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BOUNDARY VALUE PROBLEM FOR A COUPLED SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN OPERATOR AT RESONANCE

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Abstract. In this article, we discuss the existence of solutions to boundary-value problems for a coupled system of fractional differential equations with p-Laplacian operator at resonance. We prove the existence of solutions when $\dim\ker L \geq 2$, using the coincidence degree theory by Mawhin.

1. Introduction

Along with the development of sciences and technology, the subject of fractional differential equations (FDEs for short) has emerged as an important area of investigation. Indeed, we can find a large number of applications in physics, electrochemistry, control, biology, etc. (see [10, 20]). Recently, many results on FDEs have been obtained; see for example [1, 3, 4, 5, 12, 13, 18]. Many authors have studied boundary value problems (BVPs for short) of FDEs; see [2, 6, 7, 14, 24, 25, 26, 27].

The papers [8, 9, 15, 16] considered the BVPs of FDEs with p-Laplacian operator. In 2012, Chen et al. [9] showed the existence solutions by coincidence degree for the Caputo fractional p-Laplacian equations

$$
\begin{gathered}
D^{\beta}_{0+}\phi_p(D^{\alpha}_{0+}x(t)) = f(t, x(t), D^{\alpha}_{0+}x(t)), \quad 0 < t < 1, \\
D^{\alpha}_{0+}x(0) = D^{\alpha}_{0+}x(1) = 0,
\end{gathered}
$$

where $0 < \alpha,\ \beta \leq 1$, $1 < \alpha + \beta \leq 2$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $f : [0, 1] \times R^2 \to \mathbb{R}$ is continuous, D^{α}_{0+} and D^{β}_{0+} are Caputo fractional derivatives. They used the operator $Lu = D^{\beta}_{0+}\phi_p(D^{\alpha}_{0+}x(t))$ with $D^{\alpha}_{0+}x(0) = D^{\alpha}_{0+}x(1) = 0$ and obtained $\dim\ker L = 1$.

Articles [11, 22] considered BVPs for a coupled system of FDEs. In 2009, Su [22] showed the existence result by Schauder fix-point theorem for the coupled system of FDEs:

$$
\begin{gathered}
D^{\alpha}u(t) = f(t, v(t), D^{\mu}v(t)), \quad 0 < t < 1, \\
D^{\beta}v(t) = f(t, u(t), D^{\nu}u(t)), \quad 0 < t < 1, \\
u(0) = u(1) = v(0) = v(1) = 0,
\end{gathered}
$$

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where $1 < \alpha$, $\beta < 2$, $\mu, \nu > 0$, $\alpha - \nu \geq 1$, $\beta - \mu \geq 1$, $f, g : [0,1] \times R^2 \to \mathbb{R}$ are given functions and D is the standard Riemann-Liouville dervative. In 2012 Jiang [11] considered the existence results for a coupled system of FDEs:

$$D^\alpha u(t) = f(t, u(t), v(t)), \quad u(0) = 0, \quad D^\gamma u(t)|_{t=1} = \sum_{i=1}^{n} a_i D^\gamma u(t)|_{t=\xi_i},$$
$$D^\beta v(t) = g(t, u(t), v(t)), \quad v(0) = 0, \quad D^\delta v(t)|_{t=1} = \sum_{i=1}^{m} b_i D^\delta v(t)|_{t=\eta_i},$$

where $t \in [0,1]$, $1 < \alpha$, $\beta \leq 2$, $0 < \gamma \leq \alpha - 1$, $0 < \delta \leq \beta - 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$, and proved that $\dim \ker L = 1$.

As we know, there are only a few papers devoted to investigate the BVPs for a coupled system of FDEs with p-Laplacian operator at resonance. What is more, the case of $\dim \ker L \geq 2$ have not been studied. In this paper we will study the BVPs for higher order FDEs as follows:

$$\begin{gathered}
D_{0+}^\gamma \phi_p(D_{0+}^\alpha u(t)) = f(t, v(t)), \\
D_{0+}^\gamma \phi_p(D_{0+}^\beta v(t)) = g(t, u(t)), \\
D_{0+}^\alpha u(0) = D_{0+}^\alpha u(1) = D_{0+}^\beta v(0) = D_{0+}^\beta v(1) = 0,
\end{gathered} \tag{1.1}$$

where $t \in [0,1]$, $n-1 < \alpha, \beta \leq n$, $0 < \gamma \leq 1$, $f, g : [0,1] \times R \to \mathbb{R}$ are continuous functions, $D_{0+}^\alpha, D_{0+}^\beta$ and D_{0+}^γ are Caputo derivatives, and $\phi_p(s) = \begin{cases} |s|^{p-2}s & s \neq 0, \\ 0 & s = 0 \end{cases}$ is a p-Laplacian operator with $p > 1$. Hence, if $L(u,v) = (D_{0+}^\gamma \phi_p(D_{0+}^\alpha u), D_{0+}^\gamma \phi_p(D_{0+}^\beta v))$ and

$$\begin{aligned}
\operatorname{dom} L = \big\{&(u,v) \in X | (D_{0+}^\gamma \phi_p(D_{0+}^\alpha u), D_{0+}^\gamma \phi_p(D_{0+}^\beta v)) \in Y, \\
&D_{0+}^\alpha u(0) = D_{0+}^\alpha u(1) = D_{0+}^\beta v(0) = D_{0+}^\beta v(1) = 0\big\},
\end{aligned}$$

then $\dim \ker L = n, n \geq 2$.

2. Preliminaries

For convenience, we present here some necessary basic knowledge and a theorem, which can be found in [19].

Let X and Y be real Banach spaces and $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator with index zero, $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad X = \ker L \oplus \ker P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L,$$

is invertible. We denote the inverse by K_p.

If Ω is an open bounded subset of X, $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.1 ([19]). *Let $L : \operatorname{dom} \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be called L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(1) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$;*

(2) *$Nx \notin \operatorname{Im} L$ for every $x \in \ker L \cap \partial\Omega$;*
(3) *$\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, where $Q : Y \to Y$ is a projection such that $\operatorname{Im} L = \ker Q$.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

In this article, we take $X = C^{\alpha-1}[0,1] \times C^{\beta-1}[0,1]$ with norm

$$\|(u,v)\| = \max\{\|u\|_\infty, \|v\|_\infty, \|D_{0+}^{\alpha-1}u\|_\infty, \|D_{0+}^{\beta-1}v\|_\infty\},$$

and $Y = C[0,1] \times C[0,1]$ with norm

$$\|(f,g)\| = \max\{\|f(x)\|_\infty, \|g(x)\|_\infty\},$$

where $C^{\alpha-1}[0,1] = \{u|u, D_{0+}^{\alpha}u \in C[0,1]\}$, $C^{\beta}[0,1] = \{v|v, D_{0+}^{\beta}v \in C[0,1]\}$.

Define the operator $L : \operatorname{dom} L \cap X \to Y$,by

$$L(u(t),v(t)) = (D_{0+}^{\gamma}\phi_p(D_{0+}^{\alpha}u(t)), D_{0+}^{\gamma}\phi_p(D_{0+}^{\beta}v(t))), \tag{2.1}$$

where

$$\begin{aligned}\operatorname{dom} L = \big\{(u,v) \in X|(D_{0+}^{\gamma}\phi_p(D_{0+}^{\alpha}u(t)), D_{0+}^{\gamma}\phi_p(D_{0+}^{\beta}v(t))) \in Y,\\ D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha}u(1) = D_{0+}^{\beta}v(0) = D_{0+}^{\beta}v(1) = 0\}.\end{aligned}$$

Define the operator $N : X \to Y$, by

$$N(u(t),v(t)) = (N_1u(t), N_2v(t)), t \in [0,1],$$

where $N_1u(t) = f(t,v(t)), N_2v(t) = g(t,u(t))$.

It is easy to see that X is a Banach space, and problem (1.1) is equivalent to the operator equation

$$L(u,v) = N(u,v), (u,v) \in \operatorname{dom} L.$$

The following definitions can be found in [20, 23].

Definition 2.2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,1) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

provided that the right side integral is pointwise defined on $(0,+\infty)$.

Definition 2.3. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u : (0,1) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha}u(t) = I_{0+}^{n-\alpha}\frac{d^n u(t)}{dt^n} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^n(s)ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0,+\infty)$.

Lemma 2.4 ([17])**.** Let $\alpha > 0$.The fractional differential equation $D_{0+}^{\alpha}u(t) = 0$ has solution

$$u(t) = C_1 + C_2t + C_3t^2 + \dots + C_nt^{n-1}.$$

Lemma 2.5 ([12])**.** Assume that $u(t)$ with a fractional derivative of order $\alpha > 0$. Then

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + C_1 + C_2t + C_3t^2 + \dots + C_nt^{n-1}, \quad C_i \in R, i = 1,2,\dots,n,$$

where n is the smallest integer greater than or equal to α.

3. Main result

In this section, a theorem on existence of solutions for problem (1.1) will be given. Define the operators T_1 and T_2 as follows:

$$T_1 y_1(s) = \int_0^1 (1-s)^{\alpha-1} y_1(s)ds, \quad T_2 y_2(s) = \int_0^1 (1-s)^{\beta-1} y_2(s)ds.$$

Theorem 3.1. *Let $f, g : [0,1] \times R \to \mathbb{R}$ be continuous and assume that*

(H1) *there exist nonnegative functions $a(t), b(t), c(t), d(t) \in C[0,1]$, such that*
$$|f(t,v)| \le a(t) + b(t)|v|^{p-1}; \quad |g(t,u)| \le c(t) + d(t)|u|^{p-1};$$

(H2) *for $(u,v) \in \operatorname{dom} L$, there exist constants $M_i > 0$, $i = 1,2$, such that, if either $|u(t)| > M_1, t \in [\xi, 1]$, or $|v(t)| > M_2, t \in [\eta, 1]$, then either*
$$T_1 N_1 u \neq 0, \quad \text{or} \quad T_2 N_2 v \neq 0;$$

(H3) *there exist a positive constant B, such that for each $(u,v) \in \ker L$, if $\min\{|\pi_i|, |\pi_i'|\} > B$, $i = 1, 2, \dots n$.*

Then either (1)

(i) $(\sum_{i=1}^n \pi_i')T_1 N_1 u > 0, (\sum_{i=1}^n \pi_i)T_2 N_2 v > 0,$
(ii) $(\sum_{i=1}^n \pi_i')T_1 N_1 u > 0, (\sum_{i=1}^n \pi_i)T_2 N_2 v < 0;$

or (2)

(i) $(\sum_{i=1}^n \pi_i')T_1 N_1 u < 0,\ (\sum_{i=1}^n \pi_i)T_2 N_2 v < 0,$
(ii) $(\sum_{i=1}^n \pi_i')T_1 N_1 u < 0,\ (\sum_{i=1}^n \pi_i)T_2 N_2 v > 0,$ *where $b(t), d(t)$ satisfy*

$$\|b\|_\infty \|d\|_\infty < \frac{(\Gamma(\gamma+1))^2}{4} \Big(\frac{\xi\eta\Gamma(\alpha+1)\Gamma(\beta+1)}{(1+\xi)(1+\eta)}\Big)^{1-q}.$$

Lemma 3.2. *Let L be defined by (2), then*

$$\begin{gathered} \ker L = \big\{(u,v) \in X : (u,v) = (\sum_{i=1}^n \pi_i t^{i-1}, \sum_{i=1}^n \pi_i' t^{i-1}), \\ \pi_i, \pi_i' \in R, i = 1, 2, \dots, n, t \in [0,1]\big\}, \end{gathered} \tag{3.1}$$

$$\operatorname{Im} L = \{(y_1, y_2) \in Y | T_1 y_1 = 0, T_2 y_2 = 0\}. \tag{3.2}$$

Proof. By Lemmas 2.4 and 2.5, and $\phi_p^{-1}(s) = \phi_q(s)$, $1/p + 1/q = 1$, the equation $D_{0+}^\gamma \phi_p(D_{0+}^\alpha u(t)) = 0$ has solution

$$u(t) = I_{0+}^\alpha \phi_q(c) + \sum_{i=1}^n \pi_i t^{i-1}, \quad \pi_i \in R,\ i = 1, 2, \dots, n,$$

which satisfies $D_{0+}^\alpha u(t) = \phi_q(c)$, combining with the boundary value condition $D_{0+}^\alpha u(0) = 0$, we can get $u(t) = \sum_{i=1}^n \pi_i t^{i-1}$, similarly $v(t) = \sum_{i=1}^n \pi_i' t^{i-1}$. So, it has (3.1) holds.

On the one hand, if $(y_1, y_2) \in \operatorname{Im} L$, then there exist two functions $u, v \in \operatorname{dom} L$ such that

$$y_1 = D_{0+}^\gamma \phi_p(D_{0+}^\alpha u(t)), y_2 = D_{0+}^\gamma \phi_p(D_{0+}^\beta v(t)).$$

Based on Lemma 2.5 and $D_{0+}^\alpha u(0) = D_{0+}^\alpha v(0) = 0$,

$$D_{0+}^\alpha u(t) = \phi_q I_{0+}^\gamma y_1, D_{0+}^\beta u(t) = \phi_q I_{0+}^\gamma y_2.$$

From condition the $D^{\alpha}_{0^+}u(1) = D^{\beta}_{0^+}v(1) = 0$, we obtain that

$$T_1y_1 = \int_0^1 (1-s)^{\alpha-1}y_1(s)ds = 0, T_2y_2 = \int_0^1 (1-s)^{\beta-1}y_2(s)ds = 0.$$

On the other hand, for each $(y_1, y_2) \in Y$ satisfying $T_iy_i = 0$, $i = 1, 2$. Let

$$u(t) = I^{\alpha}_{0^+}\phi_q(I^{\gamma}_{0^+}y_1(t)), \quad v(t) = I^{\beta}_{0^+}\phi_q(I^{\gamma}_{0^+}y_2(t)),$$

then $(u, v) \in \operatorname{dom} L$ and

$$L(u(t), v(t)) = (D^{\gamma}_{0^+}\phi_p(D^{\alpha}_{0^+}u(t)), D^{\gamma}_{0^+}\phi_p(D^{\beta}_{0^+}v(t))),$$

so that $(y_1, y_2) \in \operatorname{Im} L$. Therefore, (3.2) holds. The proof is complete. $\square$

Lemma 3.3. *Let L be defined by (2.1), then L is a Fredholm operator of index zero, and the linear continuous projector operators $P : X \to X, Q : Y \to Y$ can be defined as*

$$P(u(t), v(t)) = (P_1u(t), P_2v(t)), \tag{3.3}$$

$$Q(y_1(t), y_2(t)) = (Q_1y_1(t), Q_2y_2(t)), \tag{3.4}$$

where

$$P_1u(t) = u(0) + \sum_{i=1}^{n-1} u^{(i)}t^i, P_2v(t) = v(0) + \sum_{i=1}^{n-1} v^{(i)}t^i,$$

$$Q_1y_1(t) = \Lambda(\sum_{i=1}^{n} \Lambda_i t^{i-1})T_1y_1(t), Q_2y_2(t) = \Lambda'(\sum_{i=1}^{n} \Lambda'_i t^{i-1})T_2y_2(t),$$

$$\frac{1}{\Lambda} = \sum_{i=1}^{n} \frac{\Lambda_i\Gamma(\alpha)\Gamma(i)}{\Gamma(\alpha+i)}, \frac{1}{\Lambda'} = \sum_{i=1}^{n} \frac{\Lambda'_i\Gamma(\beta)\Gamma(i)}{\Gamma(\beta+i)}.$$

Furthermore, the operator $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ can be written as

$$\begin{aligned} K_P(y_1(t), y_2(t)) &= (K_{P_1}y_1(t), K_{P_2}y_2(t)) \\ &= (I^{\alpha}_{0^+}\phi_q(I^{\gamma}_{0^+}y_1(t)), I^{\beta}_{0^+}\phi_q(I^{\gamma}_{0^+}y_2(t))), \quad \forall t \in [0, 1]. \end{aligned}$$

Proof. For each $(y_1, y_2) \in Y$ and (3.4), we have

$$\begin{aligned} Q_1^2y_1 &= Q_1[\Lambda\Big(\sum_{i=1}^{n} \Lambda_i t^{i-1}\Big)T_1y_1(t)] \\ &= \Lambda\Big(\sum_{i=1}^{n} \Lambda_i t^{i-1}\Big)T_1\Lambda\Big(\sum_{i=1}^{n} \Lambda_i t^{i-1}\Big)T_1y_1(t) \\ &= \Lambda\Big(\sum_{i=1}^{n} \Lambda_i t^{i-1}\Big)\sum_{i=1}^{n} \frac{\Lambda_i\Gamma(\alpha)\Gamma(i)}{\Gamma(\alpha+i)}T_1y_1(t) \\ &= \Lambda\sum_{i=1}^{n} \frac{\Lambda_i\Gamma(\alpha)\Gamma(i)}{\Gamma(\alpha+i)}Q_1y_1. \end{aligned}$$

From $\frac{1}{\Lambda} = \sum_{i=1}^{n} \frac{\Lambda_i\Gamma(\alpha)\Gamma(i)}{\Gamma(\alpha+i)}$, we obtain

$$Q_1^2y_1 = Q_1y_1. \tag{3.5}$$

Similarly, we can derive

$$Q_2^2y_2 = Q_1y_1. \tag{3.6}$$

So, for each $(y_1, y_2) \in Y$ and $t \in [0,1]$, it follows from (3.5) (3.6) that

$$Q^2(y_1, y_2) = Q(Q_1y_1, Q_1y_1) = (Q_1^2y_1, Q_2^2y_2) = (Q_1y_1, Q_1y_1) = Q(y_1, y_2).$$

Obviously,

$$\ker Q = \{(y_1, y_2) \in Y | T_1y_1 = T_2y_2 = 0\} = \operatorname{Im} L.$$

Let $(y_1, y_2) = [(y_1, y_2) - Q(y_1, y_2)] + (y_1, y_2)$, then $(y_1, y_2) - Q(y_1, y_2) \in \ker Q = \operatorname{Im} L, Q(y_1, y_2) \in \operatorname{Im} Q$. For $(y_1, y_2) \in \operatorname{Im} L \cap \operatorname{Im} Q$, we can get $(y_1, y_2) = (0, 0)$, then we have

$$Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

For each $(u, v) \in X$ by (3.3), we have

$$\begin{aligned}
P_1^2u(t) &= P_1(u(0) + \sum_{i=1}^{n-1} u^{(i)}t^i) \\
&= u(0) + \sum_{i=1}^{n-1}(u(0) + \sum_{i=1}^{n-1} u^{(i)}t^i)^{(i)}|_{t=0}t^i \\
&= u(0) + \sum_{i=1}^{n-1} u^{(i)}t^i \\
&= P_1u(t);
\end{aligned}$$

that is,

$$P_1^2u(t) = P_1u(t). \tag{3.7}$$

Similarly, we can derive that

$$P_2^2u(t) = P_2u(t). \tag{3.8}$$

So, for each $(u, v) \in X$ and $t \in [0, 1]$, it follows from (3.7) (3.8) that

$$P^2(u(t), v(t)) = P(u(t), v(t)).$$

Obviously, $\operatorname{Im} P = \ker L$,

$$\ker P = \{(u, v) \in X : u(0) = v(0) = u^{(i)}(0) = v^{(i)}(0) = 0, i = 1, 2, \dots, n-1\}.$$

Let $(u, v) = [(u, v) - P(u, v)] + P(u, v)$, we can get $(u, v) - P(u, v) \in \ker P$, $P(u, v) \in \operatorname{Im} P$, so $X = \ker P + \ker L$. By simple calculation, we can get $\ker L \cap \ker P = (0, 0)$, then

$$X = \ker L \oplus \ker P.$$

Thus

$$\dim \ker L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = n, \quad n \geq 2.$$

This means that L is a Fredholm operator of index zero.

From the definitions of P, K_p, it is easy to see that the generalized inverse of L is K_P. In fact, for $(y_1, y_2) \in \operatorname{Im} L$, we have

$$LK_P(y_1, y_2) = L(I_{0+}^{\alpha}\phi_q(I_{0+}^{\gamma}y_1(t)), I_{0+}^{\beta}\phi_q(I_{0+}^{\gamma}y_2(t))) = (y_1, y_2). \tag{3.9}$$

Moreover, for $(u, v) \in \operatorname{dom} L \cap \ker P$, we get $u(0) = v(0) = u^{(i)}(0) = v^{(i)}(0) = 0$, $i = 1, 2, \dots, n-1$. Hence

$$K_PL(u, v) = K_P(D_{0+}^{\gamma}\phi_p(D_{0+}^{\alpha}u(t)), D_{0+}^{\gamma}\phi_p(D_{0+}^{\beta}v(t))) = (u, v). \tag{3.10}$$

Combining (3.9) and (3.10), we know that $K_P = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$. The proof is complete. $\square$

Lemma 3.4. *Assume $\Omega \subset X$ is an open boundary subset such that* $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, *then N is L-compact on $\overline{\Omega}$.*

Proof. By the continuity of f, g, we can get that $QN(\overline{\Omega})$ and $K_P(I-Q)N(\overline{\Omega})$ are bounded. So, in view of the Arzela-Ascoli theorem, we need only prove that $K_P(I-Q)(\overline{\Omega}) \subset X$ is equicontinuous.

From the continuity of f, g, there exists a constant $M > 0$ such that

$$|(I - Q_i)N_i(u, v)| \leq M, \quad \forall t \in [0, 1],\ (u, v) \in \overline{\Omega},\ i = 1, 2,$$

where $I : C[0, 1] \to C[0, 1]$ is the indentity mapping. Furthermore, denote $K_{P,Q} = K_P(I - Q)N$ and for $0 \leq t_1 < t_2 \leq 1$,$(u, v) \in \overline{\Omega}$, we have

$$\begin{aligned}
&K_{P,Q}(u(t_2), v(t_2)) - K_{P,Q}(u(t_1), v(t_1))\\
&= (K_{P_1}(I - Q_1)N_1u(t_2) - K_{P_1}(I - Q_1)N_1u(t_1),\\
&\quad K_{P_2}(I - Q_2)N_2u(t_2) - K_{P_2}(I - Q_2)N_2u(t_1)),
\end{aligned}$$

From

$$\begin{aligned}
&|K_{P_1}(I - Q_1)N_1u(t_2) - K_{P_1}(I - Q_1)N_1u(t_1)|\\
&= \frac{1}{\Gamma(\alpha)}|\int_0^{t_2}(t_2 - s)^{\alpha-1}\phi_q(\frac{1}{\Gamma(\gamma)}\int_0^s(s - \tau)^{\gamma-1}I - Q_1)N_1u(\tau)d\tau)ds\\
&\quad - \int_0^{t_1}(t_1 - s)^{\alpha-1}\phi_q(\frac{1}{\Gamma(\gamma)}\int_0^s(s - \tau)^{\gamma-1}I - Q_1)N_1u(\tau)d\tau)ds|\\
&\leq \frac{\phi_q(M)}{\Gamma(\alpha)}|\int_0^{t_1}[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]ds + \int_{t_1}^{t_2}(t_2 - s)^{\alpha-1}ds|\\
&\leq \frac{\phi_q(M)}{\Gamma(\alpha)}(t_2^\alpha - t_1^\alpha),
\end{aligned}$$

and

$$\begin{aligned}
&|D_{0^+}^{\alpha-1}K_{P_1}(I - Q_1)N_1u(t_2) - D_{0^+}^{\alpha-1}K_{P_1}(I - Q_1)N_1u(t_1)|\\
&= |\int_0^{t_2}\phi_q(\frac{1}{\Gamma(\gamma)}\int_0^s(s - \tau)^{\gamma-1}(I - Q_1)N_1u(\tau)d\tau)ds\\
&\quad - \int_0^{t_1}\phi_q(\frac{1}{\Gamma(\gamma)}\int_0^s(s - \tau)^{\gamma-1}(I - Q_1)N_1u(\tau)d\tau)ds|\\
&= |\int_{t_1}^{t_2}\phi_q(\frac{1}{\Gamma(\gamma)}\int_0^s(s - \tau)^{\gamma-1}(I - Q_1)N_1u(\tau)d\tau)ds|\\
&\leq \phi_q(M)(t_2 - t_1).
\end{aligned}$$

Similarly,

$$\begin{gathered}
|K_{P_2}(I - Q_2)N_1u(t_2) - K_{P_2}(I - Q_2)N_1u(t_1)| \leq \frac{\phi_q(M)}{\Gamma(\beta)}(t_2^\beta - t_1^\beta),\\
|D_{0^+}^{\beta-1}K_{P_2}(I - Q_2)N_1u(t_2) - D_{0^+}^{\beta-1}K_{P_2}(I - Q_1)N_2u(t_1)| \leq \phi_q(M)(t_2 - t_1),
\end{gathered}$$

and since t^α, t^β are uniformly continuous on $[0, 1]$, we can get that $K_P(I-Q)N(\overline{\Omega}) \subset X$ is equicontinuous. Thus, we get that $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. The proof is complete. $\square$

Lemma 3.5. *Suppose* (H1)–(H2) *hold. Then the set*

$$\Omega_1 = \{(u,v) | (u,v) \in \operatorname{dom} L \setminus \ker L, L(u,v) = \lambda N(u,v), \lambda \in (0,1)\}$$

is bounded.

Proof. Take $(u,v) \in \Omega_1$, then $N(u,v) \in \operatorname{Im} L$. By (3.2),we have

$$T_1 N_1 u = 0, \quad T_2 N_2 v = 0.$$

By $L(u,v) = \lambda N(u,v)$ and $D^{\alpha}_{0+} u(0) = D^{\beta}_{0+} v(0) = 0$, we have

$$\begin{aligned}
&(u(t), v(t)) \\
&= \lambda \Big(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \big(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} f(\tau, v(\tau)) d\tau \big) ds + \sum_{i=0}^{n-1} c_i t^i, \\
&\quad \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \big(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} g(\tau, u(\tau)) d\tau \big) ds + \sum_{i=0}^{n-1} c_i' t^i \Big).
\end{aligned} \tag{3.11}$$

Together with (H2) means that there exist constants $t_0 \in [\xi, 1]$, $t_1 \in [\eta, 1]$ such that $|u(t_0)| \leq M_1$, $|v(t_1)| \leq M_2$. By (3.11), we have

$$\sum_{i=0}^{n-1} |c_i| t_0^i \leq M_1 + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \big(\frac{1}{\Gamma(\gamma)} \int_0^1 (1-\tau)^{\gamma-1} f(\tau, v(\tau)) d\tau \big) ds, \tag{3.12}$$

$$\sum_{i=0}^{n-1} |c_i'| t_1^i \leq M_2 + \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \phi_q \big(\frac{1}{\Gamma(\gamma)} \int_0^1 (1-\tau)^{\gamma-1} g(\tau, u(\tau)) d\tau \big) ds. \tag{3.13}$$

It follows from (H1) and (3.11) (3.12) that

$$\begin{aligned}
&|u(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \big(\frac{1}{\Gamma(\gamma)} \int_0^1 (1-\tau)^{\gamma-1} |f(\tau, v(\tau))| d\tau \big) ds + |c_0| + \frac{1}{\xi} \big(\sum_{i=1}^{n-1} |c_i| t_0^i \big) \\
&\leq \frac{M_1}{\xi} + \frac{1+\xi}{\xi \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \big(\frac{1}{\Gamma(\gamma)} \int_0^1 (1-\tau)^{\gamma-1} (a(t) + b(t) |v(t)|^{p-1}) d\tau \big) ds \\
&\leq \frac{M_1}{\xi} + \frac{1+\xi}{\xi \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \big(\frac{1}{\Gamma(\gamma)} \int_0^1 (1-\tau)^{\gamma-1} (\|a\|_\infty + \|b\|_\infty \|v\|_\infty^{p-1}) d\tau \big) ds \\
&= \frac{M_1}{\xi} + \frac{1+\xi}{\xi \Gamma(\alpha+1)} \phi_q \big(\frac{1}{\Gamma(\gamma+1)} (\|a\|_\infty + \|b\|_\infty \|v\|_\infty^{p-1}) \big) \\
&\leq \frac{M_1}{\xi} + \frac{2^{q-1}(1+\xi)}{\xi \Gamma(\alpha+1)} \big(\phi_q \big(\frac{\|a\|_\infty}{\Gamma(\gamma+1)} \big) + \big(\phi_q \frac{\|b\|_\infty \|v\|_\infty^{p-1}}{\Gamma(\gamma+1)} \big) \big) \\
&\leq \frac{M_1}{\xi} + \frac{2^{q-1}(1+\xi)}{\xi \Gamma(\alpha+1)} \big(\big(\frac{\|a\|_\infty}{\Gamma(\gamma+1)} \big)^{q-1} + \big(\frac{\|b\|_\infty}{\Gamma(\gamma+1)} \big)^{q-1} \|v\|_\infty \big);
\end{aligned}$$

that is,

$$\|u(t)\|_\infty \leq \frac{M_1}{\xi} + \frac{2^{q-1}(1+\xi)}{\xi \Gamma(\alpha+1)} \big(\big(\frac{\|a\|_\infty}{\Gamma(\gamma+1)} \big)^{q-1} + \big(\frac{\|b\|_\infty}{\Gamma(\gamma+1)} \big)^{q-1} \|v\|_\infty \big).$$

Similarly, from (H1), (3.11), (3.13) and $\phi_p(s+t) \le 2^p(\phi_p(s)+\phi_p(t))$, $s,t>0$, we obtain

$$\|v(t)\|_\infty \le \frac{M_2}{\eta} + \frac{2^{q-1}(1+\eta)}{\xi\Gamma(\beta+1)}\Big(\big(\frac{\|c\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1} + \big(\frac{\|d\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1}\|u\|_\infty\Big).$$

Let

$$\begin{gathered}
\frac{M_1}{\xi} + \frac{2^{q-1}(1+\xi)}{\xi\Gamma(\alpha+1)}\big(\frac{\|a\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1} = A, \quad \frac{2^{q-1}(1+\xi)}{\xi\Gamma(\alpha+1)}\big(\frac{\|b\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1} = B, \\
\frac{M_2}{\eta} + \frac{2^{q-1}(1+\eta)}{\eta\Gamma(\beta+1)}\big(\frac{\|c\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1} = A', \quad \frac{2^{q-1}(1+\eta)}{\eta\Gamma(\beta+1)}\big(\frac{\|d\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1} = B',
\end{gathered}$$

then, the condition

$$\|b\|_\infty\|d\|_\infty < \frac{(\Gamma(\gamma+1))^2}{4}\big(\frac{\xi\eta\Gamma(\alpha+1)\Gamma(\beta+1)}{(1+\xi)(1+\eta)}\big)^{1-q},$$

which by Theorem 3.1 could written as $BB' < 1$, so, we obtain

$$\|u(t)\|_\infty \le \frac{A+A'B}{1-BB'}, \quad \|v(t)\|_\infty \le \frac{A'+AB'}{1-BB'}.$$

By (3.12) and (3.13) we have

$$\begin{aligned}
|c_{n-1}| &\le \frac{M_1}{\xi} + \frac{1}{\xi\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\big(\frac{1}{\Gamma(\gamma)}\int_0^1 (1-\tau)^{\gamma-1}|f(\tau,v(\tau))|d\tau\big)ds \\
&\le \frac{M_1}{\xi} + \frac{2^{q-1}}{\xi\Gamma(\alpha+1)}\Big(\big(\frac{\|a\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1} + \big(\frac{\|b\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1}\|v\|_\infty\Big),
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
|c'_{n-1}| &\le \frac{M_2}{\eta} + \frac{1}{\eta\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\phi_q\big(\frac{1}{\Gamma(\gamma)}\int_0^1 (1-\tau)^{\gamma-1}|f(\tau,u(\tau))|d\tau\big)ds \\
&\le \frac{M_2}{\eta} + \frac{2^{q-1}}{\xi\Gamma(\beta+1)}\Big(\big(\frac{\|c\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1} + \big(\frac{\|d\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1}\|u\|_\infty\Big).
\end{aligned} \tag{3.15}$$

Then, by (3.11), (3.12) and (3.13) we obtain

$$\begin{aligned}
|D_0^{\alpha-1}u(t)| &\le \int_0^1 \phi_q\big(\frac{1}{\Gamma(\gamma)}\int_0^1 (1-\tau)^{\gamma-1}|f(\tau,v(\tau))|d\tau\big)ds + \frac{|c_{n-1}|t^{n-\alpha}}{\Gamma(n+1-\alpha)} \\
&\le \frac{M_1}{\xi} + \frac{2^{q-1}(1+\xi\Gamma(\alpha+1))}{\xi\Gamma(\alpha+1)}\Big(\big(\frac{\|a\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1} + \big(\frac{\|b\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1}\|v\|_\infty\Big),
\end{aligned}$$

$$\begin{aligned}
|D_0^{\beta-1}u(t)| &\le \int_0^1 \phi_q\big(\frac{1}{\Gamma(\gamma)}\int_0^1 (1-\tau)^{\gamma-1}|f(\tau,u(\tau))|d\tau\big)ds + \frac{|c'_{n-1}|t^{n-\beta}}{\Gamma(n+1-\beta)} \\
&\le \frac{M_2}{\eta} + \frac{2^{q-1}(1+\eta\Gamma(\beta+1))}{\xi\Gamma(\beta+1)}\Big(\big(\frac{\|a\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1} + \big(\frac{\|d\|_\infty}{\Gamma(\gamma+1)}\big)^{q-1}\|u\|_\infty\Big).
\end{aligned}$$

Hence the Ω_1 is bounded in X. The proof is complete. $\square$

Lemma 3.6. *Suppose that* (H3) *hold. Then the set*

$$\Omega_2 = \{(u,v) | (u,v) \in \ker L, N(u,v) \in \operatorname{Im} L\}$$

is bounded in X.

Proof. For $(u,v) \in \Omega_2$, we have $(u(t),v(t)) = (\sum_1^n \pi_i t^{i-1}, \sum_1^n \pi_i' t^{i-1}), \pi_i, \pi_i' \in R, i = 1,2,\dots,n$ and $T_1N_1(\sum_1^n \pi_i t^{i-1}) = T_2N_2(\sum_1^n \pi_i' t^{i-1}) = 0$. By (H3), we obtain that $\max\{|\pi_i|, |\pi_i'|\} \le B, i = 1,2,\dots,n$, so $\max\{\|u\|_\infty, \|v\|_\infty\} \le 2B$. Furthermore,

$$|D_{0^+}^{\alpha-1}u(t)| = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-1-\alpha}|\pi_n|ds \le \frac{|\pi|_n}{\Gamma(n+1-\alpha)} \le \frac{B}{\Gamma(n+1-\alpha)},$$

$$|D_{0^+}^{\beta-1}v(t)| \le \frac{B}{\Gamma(n+1-\beta)}.$$

Hence, Ω_2 is bounded in X. The proof is complete. □

Lemma 3.7. *Suppose that* (H3)(1) *holds. Then the set*

$$\Omega_3 = \{(u,v) \in \ker L | \lambda J(u,v) + (1-\lambda)Q(N_1u, \theta N_2 v) = (0,0), \lambda \in [0,1]\}$$

is bounded in X. If (H3)(1)(i) *holds, then $\theta = 1$, if* (H3)(1)(ii) *hold, then $\theta = -1$, where, $J : \ker L \to \operatorname{Im} Q$ is a linear isomorphism given by*

$$J(\sum_1^n \pi_i t^{i-1}, \sum_1^n \pi_i' t^{i-1}) = (\Lambda(\sum_1^n \Lambda_i)(\sum_1^n \pi_i' t^{i-1}), \Lambda'(\sum_1^n \Lambda_i')(\sum_1^n \pi_i t^{i-1})),$$

where $\Lambda(\sum_1^n \Lambda_i) \ne 0, \Lambda'(\sum_1^n \Lambda_i') \ne 0$.

Proof. For $(u,v) \in \Omega_3$, we have $(u(t),v(t)) = (\sum_1^n \pi_i t^{i-1}, \sum_1^n \pi_i' t^{i-1}), \pi_i, \pi_i' \in R, i = 1,2,\dots,n$, by (H3)(1)(i), there exists $\lambda \in [0,1]$ such that

$$\begin{aligned} &\lambda J(\sum_1^n \pi_i t^{i-1}, \sum_1^n \pi_i' t^{i-1}) + (1-\lambda)(\Lambda(\sum_1^n \Lambda_i)T_1N_1(\sum_1^n \pi_i t^{i-1}), \\ &\Lambda'(\sum_1^n \Lambda_i')T_2N_2(\sum_1^n \pi_i' t^{i-1}))) = (0,0). \end{aligned} \tag{3.16}$$

If $\lambda = 0$, we can get that $\max\{|\pi_i|, |\pi_i'|\} \le B$, $i = 1,2$, then $\max\{\|u\|_\infty, \|v\|_\infty\} \le 2B$. Hence, Ω_3 is bounded.

If $\lambda = 1$, then $u = v = 0$.

For $\lambda(0,1)$, let $\Lambda_i = \pi_i'$, $\Lambda_i' = \pi_i$, $i = 1,2,\dots,n$, if $\min\{|\pi_i|, |\pi_i'|\} > B$, $i = 1,2,\dots,n$, we have the following inequalities:

$$\lambda(\sum_1^n \pi_i')^2 + (1-\lambda)(\sum_1^n \pi_i')T_1N_1(\sum_1^n \pi_i) > 0,$$
$$\lambda(\sum_1^n \pi_i)^2 + (1-\lambda)(\sum_1^n \pi_i)T_2N_2(\sum_1^n \pi_i') > 0,$$

this contradicts (3.16), so, Ω_3 is bounded in X.

Similarly, if (H3)(1)(ii) holds, we have Ω_3 is bounded in X. The proof is complete. □

Lemma 3.8. *If* (H3)(2) *hold, then the set*

$$\Omega_3 = \{(u,v) \in \ker L | -\lambda J(u,v) + (1-\lambda)Q(N_1u, \theta N_2 v) = (0,0), \lambda \in [0,1]\}$$

is bounded in X.

The proof of the above lemma is similarly with Lemma 3.7, and it is omitted. Now with Lemmas 3.2–3.8 in hand, we prove our main result.

Proof the Theorem 3.1. Let Ω is a bounded open set of X with $\cup_{i=1}^{3} \subset \Omega$. By Lemma 3.4, we can get that N is L-compact on $\overline{\Omega}$. Then by Lemmas 3.5 and 3.6, we have (1) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0, 1)$; (2) $Nx \notin \operatorname{Im} L$ for every $x \in \ker L \cap \partial\Omega$; we need to prove only (3) $\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$.

Take
$$H(u, v, \lambda) = \pm\lambda J(u, v) + (1 - \lambda)Q(N_1 u, \theta N_2 v),$$
according to Lemma 3.7, we have $H(u, v, \lambda) \neq 0$ for $(u, v) \in \partial\Omega \cap \ker L$. By the homotopy property of degree, we can get
$$\begin{aligned}
\deg(QN|_{\ker L}, \Omega \cap \ker L, (0, 0)) &= \deg(H(\cdot, 0), \Omega \cap \ker L, (0, 0)) \\
&= \deg(H(\cdot, 1), \Omega \cap \ker L, (0, 0)) \\
&= \deg(\pm J, \Omega \cap \ker L, (0, 0)) \neq 0.
\end{aligned}$$
By Theorem 2.1, we obtain that $L(u, v) = N(u, v)$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$; i.e, problem (1.1) has at least one solution in X, The proof is complete. □

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