s)|ds.
\end{aligned} \tag{2.5}$$

On the other hand, by the mean value theorem,

$$|u_1'(t) - u_2'(t)| = \frac{1}{p - 1} |\xi(t)|^{\frac{2-p}{p-1}} |\varphi_p(u_1'(t)) - \varphi_p(u_2'(t))|,$$

where $\xi(t)$ is some value between $\varphi_p(u_1'(t))$ and $\varphi_p(u_2'(t))$. In fact, since $u_1'(t)$ and $u_2'(t)$ are in $[-c, c]$ for all $t \in [a, b]$, we see that $\xi(t)$ belongs to the interval $[-\varphi_p(c), \varphi_p(c)]$. Note that

$$|u_1(t) - u_2(t)| \le \int_a^t |u_1'(s) - u_2'(s)|ds, \tag{2.6}$$

for all $t \in [a, b]$. Combining inequalities (2.5) and (2.6), we conclude that

$$|u_1'(t) - u_2'(t)| \le \frac{\pi^2 K_c}{(p - 1)(b - a)^2} |\xi(t)|^{\frac{2-p}{p-1}} \int_a^t |u_1'(s) - u_2'(s)|ds,$$

for all $t \in [a, b]$. Since $1 < p < 2$, it follows that $|\xi(t)|^{\frac{2-p}{p-1}}$ is bounded by c^{2-p} for all $t \in [a, b]$. Thus we have

$$|u_1'(t) - u_2'(t)| \le \frac{\pi^2 c^{2-p} K_c}{(p - 1)(b - a)^2} \int_a^t |u_1'(s) - u_2'(s)|ds \le \frac{\pi^2}{(b - a)^2} \int_a^t |u_1'(s) - u_2'(s)|ds,$$

for all $t \in [a, b]$. Thus, using Remark 2.3, we conclude that $u_1' - u_2' \equiv 0$ on $[a, b]$. Finally, since the functions $u_1(t)$ and $u_2(t)$ take the same value at $t = a$, we have $u_1 \equiv u_2$, which completes the proof. If $\zeta = b$, the proof is similar. $\square$

Remark 2.6. Note that, when $p = 2$, we can take $K_c = 1$ (which is independent of c), and so we recover [9, Lemma 3.2].

The next result establishes a bound on the number of solutions of (1.1) which belong to the set S.

Lemma 2.7. *Assume that* (H1) *holds and that* $h(t) \geq m$, *with strict inequality on a subinterval of* $[a, b]$. *If* $g'(x) < \frac{\pi^2 K_c}{(b-a)^2}$, $x \in \mathbb{R}$, *then* (1.1) *has at most two solutions in the set* S.

Proof. Lemma 2.5 tell us that under the condition that $g \in C^1(\mathbb{R})$ with $g'(x) < \frac{\pi^2 K_c}{(b-a)^2}$, $x \in \mathbb{R}$, the solutions of (1.1) are ordered. Suppose that (1.1) has solutions $u_i(t)$ for $i = 1, 2, 3$. Then we may assume that $u_1(t) < u_2(t) < u_3(t)$ for all $t \in [a, b]$. Now we note that if $u_i \in S$ for $i = 1, 2, 3$, then at least two of these functions belong to the same set $\{c_1 < u < \theta\}$ or else to $\{\theta < u < c_2\}$. Without loss of generality we may assume that $u_1, u_2 \in \{c_1 < u < \theta\}$. We have

$$\begin{gathered}(\varphi_p(u_i'(t)))' + g(u_i(t)) = h(t) \quad \text{in } (a, b),\\ u_i'(a) = u_i'(b) = 0\end{gathered}$$

for $i = 1, 2$. Therefore,

$$(\varphi_p(u_1'(t)))' - (\varphi_p(u_2'(t)))' = g(u_2(t)) - g(u_1(t)),$$

for almost all $t \in (a, b)$. Integrating over $[a, b]$ this equality and using the boundary conditions, we obtain

$$0 = \int_a^b [(\varphi_p(u_1'(t)))' - (\varphi_p(u_2'(t)))']dt = \int_a^b [g(u_2(t)) - g(u_1(t))]dt. \tag{2.7}$$

By the mean value theorem, there exists $\eta(t) \in (u_1(t), u_2(t))$ such that $g(u_2(t)) - g(u_1(t)) = g'(\eta(t))(u_2(t) - u_1(t))$. Set $v(t) = u_2(t) - u_1(t)$, for all $t \in [a, b]$. Then from (2.7) we have

$$\int_a^b g'(\eta(t))v(t)dt = 0.$$

This is a contradiction since $g'(\eta(t)) < 0$ for all $\eta(t) \in (u_1(t), u_2(t)) \subset (c_1, \theta)$ and $v(t) > 0$, for all $t \in [a, b]$. Therefore, there exist at most two solutions to (1.1) in the set S. $\square$

Remark 2.8. It follows from the above proof that, if there exist at least two solutions of (1.1) in S, then there exist exactly two solutions of (1.1) in S, one to the left of θ and the other to the right of θ.

3. Existence and exact number of solutions

This section is devoted to prove our main result, Theorem 1.1.

Proof of the Theorem 1.1. (1) If $h(t) \leq m$, with strict inequality on a subinterval of $[a, b]$, then

$$\int_a^b h(t)dt < m(b - a).$$

Suppose (1.1) has a solution $u(t)$. Since $g(u(t)) \geq m$, for all $t \in [a, b]$, we have

$$\int_a^b g(u(t))dt \geq m(b - a). \tag{3.1}$$

At the same time, integrating the equation in (1.1) over $[a,b]$ and using the boundary conditions, we obtain

$$\int_a^b g(u(t))dt = \int_a^b (\varphi_p(u'(t)))'dt + \int_a^b g(u(t))dt = \int_a^b h(t)dt < m(b-a),$$

which contradicts (3.1).

(2) If $h(t) \equiv m$, then $v(t) \equiv \theta$ is a solution of (1.1). Assume $u(t)$ is also a solution of problem (1.1) for $h(t) \equiv m$; i.e.

$$\begin{gathered}(\varphi_p(u'(t)))' + g(u(t)) = g(\theta), \quad a < t < b,\\ u'(a) = u'(b) = 0.\end{gathered}$$

As $g(u(t)) \geq m$ for all $t \in [a,b]$, we have $(\varphi_p(u'(t)))' \leq 0$ for all $t \in [a,b]$. By the boundary conditions, we conclude that $u'(t) \equiv 0$ for all $t \in [a,b]$, so $g(u(t)) \equiv g(\theta)$ on $[a,b]$. Since $g(x)$ is strictly convex, $g(x)$ has a unique minimum point, which implies $u(t) \equiv \theta$. Hence $v(t) \equiv \theta$ is the unique solution of (1.1).

(3) If $h(t) \geq m$, with strict inequality on a subinterval of $[a,b]$, then

$$\int_a^b h(t)dt > m(b-a) = \int_a^b g(\theta)dt,$$

which means $v(t) \equiv \theta$ is not a solution of problem (1.1).

Recall that θ is an upper solution while c_1 and c_2 are lower solutions of problem (1.1), where c_1, c_2 are as in (1.4). Moreover, c_1 and θ are well ordered but θ and c_2 are given in the reversed order, i.e. $c_2 \geq \theta$.

To prove that (1.1) has exactly two solutions in the set S, we proceed in three steps.

Step 1. Problem (1.1) has at least one solution in $[\theta, c_2] = [\beta, \alpha]$. Since $g'(x) < \pi^2 K_k/4(b-a)^2$, $x \in \mathbb{R}$ (here k is given by (2.2)), there exists a positive constant M such that $g'(x) < M < \pi^2 K_k/4(b-a)^2$ for every $\theta \leq x \leq c_2$. Then the function f_+ satisfies for M condition (L) in [3]. In fact, let $u, v \in [\theta, c_2]$ such that $u \leq v$. Then $g'(x)$ is nonnegative for all $\theta \leq x \leq c_2$ and by the mean value theorem

$$g(v) - g(u) = g'(c)(v-u), \quad u \leq c \leq v.$$

Thus $g(v)-g(u) \leq M(v-u)$ or, equivalently, $g(u)-g(v) \geq M(u-v)$. Consequently $g(u) - h(t) - Mu \geq g(v) - h(t) - Mv$; i.e., $f_+(t,u) - Mu \geq f_+(t,v) - Mv$. Finally, recall that the lower and upper solution are given in the reversed order. We are thus in a position to apply [3, Theorem 4.1], and deduce the existence of at least one solution u_1 of problem (1.1) such that $\theta \leq u_1(t) \leq c_2$ for all $t \in [a,b]$. Note that our result is optimal in the sense that if $p = 2$, we obtain the best possible estimate on M given in [1, Theorem 3.2 part 2].

Step 2. Problem (1.1) has at least one solution in $[c_1, \theta] = [\alpha, \beta]$. In this case the lower and the upper solutions are well ordered and we may apply [2, Theorem 2.1] with $\phi \equiv \varphi_p, f(t,u,u') \equiv f_-(t,u), A = B = 0$ to obtain at least one solution u_2 of problem (1.1) such that $c_1 \leq u_2(t) \leq \theta$ for all $t \in [a,b]$. In fact, it can easily be checked that the hypotheses (H2) and (H3) of that theorem hold.

Step 3. Problem (1.1) has exactly two solutions in S. Since $v(t) \equiv \theta$ is not a solution of problem (1.1), this problem has at least two solutions by Steps 1 and 2. Finally, problem (1.1) has exactly two solutions in S by Remark 2.8. $\square$

Remark 3.1. All of the results of this article can be deduced for positive solutions as well, with only minor modifications. Thus, we can obtain a generalization of [9, Theorem 6.1].

4. An example

Let α be a (small) positive number. Set $g(x) = \alpha x + e^{-x}$ for $x \in \mathbb{R}$. Then $g'(x) = \alpha - e^{-x}$ is strictly increasing and g is strictly convex. Therefore, according to Theorem 1.1, the problem

$$(\varphi_p(u'(t)))' + \alpha u(t) + e^{-u(t)} = h(t) \quad \text{in } (a,b),$$
$$u'(a) = u'(b) = 0,$$

has

(1) No solution if $h(t) \le \alpha(1 - \ln\alpha)$ with strict inequality on a subinterval of $[a,b]$.
(2) Exactly one solution if $h(t) \equiv \alpha(1 - \ln\alpha)$, which is $u(t) \equiv -\ln\alpha$.
(3) Exactly two solutions u_1, u_2 in the set S if $1 < p \le 2$ and $h(t) \ge \alpha(1-\ln\alpha)$, with strict inequality on a subinterval of $[a,b]$.

Next, we give more information on the solutions in case (3). For example, if $h(t) \equiv m+1$ and $\widetilde{m} = m+2$ in (1.4), where $m = g(\theta) = \alpha(1 - \ln\alpha)$ with $\theta = -\ln\alpha$, then we can estimate the values of c_1 and c_2 such that $g(c_1) = g(c_2) = \widetilde{m}$; i.e., the two roots of the equation

$$\alpha x + e^{-x} - \alpha(1 - \ln\alpha) - 2 = 0. \tag{4.1}$$

On the other hand, $c = (b-a)^{\frac{1}{p-1}}$, $k = [(2m+3)(b-a)]^{\frac{1}{p-1}}$. Since $k > c$, we have $K_k = (p-1)^{p-2}k^{p-2} < K_c = (p-1)c^{p-2}$ (these are the largest values of such constants). At this time we take the interval $[a,b]$ of length one for simplicity. Thus, the main condition in Theorem 1.1 reads

$$0 < \alpha < \frac{\pi^2}{4}K_k = \frac{\pi^2}{4}(p-1)(2m+3)^{\frac{p-2}{p-1}}$$

or, equivalently,

$$\alpha[2\alpha(1-\ln\alpha)+3]^{\frac{2-p}{p-1}} < (p-1)\frac{\pi^2}{4}\cdot \tag{4.2}$$

Note that when α tends to zero, the left-hand side of (4.2) also tends to zero (independent of $1 < p \le 2$), so we can always find an α small enough which satisfies this inequality. In particular, if $p = \frac{3}{2}$, then $\alpha = 0.2$ satisfies (4.2) and $\theta = -\ln(0.2) = \ln 5$. On the other hand, numerical approximations of the roots of equation (4.1) give the values $c_1 \approx -1.001430$ and $c_2 \approx 12.609421$. Hence, in this case the problem has exactly two solutions u_1, u_2 satisfying

$$\ln 5 > u_1(t) > c_1 \approx -1.001430, \quad t \in [a,b]$$
$$\ln 5 < u_2(t) < c_2 \approx 12.609421, \quad t \in [a,b].$$

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