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SOLVABILITY OF A QUADRATIC INTEGRAL EQUATION OF FREDHOLM TYPE IN HÖLDER SPACES

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Abstract. In this article, we prove the existence of solutions of a quadratic integral equation of Fredholm type with a modified argument, in the space of functions satisfying a Hölder condition. Our main tool is the classical Schauder fixed point theorem.

1. Introduction

Differential equations with a modified arguments arise in a wide variety of scientific and technical applications, including the modelling of problems from the natural and social sciences such as physics, biological and economics sciences. A special class of these differential equations have linear modifications of their arguments, and have been studied by several authors, see [1]–[9] and their references.

The aim of this article is to investigate the existence of solutions of the following integral equation of Fredholm type with a modified argument,

$$x(t) = p(t) + x(t) \int_0^1 k(t,\tau)\, x(r(\tau))\, d\tau, \quad t \in [0,1]. \tag{1.1}$$

Our solutions are placed in the space of functions satisfying the Hölder condition. A sufficient condition for the relative compactness in these spaces and the classical Schauder fixed point theorem are the main tools in our study.

2. Preliminaries

Our starting point in this section is to introduce the space of functions satisfying the Hölder condition and some properties in this space. These properties can be found in [2].

Let $[a,b]$ be a closed interval in $\mathbb{R}$, by $C[a,b]$ we denote the space of continuous functions on $[a,b]$ equipped with the supremum norm; i.e., $\|x\|_\infty = \sup\{|x(t)| : t \in [a,b]\}$ for $x \in C[a,b]$. For $0 < \alpha \le 1$ fixed, by $H_\alpha[a,b]$ we will denote the space of the real functions x defined on $[a,b]$ and satisfying the Hölder condition; that is, those functions x for which there exists a constant H_x^α such that

$$|x(t) - x(s)| \le H_x^\alpha |t-s|^\alpha, \tag{2.1}$$

for all $t, s \in [a,b]$. It is easily proved that $H_\alpha[a,b]$ is a linear subspace of $C[a,b]$.

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In the sequel, for $x \in H_\alpha[a,b]$, by H_x^α we will denote the least possible constant for which inequality (2.1) is satisfied. More precisely, we put

$$H_x^\alpha = \sup\Big\{\frac{|x(t)-x(s)|}{|t-s|^\alpha} : t \in [a,b],\ t \neq s\Big\}. \tag{2.2}$$

The spaces $H_\alpha[a,b]$ with $0 < \alpha \leq 1$ can be equipped with the norm

$$\|x\|_\alpha = |x(a)| + \sup\Big\{\frac{|x(t)-x(s)|}{|t-s|^\alpha} : t \in [a,b],\ t \neq s\Big\},$$

for $x \in H_\alpha[a,b]$. In [2], the authors proved that $(H_\alpha[a,b], \|\cdot\|_\alpha)$ with $0 < \alpha \leq 1$ is a Banach space. The following lemmas appear in [2].

Lemma 2.1. *For $x \in H_\alpha[a,b]$ with $0 < \alpha \leq 1$, the following inequality is satisfied*

$$\|x\|_\infty \leq \max\big(1, (b-a)^\alpha\big)\|x\|_\alpha. \tag{2.3}$$

Lemma 2.2. *For $0 < \alpha < \gamma \leq 1$, we have*

$$H_\gamma[a,b] \subset H_\alpha[a,b] \subset C[a,b]. \tag{2.4}$$

Moreover, for $x \in H_\gamma[a,b]$ the following inequality holds

$$\|x\|_\alpha \leq \max\big(1, (b-a)^{\gamma-\alpha}\big)\|x\|_\gamma. \tag{2.5}$$

Now, we present the following sufficient condition for relative compactness in the spaces $H_\alpha[a,b]$ with $0 < \alpha \leq 1$ which appears in Example 6 of [2] and it is an important result for our study.

Theorem 2.3. *Suppose that $0 < \alpha < \beta \leq 1$ and that A is a bounded subset in $H_\beta[a,b]$ (this means that $\|x\|_\beta \leq M$ for certain constant $M > 0$, for any $x \in A$) then A is a relatively compact subset of $H_\alpha[a,b]$.*

3. Main results

In this section, we will study the solvability of (1.1) in the Hölder spaces. We will use the following assumptions:

(i) $p \in H_\beta[0,1]$, $0 < \beta \leq 1$.

(ii) $k : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function such that it satisfies the Hölder condition with exponent β with respect to the first variable, that is, there exists a constant K_β such that

$$|k(t,\tau) - k(s,\tau)| \leq K_\beta\,|t-s|^\beta,$$

for any $t, s, \tau \in [0,1]$.

(iii) $r : [0,1] \to [0,1]$ is a measurable function.

(iv) The following inequality is satisfied

$$\|p\|_\beta(2K + K_\beta) < \frac{1}{4},$$

where the constant K is defined by

$$K = \sup\Big\{\int_0^1 |k(t,\tau)|\,d\tau : t \in [0,1]\Big\},$$

which exists by (ii).

Theorem 3.1. *Under assumptions* (i)–(iv), *Equation* (1.1) *has at least one solution belonging to the space* $H_\alpha[0,1]$, *where* α *is arbitrarily fixed number satisfying* $0 < \alpha < \beta$.

Proof. Consider the operator $\mathcal{T}$ defined on $H_\beta[0,1]$ by

$$(\mathcal{T}x)(t) = p(t) + x(t)\int_0^1 k(t,\tau)\, x(r(\tau))\, d\tau, \quad t \in [0,1].$$

In the sequel, we will prove that $\mathcal{T}$ transforms the space $H_\beta[0,1]$ into itself. In fact, we take $x \in H_\beta[0,1]$ and $t,s \in [0,1]$ with $t \neq s$. Then, by assumptions (i) and (ii), we obtain

$$\begin{aligned}
&\frac{|(\mathcal{T}x)(t) - (\mathcal{T}x)(s)|}{|t-s|^\beta}\\
&= \frac{\left|p(t) + x(t)\int_0^1 k(t,\tau)\, x(r(\tau))\, d\tau - p(s) - x(s)\int_0^1 k(s,\tau)\, x(r(\tau))\, d\tau\right|}{|t-s|^\beta}\\
&\le \frac{|p(t)-p(s)|}{|t-s|^\beta} + \frac{\left|x(t)\int_0^1 k(t,\tau)\, x(r(\tau))\, d\tau - x(s)\int_0^1 k(t,\tau)\, x(r(\tau))\, d\tau\right|}{|t-s|^\beta}\\
&\quad + \frac{\left|x(s)\int_0^1 k(t,\tau)\, x(r(\tau))\, d\tau - x(s)\int_0^1 k(s,\tau)\, x(r(\tau))\, d\tau\right|}{|t-s|^\beta}\\
&\le \frac{|p(t)-p(s)|}{|t-s|^\beta} + \frac{|x(t)-x(s)|}{|t-s|^\beta}\int_0^1 |k(t,\tau)|\, |x(r(\tau))|\, d\tau\\
&\quad + \frac{|x(s)|\int_0^1 |k(t,\tau)-k(s,\tau)|\, |x(r(\tau))|\, d\tau}{|t-s|^\beta}\\
&\le \frac{|p(t)-p(s)|}{|t-s|^\beta} + \frac{|x(t)-x(s)|}{|t-s|^\beta}\|x\|_\infty \int_0^1 |k(t,\tau)|\, d\tau\\
&\quad + \frac{\|x\|_\infty \cdot \|x\|_\infty \int_0^1 |k(t,\tau)-k(s,\tau)|\, d\tau}{|t-s|^\beta}\\
&\le \frac{|p(t)-p(s)|}{|t-s|^\beta} + K\|x\|_\infty \frac{|x(t)-x(s)|}{|t-s|^\beta} + \frac{\|x\|_\infty^2 \int_0^1 K_\beta |t-s|^\beta\, d\tau}{|t-s|^\beta}\\
&\le H_p^\beta + K\|x\|_\infty H_x^\beta + K_\beta \|x\|_\infty^2.
\end{aligned}$$

By Lemma 2.1, since $\|x\|_\infty \le \|x\|_\beta$ and, as $H_x^\beta \le \|x\|_\beta$, we infer that

$$\frac{|(\mathcal{T}x)(t) - (\mathcal{T}x)(s)|}{|t-s|^\beta} \le H_p^\beta + (K + K_\beta)\|x\|_\beta^2.$$

Therefore,

$$
\begin{aligned}
\|\mathcal{T}x\|_\beta &= |(\mathcal{T}x)(0)| + \sup\Big\{\frac{|(\mathcal{T}x)(t)-(\mathcal{T}x)(s)|}{|t-s|^\beta} : t,s\in[0,1], t\neq s\Big\}\\
&\le |(\mathcal{T}x)(0)| + H_p^\beta + (K+K_\beta)\|x\|_\beta^2\\
&\le |p(0)| + |x(0)|\int_0^1 |k(0,\tau)|\,|x(r(\tau))|\,d\tau + H_p^\beta + (K+K_\beta)\|x\|_\beta^2\\
&\le \|p\|_\beta + \|x\|_\infty\cdot\|x\|_\infty\int_0^1 |k(0,\tau)|\,d\tau + (K+K_\beta)\|x\|_\beta^2\\
&\le \|p\|_\beta + K\|x\|_\beta^2 + (K+K_\beta)\|x\|_\beta^2\\
&= \|p\|_\beta + (2K+K_\beta)\|x\|_\beta^2 < \infty.
\end{aligned}
\tag{3.1}
$$

This proves that the operator $\mathcal{T}$ maps $H_\beta[0,1]$ into itself.

Taking into account that the inequality

$$\|p\|_\beta + (2K+K_\beta)r^2 < r$$

is satisfied for values between the numbers

$$r_1 = \frac{1-\sqrt{1-4\|p\|_\beta(2K+K_\beta)}}{2(2K+K_\beta)}$$

and

$$r_2 = \frac{1-\sqrt{1+4\|p\|_\beta(2K+K_\beta)}}{2(2K+K_\beta)}$$

which are positive by assumption (iv), consequently, from (3.1) it follows that $\mathcal{T}$ transforms the ball $B_{r_0}^\beta = \{x\in H_\beta[0,1] : \|x\|_\beta \le r_0\}$ into itself, for any $r_0\in[r_1,r_2]$; i.e., $\mathcal{T} : B_{r_0}^\beta \to B_{r_0}^\beta$, where $r_1\le r_0\le r_2$.

By Theorem 2.3, we have that the set $B_{r_0}^\beta$ is relatively compact in $H_\alpha[0,1]$ for any $0<\alpha<\beta\le 1$. Moreover, we can prove that $B_{r_0}^\beta$ is a compact subset in $H_\alpha[0,1]$ for any $0<\alpha<\beta\le1$ (see Appendix).

Next, we will prove that the operator $\mathcal{T}$ is continuous on $B_{r_0}^\beta$, where in $B_{r_0}^\beta$ we consider the induced norm by $\|\cdot\|_\alpha$, where $0<\alpha<\beta\le 1$. To do this, we fix $x\in B_{r_0}^\beta$ and $\varepsilon>0$. Suppose that $y\in B_{r_0}^\beta$ and $\|x-y\|_\alpha\le\delta$, where δ is a positive number such that $\delta < \frac{\varepsilon}{2(2K+3K_\beta)r_0}$.

Then, for any $t,s\in[0,1]$ with $t\neq s$, we have

$$
\begin{aligned}
&\frac{|[(\mathcal{T}x)(t)-(\mathcal{T}y)(t)]-[(\mathcal{T}x)(s)-(\mathcal{T}y)(s)]|}{|t-s|^\alpha}\\
&= \Big|\frac{\big[x(t)\int_0^1 k(t,\tau)\,x(r(\tau))\,d\tau - y(t)\int_0^1 k(t,\tau)\,y(r(\tau))\,d\tau\big]}{|t-s|^\alpha}\\
&\quad - \frac{\big[x(s)\int_0^1 k(s,\tau)\,x(r(\tau))\,d\tau - y(s)\int_0^1 k(s,\tau)\,y(r(\tau))\,d\tau\big]}{|t-s|^\alpha}\Big|\\
&\le \Big|\frac{\big[x(t)\int_0^1 k(t,\tau)\,x(r(\tau))\,d\tau - y(t)\int_0^1 k(t,\tau)\,x(r(\tau))\,d\tau\big]}{|t-s|^\alpha}\\
&\quad + \frac{\big[y(t)\int_0^1 k(t,\tau)\,x(r(\tau))\,d\tau - y(t)\int_0^1 k(t,\tau)\,y(r(\tau))\,d\tau\big]}{|t-s|^\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\big[x(s)\int_0^1 k(s,\tau)\ x(r(\tau))\,d\tau - y(s)\int_0^1 k(s,\tau)\ x(r(\tau))\,d\tau\big]}{|t-s|^\alpha}\\
&- \frac{\big[y(s)\int_0^1 k(s,\tau)\ x(r(\tau))\,d\tau - y(s)\int_0^1 k(s,\tau)\ y(r(\tau))\,d\tau\big]}{|t-s|^\alpha}\Big|\\
&= \frac{1}{|t-s|^\alpha}\Big|(x(t)-y(t))\int_0^1 k(t,\tau)\ x(r(\tau))\,d\tau + y(t)\int_0^1 k(t,\tau)(x(r(\tau)-y(r(\tau)))\,d\tau\\
&- (x(s)-y(s))\int_0^1 k(s,\tau)\ x(r(\tau))\,d\tau - y(s)\int_0^1 k(s,\tau)\ (x(r(\tau))-y(r(\tau)))\,d\tau\Big|\\
&\le \frac{1}{|t-s|^\alpha}\Big\{|(x(t)-y(t))-(x(s)-y(s))|\cdot\Big|\int_0^1 k(t,\tau)\ x(r(\tau))\,d\tau\Big|\\
&+ |x(s)-y(s)|\cdot\Big|\int_0^1 (k(t,\tau)-k(s,\tau))\ x(r(\tau))\,d\tau\Big|\\
&+ \Big|y(t)\int_0^1 k(t,\tau)(x(r(\tau)-y(r(\tau)))\,d\tau - y(s)\int_0^1 k(s,\tau)(x(r(\tau)-y(r(\tau)))\,d\tau\Big|\Big\}\\
&\le \frac{|(x(t)-y(t))-(x(s)-y(s))|}{|t-s|^\alpha}\|x\|_\infty\int_0^1 |k(t,\tau)|\,d\tau\\
&+ \big[|(x(s)-y(s))-(x(0)-y(0))| + |x(0)-y(0)|\big]\|x\|_\infty\\
&\times\int_0^1 \frac{|k(t,\tau)-k(s,\tau)|}{|t-s|^\alpha}d\tau + \frac{1}{|t-s|^\alpha}\Big|y(t)\\
&\times\int_0^1 k(t,\tau)(x(r(\tau)-y(r(\tau)))\,d\tau - y(s)\int_0^1 k(t,\tau)(x(r(\tau)-y(r(\tau)))\,d\tau\Big|\\
&+ \frac{1}{|t-s|^\alpha}\Big|y(s)\int_0^1 k(t,\tau)(x(r(\tau)-y(r(\tau)))\,d\tau\\
&- y(s)\int_0^1 k(s,\tau)(x(r(\tau)-y(r(\tau)))\,d\tau\Big|\\
&\le K\|x-y\|_\alpha\|x\|_\infty + \sup_{p,q\in[0,1]}|(x(p)-y(p))-(x(q)-y(q))|\\
&\times\|x\|_\infty\int_0^1 \frac{K_\beta|t-s|^\beta}{|t-s|^\alpha}d\tau + |x(0)-y(0)|\|x\|_\infty\int_0^1 \frac{K_\beta|t-s|^\beta}{|t-s|^\alpha}d\tau\\
&+ \frac{|y(s)-x(s)|}{|t-s|^\alpha}\int_0^1 |k(t,\tau)|\ |x(r(\tau)-y(r(\tau))|\,d\tau\\
&+ |y(s)|\int_0^1 \frac{|k(t,\tau)-k(s,\tau)|}{|t-s|^\alpha}|x(r(\tau)-y(r(\tau))|\,d\tau\\
&\le K\|x\|_\infty\|x-y\|_\alpha + \|x\|_\infty K_\beta|t-s|^{\beta-\alpha}\\
&\times\sup_{p,q\in[0,1],\ p\ne q}\Big\{\frac{|(x(p)-y(p))-(x(q)-y(q))|}{|p-q|^\alpha}|p-q|^\alpha\Big\}\\
&+ K_\beta\|x\|_\beta|t-s|^{\beta-\alpha}|x(0)-y(0)|\\
&+ KH_y^\alpha\|x-y\|_\infty + \|y\|_\infty\ \|x-y\|_\infty\int_0^1 \frac{K_\beta\ |t-s|^\beta}{|t-s|^\alpha}d\tau\\
&\le K\|x\|_\beta\|x-y\|_\alpha + 2K_\beta\|x\|_\beta\|x-y\|_\alpha + K\|y\|_\alpha\|x-y\|_\alpha + K_\beta\|y\|_\alpha\|x-y\|_\alpha
\end{aligned}
$$

$$\leq \big(K\|x\|_\beta + 2K_\beta\|x\|_\beta + K\|y\|_\alpha + K_\beta\|y\|_\alpha\big)\|x-y\|_\alpha.$$

Since $\|y\|_\alpha \leq \|y\|_\beta$ (see, Lemma 2.2) and $x, y \in B^\beta_{r_0}$, from the above inequality we infer that

$$\begin{aligned}\frac{|[(\mathcal{T}x)(t)-(\mathcal{T}y)(t)]-[(\mathcal{T}x)(s)-(\mathcal{T}y)(s)]|}{|t-s|^\alpha} &\leq (2Kr_0 + 3K_\beta r_0)\|x-y\|_\alpha \\ &\leq (2Kr_0+3K_\beta r_0)\delta < \frac{\varepsilon}{2}.\end{aligned} \tag{3.2}$$

On the other hand,

$$\begin{aligned}|(\mathcal{T}x)(0)-(\mathcal{T}y)(0)| &= \Big|x(0)\int_0^1 k(0,\tau)\; x(r(\tau))\,d\tau - y(0)\int_0^1 k(0,\tau)y(r(\tau))\,d\tau\Big| \\ &\leq \Big|x(0)\int_0^1 k(0,\tau)\; x(r(\tau))\,d\tau - x(0)\int_0^1 k(0,\tau)\; y(r(\tau))\,d\tau\Big| \\ &\quad + \Big|x(0)\int_0^1 k(0,\tau)\; y(r(\tau))\,d\tau - y(0)\int_0^1 k(0,\tau)\; y(r(\tau))\,d\tau\Big| \\ &\leq \Big|x(0)\int_0^1 k(0,\tau)(x(r(\tau))-y(r(\tau)))\,d\tau\Big| \\ &\quad + \Big|(x(0)-y(0))\int_0^1 k(0,\tau)\; y(r(\tau)))\,d\tau\Big| \\ &\leq K\|x\|_\infty\|x-y\|_\infty + K\|y\|_\infty\|x-y\|_\alpha \\ &\leq K\|x\|_\beta\|x-y\|_\alpha + K\|y\|_\beta\|x-y\|_\alpha \\ &\leq 2Kr_0\|x-y\|_\alpha < 2Kr_0\delta < \frac{\varepsilon}{2}.\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), it follows that

$$\begin{aligned}&\|\mathcal{T}x-\mathcal{T}y\| \\ &= |(\mathcal{T}x)(0)-(\mathcal{T}y)(0)| \\ &\quad + \sup\Big\{\frac{|((\mathcal{T}x)(t)-(\mathcal{T}y)(t))-((\mathcal{T}x)(s)-(\mathcal{T}y)(s))|}{|t-s|^\alpha} : t,s\in[0,1],\; t\neq s\Big\} \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This proves that the operator $\mathcal{T}$ is continuous at the point $x \in B^\delta_{r_0}$ for the norm $\|\|_\alpha$. Since $B^\delta_{r_0}$ is compact in $H_\alpha[0,1]$, applying the classical Schauder fixed point theorem we obtain the desired result. □

4. Example

To present an example illustrating our result we need some previous results.

Definition 4.1. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be subadditive if $f(x+y) \leq f(x) + f(y)$ for any $x, y \in \mathbb{R}_+$.

Lemma 4.2. *Suppose that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is subadditive and $y \leq x$ then $f(x) - f(y) \leq f(x-y)$.*

Proof. Since $f(x) = f(x-y+y) \leq f(x-y) + f(y)$ the result follows. □

Remark 4.3. From Lemma 4.2, we infer that if $f:\mathbb{R}_+\to\mathbb{R}_+$ is subadditive then $|f(x)-f(y)|\le f(|x-y|)$ for any $x,y\in\mathbb{R}_+$.

Lemma 4.4. *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be a concave function with $f(0)=0$. Then f is subadditive.*

Proof. For $x,y\in\mathbb{R}_+$ and, since f is concave and $f(0)=0$, we have

$$\begin{aligned} f(x) &= f\left(\frac{x}{x+y}(x+y)+\frac{y}{x+y}\cdot 0\right)\\ &\ge \frac{x}{x+y}f(x+y)+\frac{y}{x+y}f(0)\\ &= \frac{x}{x+y}f(x+y) \end{aligned}$$

and

$$\begin{aligned} f(y) &= f\left(\frac{x}{x+y}\cdot 0+\frac{y}{x+y}(x+y)\right)\\ &\ge \frac{x}{x+y}f(0)f(x+y)+\frac{y}{x+y}f(x+y)\\ &= \frac{y}{x+y}f(x+y). \end{aligned}$$

Adding these inequalities, we obtain

$$f(x)+f(y)\ge \frac{x}{x+y}f(x+y)+\frac{y}{x+y}f(x+y)=f(x+y).$$

This completes the proof. □

Remark 4.5. Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be the function defined by $f(x)=\sqrt[p]{x}$, where $p>1$. Since this function is concave (because $f''(x)\le 0$ for $x>0$) and $f(0)=0$, Lemma 4.4 says us that f is subadditive. By Remark 4.3, we have

$$|f(x)-f(y)|=|\sqrt[p]{x}-\sqrt[p]{y}|\le \sqrt[p]{|x-y|}$$

for any $x,y\in\mathbb{R}_+$.

Example 4.6. Let us consider the quadratic integral equation

$$x(t)=\arctan\sqrt[5]{q\sin t+\hat{q}}+x(t)\int_0^1\sqrt[4]{mt^2+\tau}\;x\Big(\frac{\tau}{\tau+1}\Big)\,d\tau,\quad t\in[0,1], \tag{4.1}$$

where, q, $\hat{q}$ and m are nonnegative constants. Notice that (4.1) is a particular case of (1.1), where $p(t)=\arctan\sqrt[5]{q\sin t+\hat{q}}$, $k(t,\tau)=\sqrt[4]{mt^2+\tau}$ and $r(\tau)=\frac{\tau}{\tau+1}$.

In what follows, we will prove that assumptions (i)-(iv) of Theorem 3.1 are satisfied. Since the inverse tangent function is concave (because its second derivative is nonpositive) and its value at zero is zero, taking into account Remarks 4.3 and 4.5, we have

$$\begin{aligned} |p(t)-p(s)| &= \left|\arctan\sqrt[5]{q\sin t+\hat{q}}-\arctan\sqrt[5]{q\sin s+\hat{q}}\right|\\ &\le \arctan\left(\left|\sqrt[5]{q\sin t+\hat{q}}-\sqrt[5]{q\sin s+\hat{q}}\right|\right)\\ &\le \left|\sqrt[5]{q\sin t+\hat{q}}-\sqrt[5]{q\sin s+\hat{q}}\right|\\ &\le \sqrt[5]{q|\sin t-\sin s|}\\ &\le \sqrt[5]{q}\;|t-s|^{1/5}, \end{aligned}$$

where we have used that $\arctan x \le x$ for $x \ge 0$ and $|\sin x - \sin y| \le |x-y|$ for any $x, y \in \mathbb{R}$. This says that $p \in H_{\frac{1}{5}}[0,1]$ and, moreover, $H_p^{1/5} = \sqrt[5]{q}$. Therefore, assumptions (i) of Theorem 3.1 is satisfied.

Note that

$$\begin{aligned}\|p\|_{\frac{1}{5}} &= |p(0)| + \sup\Big\{\frac{|p(t)-p(s)|}{|t-s|^{1/5}} : t, s \in [0,1],\ t \ne s\Big\}\\ &\le \arctan \sqrt[5]{\hat{q}} + H_p^{1/5}\\ &= \arctan \sqrt[5]{\hat{q}} + \sqrt[5]{q}.\end{aligned}$$

Since for any $t, s, \tau \in [0,1]$, we have (see, Remark 4.5)

$$\begin{aligned}|k(t,\tau) - k(s,\tau)| &= \big|\sqrt[4]{mt^2+\tau} - \sqrt[4]{ms^2+\tau}\big|\\ &\le \sqrt[4]{|mt^2 - ms^2|}\\ &= \sqrt[4]{m}\sqrt[4]{|t^2-s^2|}\\ &= \sqrt[4]{m}\sqrt[4]{t+s}\sqrt[4]{|t-s|}\\ &\le \sqrt[4]{m}\sqrt[4]{2}\ |t-s|^{1/4}\\ &= \sqrt[4]{m}\sqrt[4]{2}\ |t-s|^{1/20}\ |t-s|^{1/5}\\ &\le \sqrt[4]{2m}\ |t-s|^{1/5},\end{aligned}$$

assumption (ii) of Theorem 3.1 is satisfied with $K_\beta = K_{\frac{1}{5}} = \sqrt[4]{2m}$.

It is clear that $r(\tau) = \frac{\tau}{\tau+1}$ satisfies assumption (iii).

In our case, the constant K is given by

$$\begin{aligned}K &= \sup\Big\{\int_0^1 |k(t,\tau)|\,d\tau : t \in [0,1]\Big\}\\ &= \sup\Big\{\int_0^1 \sqrt[4]{mt^2+\tau}\,d\tau : t \in [0,1]\Big\}\\ &= \sup\Big\{\frac{4}{5}\Big(\sqrt[4]{(mt^2+1)^5} - \sqrt[4]{m^5t^{10}}\Big)\Big\}\\ &= \frac{4}{5}\Big(\sqrt[4]{(m+1)^5} - \sqrt[4]{m^5}\Big).\end{aligned}$$

Therefore, the inequality appearing in assumption (iv) takes the form

$$\|p\|_{\frac{1}{5}}(2K + K_\beta) = \Big(\arctan \sqrt[5]{\hat{q}} + \sqrt[5]{q}\Big)\Big(\frac{8}{5}\big[\sqrt[4]{(m+1)^5} - \sqrt[4]{m^5}\big] + \sqrt[4]{2m}\Big) < \frac{1}{4}.$$

It is easily seen that the above inequality is satisfied when, for example, $\hat{q} = 0$, $q = \frac{1}{2^{20}}$ and $m = 1$. Therefore, using Theorem 3.1, we infer that (4.1) for $\hat{q} = 0$, $q = \frac{1}{2^{20}}$ and $m = 1$ has at least one solution in the space $H_\alpha[0,1]$ with $0 < \alpha < 1/5$.

Note that in (4.1), we can take as $r(\tau)$ a particular functions such as $r(\tau) = \{e^\tau\}$, where $\{\cdot\}$ denotes the fractional part.

Remark 4.7. Note that any solution $x(t)$ of (1.1), i.e.,

$$x(t) = p(t) + x(t)\int_0^1 k(t,\tau)\ x(r(\tau))\,d\tau, \quad t \in [0,1],$$

satisfies that its zeroes are also zeroes of $p(t)$. From this, we infer that if $p(t) \neq 0$ for any $t \in [0,1]$ then $x(t) \neq 0$ for any $t \in [0,1]$. By Bolzano's theorem, this means that the solution $x(t)$ of Eq(1.1) does not change of sign on $[0,1]$ when $p(t) \neq 0$ for any $t \in [0,1]$. These questions seem to be interesting from a practical standpoint.

5. Appendix

Suppose that $0 < \alpha < \beta \leq 1$ and by B_r^β we denote the ball centered at θ and radius r in the space $H_\beta[a,b]$; i.e., $B_r^\beta = \{x \in H_\beta[a,b] : \|x\|_\beta \leq r\}$. Then B_r^β is a compact subset in the space $H_\alpha[a,b]$.

In fact, by Theorem 2.3, since B_r^β is a bounded subset in $H_\beta[a,b]$, B_r^β is a relatively compact subset of $H_\alpha[a,b]$. In the sequel, we will prove that B_r^β is a closed subset of $H_\alpha[a,b]$. Suppose that $(x_n) \subset B_r^\beta$ and $x_n \xrightarrow{\|\|_\alpha} x$ with $x \in H_\alpha[a,b]$. We have to prove that $x \in B_r^\beta$.

Since $x_n \xrightarrow{\|\cdot\|_\alpha} x$, for $\varepsilon > 0$ given we can find $n_0 \in \mathbb{N}$ such that $\|x_0 - x\|_\alpha \leq \varepsilon$ for any $n \geq n_0$, or, equivalently,

$$|x_n(a) - x(a)| + \sup\Big\{\frac{|(x_n(t) - x(t)) - (x_n(s) - x(s))|}{|t-s|^\alpha} : t, s \in [a,b],\ t \neq s\Big\} < \varepsilon, \tag{5.1}$$

for any $n \geq n_0$. Particularly, this implies that $x_n(a) \to x(a)$. Moreover, if in (5.1) we put $s = a$ then we get

$$\sup\Big\{\frac{|(x_n(t) - x(t)) - (x_n(a) - x(a))|}{|t-a|^\alpha} : t, s \in [a,b],\ t \neq a\Big\} < \varepsilon, \text{ for any } n \geq n_0.$$

This says that

$$|(x_n(t) - x(t)) - (x_n(a) - x(a))| < \varepsilon |t-a|^\alpha, \quad \text{for any } n \geq n_0 \text{ and for any } t \in [a,b]. \tag{5.2}$$

Therefore, for any $n \geq n_0$ and any $t \in [a,b]$ by (5.1) and (5.2), we have

$$\begin{aligned}
|x_n(t) - x(t)| &\leq |(x_n(t) - x(t)) - (x_n(a) - x(a))| + |x_n(a) - x(a)| \\
&< \varepsilon (t-a)^\alpha + \varepsilon \\
&= \varepsilon (1 + (b-a)^\alpha).
\end{aligned}$$

From this, it follows that

$$\|x_n - x\|_\infty \to 0. \tag{5.3}$$

Next, we will prove that $x \in B_r^\beta$. In fact, as $(x_n) \subset B_r^\beta \subset H_\beta[a,b]$, we have that

$$\frac{|x_n(t) - x_n(s)|}{|t-s|^\beta} \leq r$$

for any $t, s \in [a,b]$ with $t \neq s$. Consequently,

$$|x_n(t) - x_n(s)| \leq r|t-s|^\beta$$

for any $t, s \in [a,b]$. Letting $n \to \infty$ and taking into account (5.3), we obtain

$$|x(t) - x(s)| \leq r|t-s|^\beta$$

for any $t, s \in [a,b]$. Therefore,

$$\frac{|x(t) - x(s)|}{|t-s|^\beta} \leq r$$

for any $t, s \in [a, b]$ with $t \neq s$, and this means that $x \in B_r^\beta$. This completes the proof.

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