

## BLOCK-PULSE FUNCTIONS AND THEIR APPLICATIONS TO SOLVING SYSTEMS OF HIGHER-ORDER NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

ALI EBADIAN, AMIR AHMAD KHAJEHNASIRI

ABSTRACT. The operational block-pulse functions, a well-known method for solving functional equations, is employed to solve a system of nonlinear Volterra integro-differential equations. First, we present the block-pulse operational matrix of integration, then by using these matrices, the nonlinear Volterra high-order integro-differential equation is reduced to an algebraic system. The benefits of this method is low cost of setting up the equations without applying any projection method such as Galerkin, collocation, etc. The results reveal that the method is very effective and convenient.

### 1. INTRODUCTION

Systems of integro-differential equations are a well-known mathematical tool for representing physical problems. Historically, they have achieved great popularity among the mathematicians and physicists in formulating boundary value problems of gravitation, electrostatics, fluid dynamics and scattering.

The concept of the block-pulse functions (BPFs) was first introduced to electrical engineers by Harmuth. Then several researchers (Gopalsami and Deekshatulu, 1997 [9], Sannuti, 1977 [17], Riad, 1992 [16], Chen and Tsay, 1977 [7]) discussed the BPFs and their operational matrix [8, 12].

The aim of this work is to present a numerical method for approximating the following system of nonlinear Volterra integro-differential equations of order  $r$  ( $r \geq 1$ ):

$$u_i^r(x) + \sum_{k=1}^r B_{r-k} u_i^{(r-k)}(x) = g_i(x) + \lambda \int_0^x k_i(x, t, F(u_i(x))) dt, \quad i = 1, 2, \dots, n, \quad (1.1)$$

with initial conditions

$$u_i(0) = u_i, \quad (1.2)$$

where the parameters  $\lambda$  and functions  $g_i(x)$ ,  $k_i(x, t, F(u_i(t)))$  are known and belong to  $L^2[0, 1)$ , and  $u(x)$  is an unknown function and  $B_i$  ( $i = 0, 1, \dots, n$ ) are  $n \times n$

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matrices. In this work, we consider that, the nonlinear function has the form

$$F(u_i(t)) = (u_i(t))^p,$$

where  $p$  is a positive integer.

Systems of integro-differential equations have a major role in the fields of science and engineering [3, 10, 11, 19]. The initial value problem for a nonlinear system of integro-differential equations was used to model the competition between tumor cells and the immune system [4]. There are various techniques for solving a system of integral or integro-differential equation, e.g. Galerkin method [13], Adomian decomposition method (ADM) [18, 5], rationalized Haar functions method [14] and variational iteration method (VIM) [20], He's homotopy perturbation method (HPM) [6, 21], Tau method [2], differential transform method [1], and Maleknejad in [15] have applied Bernstein operational matrix for solving a system of high order linear Volterra-Fredholm integro-differential equations, etc.

This article is organized as follows: In Section 2, we introduce BPFs and their properties. In Section 3, the operational matrix of integration is derived. Section 4 introduces Application of the method. Some numerical results has been presented in section 5 to show accuracy and advantage of the proposed method. Finally, some concluding remarks are given in section 6.

## 2. PROPERTIES OF BLOCK-PULSE FUNCTIONS

An  $m$ -set of BPFs is defined as follows:

$$\phi_i(t) = \begin{cases} 1, & ih \leq t < (i+1)h, \\ 0, & \text{otherwise,} \end{cases} \quad (2.1)$$

where  $i = 1, 2, \dots, m-1$  with positive integer values for  $m$ , and  $h = T/m$ , and  $m$  are arbitrary positive integers. There are some properties for BPFs, e.g. disjointness, orthogonality, and completeness.

**Disjointness.** The block-pulse functions are disjoint with each other; i.e.,

$$\phi_i(t)\phi_j(t) = \begin{cases} \phi_i(t), & i = j, \\ 0, & i \neq j, \end{cases} \quad (2.2)$$

where  $i, j = 0, \dots, m-1$ .

**Orthogonality.** The block-pulse functions are orthogonal with each other; i.e.,

$$\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} h, & i = j, \\ 0, & \text{otherwise,} \end{cases} \quad (2.3)$$

in the region of  $t \in [0, T)$ , where  $i, j = 1, 2, \dots, m-1$ . This property is obtained from the disjointness property.

**Completeness.** For every  $f \in L^2([0, 1))$  when  $m$  go to infinity, Parseval identity holds:

$$\int_0^1 f^2(t)dt = \sum_{i=0}^{\infty} f_i^2 \|\phi_i(t)\|^2, \quad (2.4)$$

where

$$f_i = \frac{1}{h} \int_0^1 f(t)\phi_i(t)dt.$$

The set of BPFs may be written as a  $m$ -vector  $\Phi(t)$  :

$$\Phi(t) = [\phi_0(t), \dots, \phi_{m-1}(t)]^T, \quad (2.5)$$

where  $t \in [0, 1)$ . From the above representation and disjointness property, it follows:

$$\begin{aligned} \Phi(t)\Phi^T(t) &= \begin{bmatrix} \phi_0(t) & 0 & \dots & 0 \\ 0 & \phi_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \phi_{m-1}(t) \end{bmatrix} \\ \Phi^T(t)\Phi(t) &= 1, \\ \Phi(t)\Phi(t)^T V &= \tilde{V}\Phi(t), \end{aligned}$$

where  $V$  is an  $m$ -vector and  $V = \text{diag}(V)$ . Moreover, it can be clearly concluded that for every  $m \times m$  matrix  $A$ :

$$\Phi^T(t)A\Phi(t) = \widehat{A^T}\Phi(t), \quad (2.6)$$

where  $A$  is an  $m$ -vector with elements equal to the diagonal entries of matrix  $A$ .

**2.1. Functions approximation.** A function  $f(t) \in L^2([0, 1))$  may be expanded by the BPFs as:

$$f(t) \simeq \sum_{i=0}^{m-1} f_{i1}\phi_i(t) = F^T\Phi(t) = \Phi^T(t)F, \quad (2.7)$$

where  $F$  is a  $m$ -vector given by

$$F = [f_0, \dots, f_{m-1}]^T, \quad (2.8)$$

$$\Phi(t) = [\phi_1(t), \phi_2(t), \dots, \phi_{m-1}(t)]^T, \quad (2.9)$$

the block-pulse coefficients  $f_i$  are obtained as

$$f_i = \frac{1}{h} \int_{ih}^{(i+1)h} f(t)dt, \quad (2.10)$$

such that error between  $f(t)$ , and its block-pulse expansion (2.7) in the region of  $t \in [0, 1)$

$$\varepsilon = \int_0^1 \left( f - \sum_{i=0}^{m-1} f_i\phi_i(t) \right)^2 dt, \quad (2.11)$$

is minimal. Now assume  $K(x, t) \in L^2([0, 1) \times [0, 1))$  may be approximated with respect to BPFs such as:

$$k(x, t) = \Phi^T(x)K\Phi(t), \quad (2.12)$$

where  $\Phi(x)$  and  $\Phi(t)$  are BPFs vectors of dimension  $m_1$  and  $m_2$ , respectively, and  $K$  is a  $m_1 \times m_2$  one dimensional block-pulse coefficients matrix with  $k_{ij}$ ,  $i = 0, \dots, m_1 - 1$ ,  $j = 0, \dots, m_2 - 1$  as follows:

$$k_{ij} = m_1 m_2 \int_0^1 \int_0^1 k(x, t)\phi_i(x)\phi_j(t) dx dt. \quad (2.13)$$

Also, the positive integer powers of a function  $f(s)$  may be approximated by BPFs as

$$[u(t)]^p = [\Phi^T(t)U]^p = \Phi^T(t)\Lambda,$$

where  $\Lambda$  is a column vector, whose elements are  $p$ th power of the elements of the vector  $U$ .

**2.2. Block-pulse functions series.** The function  $x^k$ ,  $x \in [0, 1]$ ,  $k \in \mathbb{N}$  can be approximated as a BPF series of size  $m$ . Indeed, from (2.7) and (2.10), we have

$$x^k \simeq \sum_{i=0}^{m-1} f_k(i)\phi(x), \quad (2.14)$$

where

$$f_k(i) = \frac{1}{h} \int_{ih}^{(i+1)h} t^k dt = \frac{1}{h(k+1)} [((i+1)h)^{k+1} - (ih)^{k+1}]. \quad (2.15)$$

Therefore,

$$x^k \simeq \frac{1}{h(k+1)} \sum_{i=0}^{m-1} [((i+1)h)^{k+1} - (ih)^{k+1}] \Phi_i(x), \quad (2.16)$$

and in matrix form

$$x^k \simeq \frac{1}{h(k+1)} Y_k^T \Phi_m(x), \quad (2.17)$$

where

$$Y_k^T = \sum_{i=0}^{m-1} [((i+1)h)^{k+1} - (ih)^{k+1}].$$

### 3. OPERATIONAL MATRIX OF INTEGRATION

We compute  $\int_0^t \Phi_i d\tau$  as

$$\int_0^t \Phi_i(\tau) d\tau = \begin{cases} 0, & t \leq ih, \\ t - ih & ih \leq t < (i+1)h, \\ h & (i+1)h \leq t < 1. \end{cases} \quad (3.1)$$

Then (3.1) can be written as

$$\int_0^t \Phi_i(\tau) d\tau = (t - ih)\Phi_i(t) + h \sum_{j=i+1}^{m-1} \Phi_j(t). \quad (3.2)$$

From (2.17) we have

$$x \simeq \frac{1}{2h} \sum_{i=0}^{m-1} [((i+1)h)^2 - (ih)^2] \Phi_i(t). \quad (3.3)$$

Substituting (3.3) and (2.2) into (3.2), and by using orthogonal property, for  $0 \leq i < m$ , we have

$$\begin{aligned} \int_0^t \Phi_i(\tau) d\tau &= \frac{1}{2h} \sum_{j=0}^{m-1} [((j+1)h)^2 - (jh)^2] \Phi_j(t)\Phi_i(t) - ih\Phi_i(t) + h \sum_{j=i+1}^{m-1} \Phi_j(t) \\ &= \frac{1}{2h} [((i+1)h)^2 - (ih)^2] \Phi_i(t) - ih\Phi_i(t) + h \sum_{j=i+1}^{m-1} \Phi_j(t) \\ &= \frac{h}{2} \Phi_i + h \sum_{j=i+1}^{m-1} \Phi_j(t). \end{aligned}$$

The integration of the vector  $\Phi(t)$  defined in (2.5) may be obtained as

$$\int_0^t \Phi(\tau) d\tau \simeq \Upsilon \Phi(t), \quad (3.4)$$

where  $\Upsilon$  is called operational matrix of integration which can be represented by

$$\Upsilon = \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

and their integrals in the matrix form

$$\begin{pmatrix} \int \Phi_0 \\ \int \Phi_1 \\ \vdots \\ \int \Phi_{m-1} \end{pmatrix} \simeq \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \vdots \\ \Phi_{m-1} \end{pmatrix},$$

or in more compact form

$$\int_0^t \Phi_m(\tau) d\tau \simeq \Upsilon \Phi_m(t), \quad (3.5)$$

By using this matrix, we can express the integral of a function  $f(t)$  into its block pulse series

$$\int_0^t f_m(\tau) d\tau \simeq \int_0^t F^T \Phi_m(\tau) d\tau \simeq F^T \Upsilon \Phi_m(t). \quad (3.6)$$

#### 4. APPLICATION OF THE METHOD

In this section, we calculate  $U_i^{k-r}(x)$  by using

$$u_i^r(x) = \sum_{i=0}^{m-1} u_i \Phi(x) = U_i^T \Phi_m(x). \quad (4.1)$$

Now integrating from 0 to  $t$  and using (3.5) we obtain

$$U_i^{r-1}(x) = U \Upsilon \Phi_m(x) + U_0^{r-1}(x) \quad (4.2)$$

The  $k$ -th integration of (4.1) yields

$$U_i^{r-k}(x) = U_i \Upsilon^k \phi_m(x) + \sum_{i=1}^k U_0^{r-i} \frac{t^{k-i}}{(k-i)!} \quad k = 1, 2, \dots, r. \quad (4.3)$$

From (2.17) we have

$$\frac{t^{k-i}}{(k-i)!} \simeq \frac{1}{h(k-i+1)!} Y_{k-i}^T \Phi_m(x) \quad (4.4)$$

Substituting (4.4) in (4.3) we obtain

$$U_i^{r-k}(x) = U_i \Upsilon^k \Phi_m(t) + Z_k \Phi_m(x), \quad (4.5)$$

where

$$Z_k = \frac{1}{h} \sum_{i=1}^k \frac{1}{h(k-i+1)!} U_0^{r-i} Y_{k-i}^T \quad (4.6)$$

is an  $n \times m$  constant matrix.

Now, we solve the system of nonlinear Volterra high-order integro-differential equations by using BPFs. As we show before, we can write

$$\begin{aligned} g_i(x) &= G_i^T \Phi_m(x), \\ u_i^r(x) &= U_i^T \Phi_m(x), \\ [u_i(x)]^p &= \Phi_m^T(x) \Lambda, \\ k(x, t) &= \Phi^T(x) K \Phi(t), \end{aligned} \quad (4.7)$$

where the  $m$ -vectors  $U, G, \Lambda$ , and matrix  $K$  are BPFs coefficients of  $u(x), g(x), [u(t)]^p$ , and  $K(x, t)$  respectively,  $\Lambda$  is a column vector whose elements are  $p$ th power of the elements of the vector  $U$ . To approximate the integral equation (1.1), from (4.7) and (4.5) we get

$$u_i^r(x) + \sum_{k=1}^r B_{r-k} u_i^{(r-k)}(x) = g_i(x) + \int_0^x k_i(x, t, F(u_i(x))) dt \quad i = 1, 2, \dots, n.$$

Now the second part of equation

$$\begin{aligned} U_i^T \Phi_m(x) + \sum_{k=1}^r B_{r-k} (U_i \Upsilon^k + Z_k) \Phi_m(x) &= G_i^T \Phi_m(x) + \Phi_m^T(x) K \int_0^x \Phi(t) \Phi^T(t) \Lambda \\ &= G_i^T \Phi_m(x) + \Phi_m^T(x) K \tilde{\Lambda} \int_0^x \Phi_m(t) dt \\ &= G_i^T \Phi_m(x) + \Phi_m^T(x) K \tilde{\Lambda} \Upsilon \Phi_m(x). \end{aligned}$$

If we put  $A = K \tilde{\Lambda} \Upsilon$  then it can be written from (2.6),

$$U_i^T \Phi_m(x) + \sum_{k=1}^r B_{r-k} u_i \Upsilon^k \Phi_m(x) = G_i^T \Phi_m(x) + \widehat{A^T} \Phi_m(x),$$

hence, we have

$$U_i^T + \sum_{k=1}^r B_{r-k} u_i \Upsilon^k = G_i^T + \widehat{A^T}. \quad (4.8)$$

It can be written as:

$$AU = F \quad (4.9)$$

where  $A$  and  $F$  are the combination of block-pulse coefficient matrix and  $U$  can be obtained from Newton-Raphson method for solving nonlinear systems.

## 5. NUMERICAL EXAMPLES

To illustrate the effectiveness of the proposed method in the present paper, several test examples are carried out in this section.

**Example 5.1.** Consider the nonlinear Volterra integro-differential equations problem with initial conditions [6],

$$\begin{aligned} u'(x) - 1 + \frac{1}{2} v'^2(x) &= \int_0^x ((x-t)v(t) + v(t)u(t)) dt, \\ v'(x) - 2x &= \int_0^x ((x-t)u(t) - v^2(t) + u^2(t)) dt, \\ u(0) &= 0, \quad v(0) = 1. \end{aligned} \quad (5.1)$$

The exact solutions are  $u(x) = \sinh(x)$ ,  $v(x) = \cosh(x)$ . The numerical results obtained with BPFs are presented in Table 1 and Figure 1.

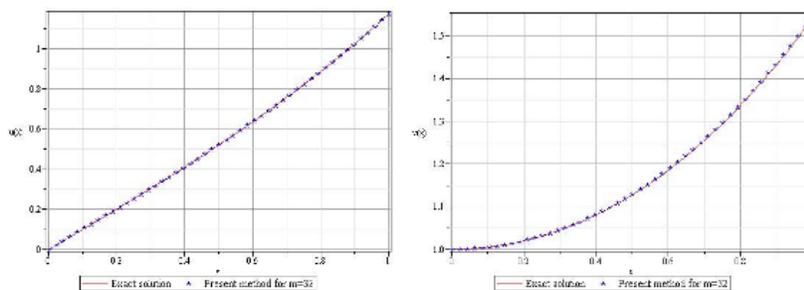


FIGURE 1. Comparison of the exact solution and the present method

**Example 5.2.** consider the system of two nonlinear integro-differential equations with boundary conditions [6],

$$\begin{aligned}
 u''(x) &= 1 - \frac{1}{3}x^3 - \frac{1}{2}v'^2(x) + \frac{1}{2} \int_0^x (u^2(t) + v^2(t))dt, \\
 v''(x) &= -1 + x^2 - xu(x) + \frac{1}{4} \int_0^x (u^2(t) - v^2(t))dt, \\
 u(0) &= 1, \quad u'(0) = 2, \quad v(0) = -1, \quad v'(0) = 0.
 \end{aligned}
 \tag{5.2}$$

TABLE 1. Numerical results of Example 5.1

$x$	$(u_{exact}(x), v_{exact}(x))$	$m = 8$	$m = 16$	$m = 32$
0.0	(0.00000, 1.00000)	(0.00145, 0.88425)	(0.00115, 0.91196)	(0.00025, 1.00056)
0.1	(0.10016, 1.00500)	(0.10727, 1.01570)	(0.101852, 1.009)	(0.10016, 1.00501)
0.3	(0.30452, 1.04533)	(0.38440, 1.00521)	(0.58562, 1.03122)	(0.30450, 1.04533)
0.5	(0.52109, 1.12762)	(0.50695, 1.108423)	(0.52012, 1.12521)	(0.52108, 1.12762)
0.7	(0.75858, 1.25516)	(0.74932, 1.20390)	(0.75125, 1.25501)	(0.75857, 1.25510)
0.9	(1.02651, 1.43308)	(1.09032, 0.30390)	(1.02541, 1.43321)	(1.02651, 1.43308)

The exact solutions are  $u(x) = x + e^x$  and  $v(x) = x - e^x$ . Numerical results for this solution is presented in Table 2.

TABLE 2. Numerical results of Example 5.2

$x$	$(u_{exact}(x), v_{exact}(x))$	$m = 8$	$m = 16$	$m = 32$
0.0	(1, -1)	(0.93915, -0.88365)	(0.96251, -0.89936)	(0.99251, -0.97936)
0.1	(1.02051, -1.00517)	(1.01107, -1.02517)	(1.02012, -1.01701)	(1.02052, -1.00701)
0.3	(1.64985, -1.04985)	(1.61852, -1.00980)	(1.64212, -1.04914)	(1.64812, -1.04984)
0.5	(2.14872, -1.14872)	(2.12211, -1.14582)	(2.14121, -1.14705)	(2.14821, -1.14725)
0.7	(2.71375, -1.31375)	(2.71175, -1.31075)	(2.71301, -1.31221)	(2.71371, -1.31321)
0.9	(3.35960, -1.55960)	(3.35255, -1.54696)	(3.35666, -1.55009)	(3.35966, -1.55909)

**Conclusion.** In this article, we approximated the solution of nonlinear Volterra integro-differential equations. To this end, we used some orthogonal functions called Block-Pulse Functions. Finally, numerical examples reveal that the present method is very accurate and convenient for solving systems of high order linear and nonlinear Volterra integro-differential equations. The benefits of this method is low cost of setting up the equations without applying any projection method such as Galerkin, collocation, etc. Also, the linear system (4.9) has a regular form which can help us for solving it.

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ALI EBADIAN

DEPARTMENT OF MATHEMATIS, URMIA UNIVERSITY, URMIA, IRAN

*E-mail address:* [a.ebadian@urmia.ac.ir](mailto:a.ebadian@urmia.ac.ir)

AMIR AHMAD KHAJEHNASIRI

DEPARTMENT OF MATHEMATIS, URMIA UNIVERSITY, URMIA, IRAN.

DEPARTMENT OF MATHEMATICS, PAYAME NOOR UNIVERSITY, PO BOX 19395-3697 TEHRAN, IRAN

*E-mail address:* [a.khajehnasiri@gmail.com](mailto:a.khajehnasiri@gmail.com)