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SOLUTIONS TO SYSTEMS OF ARBITRARY-ORDER DIFFERENTIAL EQUATIONS IN COMPLEX DOMAINS

RABHA W. IBRAHIM

Abstract. In this article, we study the existence of solutions for a three dimensional fractional system involving seven coefficients. We prove that the system has a strong global solution which is unique in an appropriate function space. We use a method based on analytic technique from the fixed point theory, along with the fractional Duhamel principle.

1. Introduction

Fractional calculus (integrals and derivatives) of any positive order can be considered as a branch of mathematical physics, associated with differential equations, integral equations and integro-differential equations, where integrals are of convolution form with weak singular kernels of power law type. It has gained more and more interest in applications in several fields of applied sciences. Fractional differential equations (real and complex) are viewed as models for nonlinear differential equations; varieties of them play important roles, not only in mathematics, but also in physics, dynamical systems, control systems and engineering, to create the mathematical modeling of many physical phenomena. Furthermore, they are employed in social science, such as, food supplement, climate and economics. Fractional models have been studied by many researchers, to sufficiently describe the operation of variety of computational, physical and biological processes and systems. Accordingly, considerable attention has been paid to the outcomes of fractional differential equations, integral equations and fractional partial differential equations of physical phenomena. Most of these fractional differential equations have analytic solutions, approximation and numerical techniques [8, 9, 11, 13, 14].

In current years, researchers have introduced and studied several types of nonlinear systems with complex variables. These systems, which involve complex variables, are employed to describe the physics of a detuned laser, rotating fluids, disk dynamos, electronic circuits, and particle beam dynamics in high energy accelerators. As special model, the chaotic complex system is used to describe and simulate the physics of detuned lasers and thermal convection of liquid flows. This model corresponds to the equilibrium state of the atmosphere, in which surfaces of constant density are not parallel to the surface of constant gravitational potential [3, 5, 6].

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Existence and uniqueness of solutions are studied widely in the field of fractional differential equations [1, 2, 4, 7, 10, 12, 15, 16]. In this work, we study fractional system involving seven coefficients in complex spaces. We show that the proposed system has a global solution in appropriate functional spaces. This is strong and unique solution. We employ a method, based on analytic methods from the fixed point theory together with the fractional Duhamel principle.

2. Fractional calculus

The idea of the fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) was planted over 300 years ago. In 1823, Abel investigated the generalized tautochrone problem, and he was the pioneer to apply fractional calculus techniques in a physical problem. Later, Liouville has applied fractional calculus to solve problems in potential theory. Since then, the fractional calculus has triggered the attention of many researchers in all area of sciences. The following section concerns with some preliminaries and notation regarding the fractional calculus.

Definition 2.1. The fractional (arbitrary) order integral of the function f of order $\alpha > 0$ is defined by

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau.$$

When $a = 0$, we write $I_a^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $(*)$ denotes the convolution product (see [11]), $\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$ and $\phi_\alpha(t) = 0$, $t \le 0$ and $\phi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 2.2. The fractional (arbitrary) order derivative of the function f of order $0 \le \alpha < 1$ is defined by

$$D_a^\alpha f(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau) d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

In the sequel, we shall use the notation ∂_t^α.

Remark 2.3. From Definitions 2.1 and 2.2, for $a = 0$, we have

$$\begin{aligned} D^\alpha t^\mu &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \quad \mu > -1;\ 0 < \alpha < 1, \\ I^\alpha t^\mu &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \quad \mu > -1;\ \alpha > 0. \end{aligned}$$

The Leibniz rule is

$$D_a^\alpha [f(t)g(t)] = \sum_{k=0}^{\infty} \binom{\alpha}{k} D_a^{\alpha-k} f(t) D_a^k g(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} D_a^{\alpha-k} g(t) D_a^k f(t),$$

where

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1-k)}.$$

3. Fractional system

In this section, we propose a one-dimensional setting, which physically corresponds to the consideration of a Laminar-Couette flow. This type of flow appropriately models flows in shear rheometers. Our three unknown fields, the velocity u, the shear stress ϕ and the fluidity f are defined as functions of a space variable $z \in U := \{z \in \mathbb{C}, |z| \leq 1\}$. They are also, functions of the time $t \geq 0$. The system can be read as

$$\rho\partial_t^\alpha u(t,z) = \eta u_{zz} + \phi_z, \tag{3.1}$$

$$\lambda\partial_t^\alpha \phi(t,z) = Gu_z - f\phi + G\ell, \tag{3.2}$$

$$\partial_t^\alpha f(t,z) = (-1 + \xi|\phi|)f^2 - \nu f^3, \tag{3.3}$$

where $\alpha \in (0,1)$, ρ is the density, η is the viscosity, λ is the characteristic relaxation time, G is the elastic modulus, ℓ is a constant scalar in $[0,\infty)$ and ξ and ν are the evolution of the fluidity f. In the sequel, we assume that u, ϕ and f are analytic with $|f| \leq 1$. System (3.1)–(3.3) is classified as a fully coupled system of three equations and seven-dimensionless coefficients. All the above coefficients are positive and constant in time. The first one is the equation of conservation of momentum for u. The second equation rules the evolution of the shear stress ϕ. The third equation is of the form evolution equations suggested by many authors. The first two equations are classical in nature, while the last equation, may differ from one model to another. Assume that system (3.1)–(3.3) supplied with initial conditions (u_0, ϕ_0, f_0) and the homogeneous boundary conditions $u(t,0) = 0$ and $u(t,1) = 0$.

The dimension of a basic physical quantity can be formulated as a product of the basic physical dimensions: length, mass, electric charge, absolute temperature and time symbolled by sans-serif symbols L, M, Q, Θ, and T, respectively, each raised to a rational power. Other physical quantities can be described in phrases of these fundamental physical dimensions. For example, speed has the dimension length (or distance) per unit of time, similarly for velocity, stress and fluidity. Usually these depending physical quantities need constant coefficients. In general, these coefficients are constant with respect to time, therefore they have positive values.

4. Existence and uniqueness

In this section, we establish the existence and uniqueness of a solution for (3.1)–(3.3).

Theorem 4.1. *Consider* (3.1)–(3.3) *with initial condition* $(u_0, \phi_0, f_0) \in H^1(U)^3$ *with* $\Re(f_0) \geq 0$. *If* $\frac{T^\alpha(1+\nu)}{\Gamma(\alpha+1)} < 1$ *then there exists a unique global solution* (u, ϕ, f) *for* (3.1)–(3.3) *subjected with the boundary condition* $u(t,0) = 0$ *and* $u(t,1) = 0$, *such that*

$$(u,\phi,f) \in \big(C([0,T];H^1) \cap L^2([0,T];H^2) \times C([0,T];H^1) \times C([0,T];H^1)\big) \tag{4.1}$$

and $\Re(f) \geq 0$ *for all* $z \in U$ *and* $t \in [0,T]$. *Moreover,*

$$(\partial_t^\alpha u, \partial_t^\alpha \phi, \partial_t^\alpha f) \in \big(L^2([0,T];L^2) \times C([0,T];L^2) \times C([0,T];L^2)\big). \tag{4.2}$$

The proof consists of eight steps. The first five steps derive the form of the solution while Step 6 describes the sequence of the approximate solution. The convergence of this sequence is proven in Step 7, thereby the existence of a solution is established to (3.1)–(3.3). Step 8 addresses uniqueness.

Step 1: Positivity. We prove that $\Re(f) \geq 0$. Define the set

$$U_0 = \{z \in U : \Re(f_0) > 0\}.$$

For $z \in U \backslash U_0$, we receive $\Re(f_0(z)) = 0$ and thus, from (3.3), $\Re(f(t,z)) = 0$ for all time $t \in [0,T]$. Now let $z \in U_0$ we proceed to prove that $\Re(f) > 0$. We dispute by contradiction and assume, by continuity of $f(.,z)$, that

$$t_m = \inf\{t \in [0,T], f(t,z) = 0\} < T.$$

The Cauchy-Lipschitz theorem employed to (3.3) with zero as initial condition at time t_m implies that $f(t,z) = 0$ for $t \in (t_m - \varepsilon, t_m + \varepsilon)$ and $\varepsilon > 0$, which contradicts the definition of t_m. Hence $\Re(f) \geq 0$.

Step 2: Boundedness. From the evolution equation (3.1) on u, we obtain

$$\frac{1}{\alpha+1}\rho\partial_t^\alpha \|u(t,.)\|_{L^2}^2 + \eta\|u_z(t,.)\|_{L^2}^2 = \int_U (\phi_z u)(t,.). \tag{4.3}$$

Similarly, the evolution equation (3.2) implies

$$\frac{1}{\alpha+1}\lambda\partial_t^\alpha \|\phi(t,.)\|_{L^2}^2 + \|\sqrt{f}\phi(t,.)\|_{L^2}^2 = G\int_U (\phi u_z \phi)(t,.) + G\ell\widehat{\phi}, \tag{4.4}$$

where

$$\widehat{g}(t) = \int_U g(t,z)dz.$$

Combining estimates (4.3) and (4.4) and using the fact that u vanishes on the boundary, yields

$$\frac{1}{\alpha+1}\partial_t^\alpha [G\rho\|u(t,.)\|_{L^2}^2 + \lambda\|\phi(t,.)\|_{L^2}^2] + \|\sqrt{f}\phi(t,.)\|_{L^2}^2 + G\eta\|u_z(t,.)\|_{L^2}^2 = G\ell\widehat{\phi}(t). \tag{4.5}$$

Now integrating (3.3) over U implies

$$\partial_t^\alpha \|f(t,.)\|_{L^1} + \|f(t,.)\|_{L^2}^2 + \nu\|f(t,.)\|_{L^3}^3 = \xi\int_U (|\phi|f^2)(t,.). \tag{4.6}$$

By the Young inequality, we have

$$\xi|\phi|f^2 = \sqrt{\nu}|f|^{3/2}\frac{\xi}{\sqrt{\nu}}|\phi||f|^{1/2} \leq \frac{\nu}{2}|f|^3 + \frac{\xi^2}{2\nu}|f|\phi^2$$

and hence we have

$$\partial_t^\alpha \|f(t,.)\|_{L^1} + \|f(t,.)\|_{L^2}^2 + \frac{\nu}{2}\|f(t,.)\|_{L^3}^3 \leq \frac{\xi^2}{2\nu}\|\sqrt{f}\phi(t,.)\|_{L^2}^2. \tag{4.7}$$

Summing (4.5) and (4.7), we obtain

$$\begin{aligned}
&\frac{1}{\alpha+1}\partial_t^\alpha [G\rho\|u(t,.)\|_{L^2}^2 + \lambda\|\phi(t,.)\|_{L^2}^2 + \frac{2\nu}{\xi^2}\|f(t,.)\|_{L^1}] \\
&+ \frac{1}{2}\|\sqrt{f}\phi(t,.)\|_{L^2}^2 + G\eta\|u_z(t,.)\|_{L^2}^2 \\
&\leq K\ell\|\phi(t,.)\|_{L^2},
\end{aligned} \tag{4.8}$$

where K is a positive constant depending on the coefficients $G, \eta, \lambda, \nu, \xi$ and ρ. By applying the fact that

$$\|\phi(t,.)\|_{L^2} \leq \frac{\|\phi(t,.)\|_{L^2}^2 + 1}{2}$$

and using the generalized Gronwall lemma to (4.8), we attain

$$\begin{aligned}\sup_{t\in[0,T]}&[\|u(t,.)\|_{L^2}^2+\|\phi(t,.)\|_{L^2}^2+\|f(t,.)\|_{L^1}]\\&+\frac{T^{\alpha-1}}{\Gamma(\alpha)}\int_0^T\Big(\|\sqrt{f}\phi(t,.)\|_{L^2}^2+\|u_z(t,.)\|_{L^2}^2\Big)dt\le\widetilde{K},\end{aligned}\tag{4.9}$$

where $\widetilde{K}$ is a positive constant depending on the seven coefficients G, T, u_0, ϕ_0, f_0, α, η, λ, ν, ξ, ρ and ℓ. Note that when $\ell=0$, in (4.8), then $\widetilde{K}$ does not depend on T and hence, we have uniform bounds in time.

Step 3: Auxiliary functions. Define a function q as follows

$$q(t,z)=\int_0^z\big(\phi(t,\zeta)-\widehat{\phi}(t)\big)d\zeta$$

satisfying the Dirichlet boundary conditions which solves

$$\frac{\partial^2 q}{\partial z^2}=\frac{\partial\phi}{\partial z}.$$

Applying (3.1) and (3.2), which respectively impose

$$\rho\partial_t^\alpha u=\eta\frac{\partial^2}{\partial z^2}(u+\frac{1}{\eta}q)$$

and

$$\lambda\partial_t^\alpha q=-\int_0^z\big(f\phi(t,\zeta)-\widehat{f\phi}(t)\big)d\zeta+Gu.$$

Define the function

$$\Lambda=u+\frac{1}{\eta}\int_0^z(\phi-\widehat{\phi})=u+\frac{1}{\eta}q.\tag{4.10}$$

A fractional derivative yields

$$\begin{aligned}\partial_t^\alpha\Lambda&=\partial_t^\alpha u+\frac{1}{\eta}\partial_t^\alpha q\\&=\frac{\eta}{\rho}\frac{\partial^2}{\partial z^2}(u+\frac{1}{\eta}q)+\frac{1}{\lambda\eta}[-\int_0^z\big(f\phi(t,\zeta)-\widehat{f\phi}(t)\big)d\zeta+Gu]\\&=\frac{\eta}{\rho}\frac{\partial^2}{\partial z^2}\Lambda-\frac{1}{\lambda\eta}\int_0^z\big(f\phi(t,\zeta)-\widehat{f\phi}(t)\big)d\zeta+\frac{G}{\eta\lambda}u.\end{aligned}\tag{4.11}$$

Multiplying by $\frac{\partial^2}{\partial z^2}\Lambda$ and integrating over U yields

$$\begin{aligned}&\frac{1}{\alpha+1}\partial_t^\alpha\|\frac{\partial\Lambda}{\partial z}(t,.)\|_{L^2}^2+\frac{\eta}{2\rho}\|\frac{\partial^2\Lambda}{\partial z^2}(t,.)\|_{L^2}^2\\&\le C\Big(\|(f\phi)(t,.)\|_{L^1}\int_U|\frac{\partial^2\Lambda}{\partial z^2}|(t,.)+\int_U|u\frac{\partial^2\Lambda}{\partial z^2}|(t,.)\Big).\end{aligned}$$

The Young and the Cauchy-Schwartz inequalities imply that

$$\partial_t^\alpha\|\frac{\partial\Lambda}{\partial z}(t,.)\|_{L^2}^2+\|\frac{\partial^2\Lambda}{\partial z^2}(t,.)\|_{L^2}^2\le C_{\alpha,\eta,\rho}\Big(\|(f(t,.)\|_{L^1}\|\sqrt{f}\phi\|_{L^2}^2+\|u(t,.)\|_{L^2}^2\Big).\tag{4.12}$$

Since

$$\frac{\partial\Lambda}{\partial z}|_{t=0}=\frac{\partial u_0}{\partial z}+\frac{1}{\eta}(\phi_0-\widehat{\phi}_0)\in L^2(U);$$

hence, we deduce from (4.12) that

$$\Lambda \in L^\infty([0,T],H^1)\cap L^2([0,T],H^2),$$

and consequently

$$u \in L^\infty([0,T],H^1)\cap L^2([0,T],H^2).$$

Step 4: $L^\infty-$bounds. By using the definition of Λ and $\widehat{\phi}$, we rewrite (3.2) as

$$\lambda \partial_t^\alpha \phi = G\frac{\partial \Lambda}{\partial z} - \big(f + \frac{G}{\eta}\big)\phi + \frac{G}{\eta}\widehat{\phi} + G\ell.$$

Multiplying the last assertion by ϕ, we conclude that

$$\frac{\lambda}{2}\partial_t^\alpha|\phi|^2 + \big(|f| + \frac{G}{\eta}\big)|\phi|^2 \le C\Big(|\phi||\frac{\partial \Lambda}{\partial z}| + |\phi|\|\phi\|_{L^2} + \ell|\phi|\Big),$$

consequently, the Young inequality yields

$$\frac{\lambda}{2}\partial_t^\alpha|\phi|^2 + \big(|f| + \frac{G}{\eta}\big)|\phi|^2 \le C\Big(|\frac{\partial \Lambda}{\partial z}|^2 + \|\phi\|_{L^2} + \ell\Big). \tag{4.13}$$

Since $\phi_0 \in H^1(U)$ (Step 2) and $\frac{\partial \Lambda}{\partial z} \in L^2([0,T],L^\infty)$ (Step 3), then the generalized Gronwall lemma to (4.13) shows that

$$\|\phi(t,.)\|_{L^\infty} \le \widetilde{K}, \tag{4.14}$$

where $\widetilde{K}$ is defined in (4.9), that is $\phi \in L^\infty([0,T],L^\infty)$.

We proceed to prove that $f \in L^\infty([0,T],L^\infty)$. For this purpose, we apply the fractional Duhamel principle which can be found in [15]. Then (3.3) reduces to

$$\partial_t^\alpha f(t,z) = (-f - \nu f^2)f + \xi|\phi|f^2.$$

Assume that F is a solution for the problem

$$\partial_t^\alpha F + (F + \nu F^2)F = 0 \tag{4.15}$$

subjected to the initial condition

$$I^{1-\alpha}F|_{t=0} = h(0), \quad \text{where } h := \xi|\phi|f^2,\ |F| \le 1.$$

Then

$$f(t) = \int_0^t F(s)ds$$

is a solution of the problem

$$\partial_t^\alpha f(t,z) + (f + \nu f^2)f = \xi|\phi|f^2.$$

It suffices to prove that $F \in L^\infty([0,T],L^\infty)$; from (4.15), we obtain

$$\Big(1 - \frac{T^\alpha}{\Gamma(\alpha+1)}(\|F\| + \nu\|F\|^2)\Big)\|F\| \le \xi f_0^2\|\phi_0\|.$$

Since $|F| \le 1$, the above inequality becomes

$$\|F\| \le \frac{\xi f_0^2\|\phi_0\|}{\big(1 - \frac{T^\alpha(1+\nu)}{\Gamma(\alpha+1)}\big)}. \tag{4.16}$$

Hence we obtain that $F \in L^\infty([0,T],L^\infty)$ (because $\phi \in L^\infty([0,T],L^\infty)$) and consequently $f \in L^\infty([0,T],L^\infty)$.

Step 5: Second estimate bounds of u. Differentiate with respect to z the evolution equation (3.2), we have

$$\begin{aligned}\lambda\partial_t^\alpha(\frac{\partial\phi}{\partial z}) &= G\frac{\partial^2 u}{\partial z^2} - \frac{\partial\phi}{\partial z}f - \frac{\partial f}{\partial z}\phi \\ &= G\frac{\partial^2\Lambda}{\partial z^2} - \frac{G}{\eta}\frac{\partial\phi}{\partial z} - \frac{\partial\phi}{\partial z}f - \frac{\partial f}{\partial z}\phi.\end{aligned} \tag{4.17}$$

Moreover, we differentiate with respect to z the evolution equation (3.3) to obtain

$$\partial_t^\alpha(\frac{\partial f}{\partial z}) = \xi\frac{\partial|\phi|}{\partial z}f^2 + 2(\xi|\phi|-1)f\frac{\partial f}{\partial z} - 3\nu f^2\frac{\partial f}{\partial z}. \tag{4.18}$$

Multiplying (4.17) and (4.18) by $\frac{\partial\phi}{\partial z}$ and $\frac{\partial f}{\partial z}$ respectively, integrating over the domain U, summing up and using that both f and ϕ are in $L^\infty([0,T],L^\infty)$, we have

$$\begin{aligned}&\partial_t^\alpha\Big(\lambda\|\frac{\partial\phi}{\partial z}(t,.)\|_{L^2}^2 + \|\frac{\partial f}{\partial z}(t,.)\|_{L^2}^2\Big) \\ &\le K_\alpha\int_U\Big(\frac{\partial^2\Lambda}{\partial z^2}\frac{\partial\phi}{\partial z} + (\frac{\partial\phi}{\partial z})^2 + \frac{\partial f}{\partial z}\frac{\partial\phi}{\partial z} + \frac{\partial|\phi|}{\partial z}\frac{\partial f}{\partial z} + (\frac{\partial f}{\partial z})^2\Big)(t,.).\end{aligned}$$

Since $\phi\in L^2([0,T],H^1)$ then in view of the Young inequality, we have

$$\begin{aligned}&\partial_t^\alpha\Big(\lambda\|\frac{\partial\phi}{\partial z}(t,.)\|_{L^2}^2 + \|\frac{\partial f}{\partial z}(t,.)\|_{L^2}^2\Big) \\ &\le K_\alpha\Big(\|\frac{\partial\phi}{\partial z}(t,.)\|_{L^2}^2 + \|\frac{\partial f}{\partial z}(t,.)\|_{L^2}^2 + \|\frac{\partial^2\Lambda}{\partial z^2}(t,.)\|_{L^2}^2\Big).\end{aligned} \tag{4.19}$$

Since $\phi, f\in L^\infty([0,T],H^1)$ and $\phi_0, f_0\in H^1(U)$, the generalized Gronwall inequality together with (4.10) imply that $u\in L^\infty([0,T],H^1)\cap L^2([0,T],H^2)$.

Step 6. Approximate solution. In this step, we construct a sequence of approximating solutions to system (3.1)–(3.3). Consider the system

$$\rho\partial_t^\alpha u_n(t,z) = \eta\frac{\partial^2 u_n}{\partial z^2} + \frac{\partial\phi_n}{\partial z}, \tag{4.20}$$

$$\lambda\partial_t^\alpha\phi_n(t,z) = G\frac{\partial u_n}{\partial z} - f_{n-1}\phi_n + G\ell, \tag{4.21}$$

$$\partial_t^\alpha f_n(t,z) = (-1+\xi|\phi_n|)f_{n-1}f_n - \nu f_{n-1}f_n^2, \tag{4.22}$$

subjected to the boundary conditions

$$u_n(t,0)=0,\quad u_n(t,1)=0,\quad \forall\, t\in[0,T]$$

and the initial condition $(u_{n0},\phi_{n0},f_{n0}) = (u_0,\phi_0,f_0)$. Our aim is to show that (4.20)–(4.22) has a unique solution (u_n,ϕ_n,f_n) in the space

$$\big(C([0,T];H^1)\cap L^2([0,T];H^2)\times C([0,T];H^1)\times C([0,T];H^1)\big).$$

For this purpose, we split the system (4.20)–(4.22) into two subsystems (4.20)-(4.21) on the one hand and (4.22) on the other hand.

First we prove the existence of unique solution. Let (u_{n1},ϕ_{n1}) and (u_{n2},ϕ_{n2}) be two solutions in the aforementioned class; the functions $u_n = u_{n1}-u_{n2}$ and $\phi_n = \phi_{n1}-\phi_{n2}$ satisfy the system

$$\rho\partial_t^\alpha u_n(t,z) = \eta\frac{\partial^2 u_n}{\partial z^2} + \frac{\partial\phi_n}{\partial z}, \tag{4.23}$$

$$\lambda\partial_t^\alpha\phi_n(t,z) = G\frac{\partial u_n}{\partial z} - f_{n-1}\phi_n. \tag{4.24}$$

Multiplying (4.23) by u_n and integrating over the domain U, we obtain

$$\frac{1}{\alpha+1}\rho\partial_t^\alpha\|u_n(t,.)\|_{L^2}^2+\eta\|\frac{\partial u_n}{\partial z}(t,.)\|_{L^2}^2=\int_U(\frac{\partial\phi_n}{\partial z}u_n)(t,.). \tag{4.25}$$

Similarly, Multiplying (4.24) by ϕ_n and and integrating over the domain U, implies

$$\frac{1}{\alpha+1}\lambda\partial_t^\alpha\|\phi_n(t,.)\|_{L^2}^2+\|\sqrt{f_{n-1}}\phi_n(t,.)\|_{L^2}^2=G\int_U(\phi_n\frac{\partial u_n}{\partial z})(t,.), \tag{4.26}$$

Adding estimates (4.25) and (4.26) and using and using integration by parts together with the fact that u_n vanishes on the boundary, yields

$$\frac{1}{\alpha+1}\partial_t^\alpha[G\rho\|u_n(t,.)\|_{L^2}^2+\lambda\|\phi_n(t,.)\|_{L^2}^2]+\|\sqrt{f_{n-1}}\phi_n(t,.)\|_{L^2}^2+G\eta\|\frac{\partial u_n}{\partial z}(t,.)\|_{L^2}^2=0, \tag{4.27}$$

which gives $u_n=0$ and $\phi_n=0$. Hence system (4.23)–(4.24) has a unique bound uniform solution in the space $(C([0,T];H^1)\cap L^2([0,T];H^2))\times C([0,T];H^1)$.

Second we show that f_n exists in $C([0,T];H^1)$ and $\Re(f_n)\ge 0$. Equation (4.22) can be reduced to

$$\partial_t^\alpha f_n(t,z)=\Theta(t,f_n,z),\, f_n|_{t=0}=f_0, \tag{4.28}$$

where $\Theta:[0,T]\times\mathbb{C}\times U\to\mathbb{C}$. The function Θ is continuous in its first two variables and locally Lipschitz in its second variable. The Cauchy-Lipschitz theorem imposes there exists a unique local solution with $f_0(z)$ as initial condition. Let $[0,T^*)$ be the interval of existence of the maximal solution for positive time. For all $t\in[0,T^*)$, we have $\Re(f_n)\ge 0$, using Step 1. Furthermore,

$$|\partial_t^\alpha f_n(t,z)|\le\xi|\phi_n||f_{n-1}||f_n|\le\xi\|\phi_n\|_{L^\infty}\|f_{n-1}\|_{L^\infty}|f_n|; \tag{4.29}$$

using that both ϕ_n and f_{n-1} belong to $C([0,T];H^1)$. The Gronwall lemma then shows that f_n remains bounded on $[0,T^*]$ and thus we have established existence and uniqueness on $[0,T^*]$.

Now we prove the boundedness of the solution. From (4.20) and (4.21), we may have

$$\begin{aligned}&\frac{1}{\alpha+1}\partial_t^\alpha[G\rho\|u_n(t,.)\|_{L^2}^2+\lambda\|\phi_n(t,.)\|_{L^2}^2]\\&+\|\sqrt{f_{n-1}}\phi_n(t,.)\|_{L^2}^2+G\eta\|\frac{\partial u_n}{\partial z}(t,.)\|_{L^2}^2=G\ell\widehat{\phi_n}(t).\end{aligned} \tag{4.30}$$

and from (4.22), we obtain

$$\partial_t^\alpha\|f_n(t,.)\|_{L^1}+\int_U(|f_{n-1}||f_n|)(t,.)+\frac{\nu}{2}\int_U(|f_{n-1}||f|_n^2)(t,.)\le\frac{\xi^2}{2\nu}\|\sqrt{f_{n-1}}\phi_n(t,.)\|_{L^2}^2. \tag{4.31}$$

Collecting (4.30) and (4.31), we obtain

$$\begin{aligned}&\frac{1}{\alpha+1}\partial_t^\alpha[G\rho\|u_n(t,.)\|_{L^2}^2+\lambda\|\phi_n(t,.)\|_{L^2}^2]+\frac{\nu}{\xi^2}\|f_n(t,.)\|_{L^1}\\&+\frac12\|\sqrt{f_{n-1}}\phi_n(t,.)\|_{L^2}^2+G\eta\|\frac{\partial u_n}{\partial z}(t,.)\|_{L^2}^2\\&\le K\ell\|\phi_n(t,.)\|_{L^2}.\end{aligned} \tag{4.32}$$

Hence the solution (u_n,ϕ_n,f_n) is bounded.

The arguments given in Step 3 to derive (4.12) and in Step 4 for the L^∞ estimates can simulate for the approximate system in (u_n,ϕ_n,f_n) instead of (u,ϕ,f), and the corresponding auxiliary functions q_n and Λ_n. In this place, we have the estimate for the solution (u_n, ϕ_n, f_n)

$$\sup_n \sup_{t\in[0,T]} \Big(\|u_n(t,.)\|_{L^2} + \|\phi_n(t,.)\|_{L^2} + \|f_n(t,.)\|_{L^1}\Big) \le \widetilde{K} \tag{4.33}$$

and

$$\sup_n \sup_{t\in[0,T]} \Big(\|\Lambda_n(t,.)\|_{H^1} + \|\phi_n(t,.)\|_{L^\infty} + \|f_n(t,.)\|_{L^\infty}\Big) + \|\Lambda_n\|_{L^2} \le \widetilde{K}, \tag{4.34}$$

where we recall that $\widetilde{K}$ denotes various constants which depends on the coefficients in system (3.1)–(3.3), the initial data u_0, ϕ_0, f_0 and the time T.

Similar arguments as the ones in Step 5, show that

$$\begin{aligned} &\partial_t^\alpha \Big(\lambda \|\frac{\partial \phi_n}{\partial z}(t,.)\|_{L^2}^2 + \|\frac{\partial f_n}{\partial z}(t,.)\|_{L^2}^2\Big) \\ &\le \widetilde{C}\Big(\|\frac{\partial \phi_n}{\partial z}(t,.)\|_{L^2}^2 + \|\frac{\partial f_n}{\partial z}(t,.)\|_{L^2}^2 + \|\frac{\partial f_{n-1}}{\partial z}(t,.)\|_{L^2}^2 + \|\frac{\partial^2 \Lambda_n}{\partial z^2}\|_{L^2}^2\Big). \end{aligned} \tag{4.35}$$

Let

$$Z_n(t) := \|\frac{\partial \phi_n}{\partial z}(t,.)\|_{L^2}^2 + \|\frac{\partial f_n}{\partial z}(t,.)\|_{L^2}^2.$$

Applying the operator I^α, yields

$$\begin{aligned} Z_n(t) &\le Z_0 + \widetilde{C}\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} Z_n(\tau)d\tau + \widetilde{C}\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} Z_{n-1}(\tau)d\tau + \widetilde{C}\|\Lambda_n\|_{L^2}^2 \\ &\le \widetilde{C}_{\alpha,0} + \widetilde{C}_{\alpha,1}\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} Z_{n-1}(\tau)d\tau \\ &\le \widetilde{M} + \widetilde{M}\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} Z_{n-1}(\tau)d\tau, \end{aligned} \tag{4.36}$$

where $\widetilde{M} := \max(\widetilde{C}_{\alpha,1}, \widetilde{C}_{\alpha,0})$. By induction, we may find that for all $t \in [0,T]$ and all n,

$$Z_n(t) \le \widetilde{M}\sum_{j=0}^{n-1} \frac{(\widetilde{M}t)^j}{\Gamma(\alpha j+1)} + \frac{(\widetilde{M}t)^n}{\Gamma(\alpha n+1)}, \tag{4.37}$$

consequently,

$$Z_n(t) \le \widetilde{M}E_{\alpha,1}(\widetilde{M}t), \tag{4.38}$$

where, $E_{\alpha,1}(.)$ is a Mittag-Leffler function. It follows that

$$\sup_n \sup_{t\in[0,T]} Z_n(t) \le \widetilde{M}E_{\alpha,1}(\widetilde{M}T). \tag{4.39}$$

Thus inequalities (4.34) and (4.39) imply the inequalities

$$\sup_n \sup_{t\in[0,T]} \Big(\|u_n(t,.)\|_{H^1} + \|\phi_n(t,.)\|_{H^1} + \|f_n(t,.)\|_{H^1}\Big) + \|u_n\|_{L^2} \le M \tag{4.40}$$

and

$$\sup_n \Big(\|\partial_t^\alpha u_n(t,.)\|_{L^2} + \|\partial_t^\alpha \phi_n(t,.)\|_{L^2} + \|\partial_t^\alpha f_n(t,.)\|_{L^2}\Big) \le M_{T,\alpha}. \tag{4.41}$$

Step 7: Convergence of the approximate solutions. The bounds introduced in the previous steps, namely (4.40) and (4.41) imply that, at least up to extraction of a subsequence, we have the weak convergence

$$(u_n, \phi_n, f_n) \to (u, \phi, f), \quad \text{weakly in } L^\infty([0,T], H^1)^3.$$

In this step, we establish a strong convergence in $L^\infty([0,T], L^2(U))^3$. Denoted by $h_n^* = h_n - h_{n-1}$ and derive the evolution equations for (u_n^*, ϕ_n^*, f_n^*),

$$\rho \partial_t^\alpha u_n^*(t,z) = \eta \frac{\partial^2 u_n^*}{\partial z^2} + \frac{\partial \phi_n^*}{\partial z}, \tag{4.42}$$

$$\lambda \partial_t^\alpha \phi_n^*(t,z) = G \frac{\partial u_n^*}{\partial z} - f_{n-1}\phi_n^* - f_{n-1}^* \phi_{n-1}, \tag{4.43}$$

$$\begin{aligned}\partial_t^\alpha f_n^*(t,z) &= (-1 + \xi|\phi_{n-1}|)(f_{n-1}^* f_n + f_{n-1} f_n^*) \\ &\quad - \nu f_{n-1} f_n^*(f_n + f_{n-1}) - \nu f_{n-1}^2 f_{n-1}^* + \xi |\phi_n^*| f_n f_{n-1},\end{aligned} \tag{4.44}$$

Since (u_n, ϕ_n, f_n) satisfies the assertions (4.1) and (4.2), then the same holds for (u_n^*, ϕ_n^*, f_n^*). We multiply (4.42), (4.43), (4.44), respectively by u_n^*, ϕ_n^*, f_n^*, integrate over U, and then sum them,

$$\partial_t^\alpha \Big(G\rho \|u_n^*(t,.)\|_{L^2}^2 + \lambda \|\phi_n^*(t,.)\|_{L^2}^2 + \|f_n^*(t,.)\|_{L^2}^2 \Big) \le \int_U \Phi\big(|\phi_{n-1}|, |\phi_n|, |f_{n-1}|, |f_n|\big),$$

where Φ is a positive valued function. Let

$$\Psi_n(t) := \|u_n^*(t,.)\|_{L^2}^2 + \|\phi_n^*(t,.)\|_{L^2}^2 + \|f_n^*(t,.)\|_{L^2}^2.$$

Now by using the $L^\infty-$ bound in (4.33) on $|\phi_{n-1}|, |\phi_n|, |f_{n-1}|, |f_n|$, Young inequality yields

$$\partial_t^\alpha \Psi_n(t) \le \widetilde{K} \Big(\Psi_n(t) + \Psi_{n-1}(t) \Big). \tag{4.45}$$

Applying the Gronwall lemma to (4.45), we may find

$$\Psi_n(t) \le \widetilde{L}_\alpha \int_0^t \Psi_{n-1}(s) ds, \tag{4.46}$$

where $\widetilde{L}_\alpha$ is a constant depending on all the coefficients of the System (3.1)–(3.3) and its initial condition. Thus we have

$$\Psi_n(t) \le \frac{(\widetilde{L}_\alpha t)^{n-1}}{(n-1)!} \sup_{s \in [0,T]} \Psi_1(s);$$

therefore, the sequence (u_n, ϕ_n, f_n) is a Cauchy sequence in $L^\infty([0,T], L^2(U))^3$ which implies that $f_{n-1} \to f$ strongly. This completes the existence proof.

Step 8: Uniqueness. Consider (u_1, ϕ_1, f_1) and (u_2, ϕ_2, f_2) satisfying (4.1) and solutions to system (3.1)–(3.3) supplied with the same initial condition $(u_0, \phi_0, f_0) \in H^1(U)$. Assume that $u = u_1 - u_2$, $\phi = \phi_1 - \phi_2$ and $f = f_1 - f_2$ satisfying the system

$$\rho \partial_t^\alpha u(t,z) = \eta \frac{\partial^2 u}{\partial z^2} + \frac{\partial \phi}{\partial z},$$

$$\lambda \partial_t^\alpha \phi(t,z) = G \frac{\partial u}{\partial z} - f\phi,$$

$$\partial_t^\alpha f(t,z) = (-1 + \xi|\phi|) f^2 - \nu f^3.$$

Multiply these three equations by u, ϕ, f, respective, then integrate over U, and then summing up, we have

$$\partial_t^\alpha\Big(G\rho\|u(t,.)\|_{L^2}^2+\lambda\|\phi(t,.)\|_{L^2}^2+\|f(t,.)\|_{L^2}^2\Big)\le\widetilde{\ell}_\alpha(\|\phi\|_{L^2}^2+\|f\|_{L^2}^2).$$

The Gronwall lemma then implies uniqueness. This completes the proof of Theorem 4.1.

5. Convergence to the steady point

In this section, we discuss the convergence of solution (u,ϕ,f) to the steady point $(0,0,0)$ in the space $H^1(U)\times L^\infty(U)\times L^\infty(U)$.

Theorem 5.1. *Consider Systen* (3.1)–(3.3). *If $\ell=0$ and $\Re(f_0)\neq 0$ then*

$$\|u(t,.)\|_{H^1}+\|\phi(t,.)\|_{L^\infty}+\|f(t,.)\|_{L^\infty}\to 0. \tag{5.1}$$

The proof will be done in three steps. The first step derive the lower bound of the fluidity f, while Step 2 proves the convergence in $L^2(U)$ and consequently Step 3, provides the convergence in $L^\infty(U)$.

Step 1: Lower bound of the fluidity f. Since $\Re(f_0)\neq 0$, there exists, by analytically of f_0 (assumed in H^1), a non-empty closed interval U_0 in U where f_0 does not vanish. We rewrite the evolution equation (3.3) on f as follows:

$$\partial_t^\alpha f^{-1}(t,z)=(1-\xi|\phi|)f^2+\nu f^3; \tag{5.2}$$

but the $L^\infty-$ bounds of f and ϕ are uniform in time (see Step 4), thus for all $z\in U_0$, we obtain that

$$\partial_t^\alpha|f^{-1}(t,z)|\le\kappa$$

and therefore,

$$|f(t,z)|\ge\frac{\Gamma(\alpha+1)}{\frac{\Gamma(\alpha+1)}{|f_0|}+\kappa t^\alpha},\quad t\in[0,T]$$

and this implies the lower bound.

Step 2: Convergence in $L^2(U)$. From (4.5) and (4.8), we have

$$\frac{1}{\alpha+1}\partial_t^\alpha[G\rho\|u(t,.)\|_{L^2}^2+\lambda\|\phi(t,.)\|_{L^2}^2]+\|\sqrt{f}\phi(t,.)\|_{L^2}^2+G\eta\|u_z(t,.)\|_{L^2}^2=0 \tag{5.3}$$

and

$$\begin{aligned}&\frac{1}{\alpha+1}\partial_t^\alpha[G\rho\|u(t,.)\|_{L^2}^2+\lambda\|\phi(t,.)\|_{L^2}^2]+\frac{\nu}{\xi^2}\|f(t,.)\|_{L^1}\\&+\frac{1}{2}\|\sqrt{f}\phi(t,.)\|_{L^2}^2+G\eta\|u_z(t,.)\|_{L^2}^2=0.\end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4) and applying the Gronwall lemma, we obtain

$$\lim_{t\to\infty}\Big(\|u(t,.)\|_{L^2}^2+\|\phi(t,.)\|_{L^2}^2+\|f(t,.)\|_{L^1}\Big)\to 0. \tag{5.5}$$

Step 3: Convergence in $L^\infty(U)$. Combining (4.12) and (4.13) and using the $L^\infty-$bound of f yield

$$\begin{aligned}&\partial_t^\alpha\Big(\|\frac{\partial\Lambda}{\partial z}(t,.)\|_{L^2}^2+\lambda|\phi(t,z)|^2\Big)+\Big(\|\frac{\partial^2\Lambda}{\partial z^2}(t,.)\|_{L^2}^2+|\phi(t,z)|^2\Big)\\&\le\sigma\Big(\|u(t,.)\|_{L^2}^2+\|\phi(t,.)\|_{L^2}^2+\|\frac{\partial\Lambda}{\partial z}(t,.)\|_{L^2}^2\Big),\end{aligned}$$

where σ is a positive constant depending on all the coefficients of (3.1)–(3.3). Employing the Gronwall lemma and using the last convergence (5.5), we obtain

$$\lim_{t\to\infty} \|\phi(t,.)\|^2_{L^\infty} = 0. \tag{5.6}$$

Using this convergence in (3.3), we have

$$\lim_{t\to\infty} \|f(t,.)\|^2_{L^\infty} = 0. \tag{5.7}$$

Finally, definition (4.10) and convergence (5.5) supply the convergence of u_z to zero in $L^2(U)$; hence we have

$$(u, \phi, f) \in H^1(U) \times L^\infty(U) \times L^\infty(U).$$

This completes the proof.

Conclusion. In this article, we had illustrated an analytic method for establishing the existence and uniqueness of solutions to fractional differential system in a complex domain. The proposed method depends on fractional Duhamel principle, which can be applied in various kinds of fractional systems. Throughout the article, we had used the homogeneous boundary value problem. For future work, one may try the non-homogeneous case.

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Rabha W. Ibrahim

Institute of Mathematical Sciences, University Malaya, 50603, Kuala Lumpur, Malaysia

E-mail address: rabhaibrahim@yahoo.com