

NON-EXISTENCE OF LIMIT CYCLES FOR PLANAR VECTOR FIELDS

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ABSTRACT. This article presents sufficient conditions for the non-existence of limit cycles for planar vector fields. Classical methods for the nonexistence of limit cycles are connected with the theory developed here.

1. INTRODUCTION

Two fundamental problems of the qualitative theory of planar differential equations are the center problem, and the determination of the number of limit cycles and their location in the phase space, see for instance [1, 11, 17]. We recall that a limit cycle is an isolated periodic solution of a differential equation, see [19]. The notion of limit cycle of a planar vector field was defined by Poincaré [16] at the end of 19th century. It was not until some decades later that van der Pol [18], Liénard [14] and Andronov [1] proved that the periodic orbit of a self-sustained oscillation occurring in a vacuum tube circuit was in fact a limit cycle. This fact had been established by Poincaré himself 20 years before, see [12]. Later on the limit cycles have been studied extensively by mathematicians and physicists focusing on properties such as nonexistence, existence, and uniqueness.

The classical method for proving the nonexistence of limit cycles in a simply connected region is the Bendixson–Dulac method, see for instance [19] where variants can be found. The method of Dulac functions also gives upper bounds for the number of closed trajectories in a multiply connected region, see also [19].

The problem of existence is the subject of the Poincaré–Bendixson theorem. The uniqueness problem is, in general, more difficult. Some criteria are known but the sufficient conditions of the known methods are very restrictive, see [19] and references therein. The Poincaré return map defined in a transversal section to the planar flow is one of the best methods for studying the nonexistence, existence and uniqueness of limit cycles, but in general such analysis is not very easy.

When the planar differential system has more than one limit cycle, the problem of their distribution on the plane appears. In fact all these problems are collected in the 16th Hilbert problem about the maximum number and distribution of limit cycles for a polynomial vector field of degree n , see [15].

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In this work we consider \mathcal{C}^1 two-dimensional autonomous systems of differential equations

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (1.1)$$

defined on an open subset U of \mathbb{R}^2 and their corresponding vector field $\mathcal{X} = P\partial/\partial x + Q\partial/\partial y$ on U . A non-constant \mathcal{C}^1 function $V : U \rightarrow \mathbb{R}$ is an *inverse integrating factor* for \mathcal{X} if it satisfies $\mathcal{X}V = V \operatorname{div} \mathcal{X}$ on U ; that is, if

$$P \frac{\partial V}{\partial x} + Q \frac{\partial V}{\partial y} = V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right), \quad (1.2)$$

holds on U . In such a case the function $1/V$ is an integrating factor for the system of differential equations on $U \setminus \Sigma$, where $\Sigma = \{(x, y) \in U : V(x, y) = 0\} = V^{-1}(0)$. Since $V^{-1}(0)$ is made up by solutions of the differential system (1.1) the set $V^{-1}(0)$ is invariant under the local flow induced by \mathcal{X} and as we will see immediately it contains any limit cycle contained in U of \mathcal{X} . The knowledge of an inverse integrating factor defined in U allows the computation of a first integral in $W = U \setminus \{V = 0\}$ by the line integral

$$H(x, y) = \int_{(x_0, y_0)}^{(x, y)} \frac{P(x, y)dy - Q(x, y)dx}{V(x, y)},$$

where $(x_0, y_0) \in W$ is any point. The inverse integrating factor has also been used to study the center problem, see [3, 4, 10].

In [9] the following result was established.

Theorem 1.1. *Let (P, Q) be a \mathcal{C}^1 vector field defined in the open subset U of \mathbb{R}^2 . Let $V = V(x, y)$ be a \mathcal{C}^1 solution of the linear partial differential equation (1.2). If γ is a limit cycle of the vector field (P, Q) in the simply connected domain of definition of V , then γ is contained in $\Sigma = \{(x, y) \in U : V(x, y) = 0\}$.*

The proof of this result is straightforward. The existence of the inverse integrating factor V defined in a simply connected region U implies that the vector field $(P/V, Q/V)$ is Hamiltonian in $U \setminus \Sigma$. Since the flow of a Hamiltonian vector field preserves area and in a neighborhood of a limit cycle a flow does not preserve the area, the theorem follows. In fact the same result is obtained for any polycycle which is a limit set of the vector field (P, Q) , see [7, 8] and the references therein. Moreover it is known that there always exists a smooth inverse integrating factor in a neighborhood of a limit cycle of a planar analytic vector fields, see [7]. From this result it is clear that the inverse integrating factor has an important role in the qualitative study of differential equations and in particular in the study of the limit cycles, see [8] and references therein. From Theorem 1.1 we can give the following corollary.

Corollary 1.2. *Let (P, Q) be a \mathcal{C}^1 vector field defined in the open subset U of \mathbb{R}^2 . Let $V = V(x, y)$ be a \mathcal{C}^1 solution of the linear partial differential equation (1.2). Let $\Sigma = \{(x, y) \in U : V(x, y) = 0\}$. The divergence $\partial(P/V)/\partial x + \partial(Q/V)/\partial y$ is identically zero in the simply connected domain of definition of $V \setminus \Sigma$ and then any limit cycle (in fact a closed trajectory) which lies entirely in the domain of definition of V must be contained in Σ .*

In this article we are interested in the study of nonexistence of limit cycles using the inverse integrating factor. From Theorem 1.1 we can study the existence and nonexistence of limit cycles of planar vector fields from the explicit knowledge of an

inverse integrating factor. For instance, in [6] it is shown that system (1.1) with a Darboux inverse integrating factor of the form $V = \exp(R)$ with rational R cannot have limit cycles. It is clear that this result is, in fact, a straightforward consequence of Theorem 1.1. First we recall the classical theorems about nonexistence of limit cycles.

Theorem 1.3 (Bendixson). *If the divergence $\partial P/\partial x + \partial Q/\partial y$ of system (1.1) has constant sign in a simply connected region U , and is not identically zero on any subregion of U , then system (1.1) does not possess any limit cycle (in fact a closed trajectory) which lies entirely in U .*

The proof is by contradiction assuming the existence of a limit cycle and applying Green's formula. In fact mainly depends on the following fact.

Proposition 1.4. *In the interior D of any closed trajectory of system (1.1) of a simply connected region we have*

$$\int \int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy = 0. \quad (1.3)$$

The classical Bendixson theorem and the inverse integrating factor are related by the following proposition.

Proposition 1.5. *Let (P, Q) be a C^1 vector field defined in the open subset U of \mathbb{R}^2 . Let $V = V(x, y)$ be a C^1 solution of the linear partial differential equation (1.2). Let $\Sigma = \{(x, y) \in U : V(x, y) = 0\}$. If the divergence $\partial P/\partial x + \partial Q/\partial y$ of system (1.1) has constant sign in the simply connected domain of definition of $V \setminus \Sigma$, then any limit cycle (in fact a closed trajectory) which lies entirely in the simply connected domain of definition of V must be contained in Σ .*

Proof. Assume that γ is a closed trajectory of system (1.1) which lies entirely in the simply connected domain of definition of V then integrating along γ the equality (1.2) we obtain

$$\oint_{\gamma} \frac{P \frac{\partial V}{\partial x} + Q \frac{\partial V}{\partial y}}{V} dt = \oint_{\gamma} \frac{d \ln |V|}{dt} dt = - \oint_{\gamma} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dt.$$

Obviously the left hand side of the above formula is equal to zero. On the other hand, the integrand on the right hand side of the equality has constant sign in the simply connected domain of definition of V . Hence its integral value should be different from zero, a contradiction with the existence of the closed trajectory except if the trajectory γ is contained in $\Sigma = \{(x, y) \in U : V(x, y) = 0\}$ because in that case the integrand of right hand side is not well-defined. \square

Theorem 1.6 (Bendixson-Dulac). *If there exists a continuously differentiable function $B(x, y)$ in a simply connected region U such that $\partial(BP)/\partial x + \partial(BQ)/\partial y$ has constant sign and is not identically zero in any subregion, then system (1.1) does not possess any limit cycle (in fact a closed trajectory) which lies entirely in U .*

The proof of Theorem 1.6 follows from the proof of Theorem 1.3 using BP and BQ instead of P and Q respectively. The function $B(x, y)$ is called *Dulac function*, and the method of proving nonexistence of closed trajectories is called the method of Dulac functions. The relation of the Bendixson-Dulac theorem and the inverse integrating factor is established through the following proposition.

Proposition 1.7. *Let (P, Q) be a C^1 vector field defined in the open subset U of \mathbb{R}^2 . Let $V = V(x, y)$ be a C^1 solution of the linear partial differential equation (1.2). Let $B(x, y)$ a continuously differentiable function in U such that $\partial(BP)/\partial x + \partial(BQ)/\partial y$ has constant sign, then any limit cycle (in fact a closed trajectory) which lies entirely in the simply connected domain of definition of V must be contained in $\Sigma = \{(x, y) \in U : V(x, y) = 0\}$.*

Proof. In the proof we use the result that if V is an inverse integrating factor of the vector field (P, Q) then the vector (BP, BQ) has the inverse integrating factor BV , see [3]. Therefore we can write the following equality

$$\oint_{\gamma} \frac{BP \frac{\partial BV}{\partial x} + BQ \frac{\partial BV}{\partial y}}{BV} dt = \oint_{\gamma} \frac{d \ln |BV|}{dt} dt = - \oint_{\gamma} \left(\frac{\partial BP}{\partial x} + \frac{\partial BQ}{\partial y} \right) dt,$$

and analogous arguments than in the proof of Proposition 1.5 are valid. \square

In fact Theorem 1.3 and 1.6 can be extended to multiply connected regions, see [19]. The next important result in order to study the nonexistence and existence of limit cycles was obtained in [5].

Theorem 1.8 (Cherkas). *Suppose that in a simply connected domain $U \subset \mathbb{R}^2$, there exists a function $\Psi(x, y)$ of class C^1 and a number $k > 0$ such that*

$$k\Psi \operatorname{div} \mathcal{X} + \mathcal{X}\Psi > 0,$$

then the domain U contains no limit cycles of system (1.1).

2. STATEMENT OF THE MAIN RESULTS

The following result provides a necessary for (1.1) to have an inverse integrating factor of the form $V(f(x, y))$.

Theorem 2.1. *The vector field $\mathcal{X} = (P, Q)$ defined in an open subset U of \mathbb{R}^2 admits an inverse integrating factor of the form $V = V(z)$ where $z = f(x, y)$ if and only if*

$$\frac{\operatorname{div} \mathcal{X}}{(Pf_x + Qf_y)} = \alpha(z), \tag{2.1}$$

where $\alpha(z)$ is a function exclusively of z and in such case the inverse integrating factor is of the form

$$V = \exp \left(\int^z \alpha(s) ds \right). \tag{2.2}$$

Moreover if γ is a limit cycle of the vector field (P, Q) in the simply connected domain of definition of (2.2), then γ is contained in $\Sigma = \{(x, y) \in U : e^{\int^z \alpha(s) ds} = 0\}$.

The following corollary to Theorem 2.1 imposes conditions for (1.1) not to have limit cycles.

Corollary 2.2. *The following statements hold:*

- (i) *Taking $z = f(x, y) = x$ if $\alpha(z)$ is a function x (or taking $z = f(x, y) = y$ if $\alpha(z)$ is a function y), then system (1.1) which has the associated vector field $\mathcal{X} = (P, Q)$ defined in the open subset U of \mathbb{R}^2 has no limit cycles in the domain of definition of (2.2).*

- (ii) Taking $z = f(x, y) = xy$ if $\alpha(z)$ is a function xy , then system (1.1) which has the associated vector field $\mathcal{X} = (P, Q)$ defined in the open subset U of \mathbb{R}^2 has no limit cycles in the domain of definition of (2.2).

In [13], these two statements of corollary 2.2 are given but they are not stated correctly. In both cases the authors of [13] state that, under the described conditions, system (1.1) does not have limit cycles. This is wrong. The correct conclusion is that system (1.1) does not have limit cycles in the simply connected domain of definition of (2.2). In general this domain of definition is not all \mathbb{R}^2 as the following example shows. Suppose that $\alpha(s) = 1/(s \ln s)$ which implies that $V = \exp(\int^x \alpha(s) ds) = \ln x$. Therefore applying Theorem 1.1 if γ is a limit cycle of the vector field (P, Q) in the domain of definition of V , which is in this case $x > 0$, then γ is contained in $\Sigma = \{(x, y) \in U : V(x, y) = 0\}$. Therefore the vector field (P, Q) has no limit cycles for $x > 0$ but it can have limit cycles in the domain $x \leq 0$. Another example is given later in the proof of Proposition 4.1 where $\alpha(s) = -1/s$ and $V = 1/x$. In fact the statement of Theorem 1.1 is not correctly stated in [13].

3. PROOF OF THEOREM 2.1 AND COROLLARY 2.2

Proof of Theorem 2.1. We assume that $V = V(z)$ where $z = f(x, y)$, then applying the chain rule, equality (1.2) is transformed into

$$P \frac{dV}{dz} \frac{\partial f}{\partial x} + Q \frac{dV}{dz} \frac{\partial f}{\partial y} = V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right). \quad (3.1)$$

We can isolate dV/dz from equation (3.1) and we have

$$\frac{dV}{dz} = \frac{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)}{P f_x + Q f_y}, \quad (3.2)$$

where f_x and f_y are the partial derivatives of f with respect to x and y . The left hand-side of equation (3.2) is a function of z , hence the right hand-side must be also a function of z and we obtain

$$\frac{dV}{dz} = \frac{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)}{P f_x + Q f_y} = \alpha(z). \quad (3.3)$$

If equality (3.3) is satisfied we get that the inverse integrating factor takes the form

$$V = e^{\int^z \frac{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)}{P f_x + Q f_y} dz} = e^{\int^z \alpha(s) ds}.$$

Now applying Theorem 1.1 the proof of Theorem 2.1 follows. \square

Proof of Corollary 2.2. (i) In the case $z = f(x, y) = x$ and $\alpha(z)$ is a function x we have that equation (3.3) takes the form $\frac{d}{dx}(\log V) = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) / P = \alpha(x)$. Hence $V = e^{\int^x \alpha(s) ds}$ and applying Theorem 1.1 the result follows. The proof is analogous for the case $z = f(x, y) = y$ and when $\alpha(z)$ is a function y .

(ii) In the case $z = f(x, y) = xy$ and $\alpha(z)$ is a function xy we have that equation (3.3) takes the form $\frac{d}{dz}(\log V) = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) / (P y + Q x) = \alpha(z)$. Hence $V = e^{\int^z \alpha(s) ds}$ and applying Theorem 1.1 we obtain the result. \square

4. EXAMPLES

In this section we give two examples where we apply the results developed in this work.

Proposition 4.1. *Consider the differential system*

$$\dot{x} = -x - 4xy - 5y^2, \quad \dot{y} = 3x + 2y + y^2. \quad (4.1)$$

System (4.1) has no limit cycles in the domain $\mathbb{R}^2 \setminus \{(x, y) \mid x + y^2 = 0\}$.

Proof. Assume that system (4.1) has an inverse integrating factor which is a function of $x + y^2$. Therefore if we take the particular case $z = x + y^2$ in equation (3.1) we obtain

$$(-x - 4xy - 5y^2) \frac{dV}{dz} + (3x + 2y + y^2) \frac{dV}{dz}(2y) = V(1 - 2y). \quad (4.2)$$

Isolating dV/dz from this equation and we have

$$\frac{\frac{dV}{dz}}{V} = \frac{1 - 2y}{(-1 + 2y)(x + y^2)} = -\frac{1}{x + y^2} = -\frac{1}{z}. \quad (4.3)$$

Integrating the differential equation (4.3) we obtain

$$V = e^{-\int^z \frac{1}{s} ds} = \frac{1}{z} = \frac{1}{x + y^2}.$$

Hence, applying Theorem 1.1 the result follows. In fact if system (4.1) has a limit cycle, this limit cycle must cut the curve $x + y^2 = 0$. \square

To prove the next result we need to recall the following proposition given in [3] whose proof is obvious from the definition of inverse integrating factor.

Proposition 4.2. *Let (P_1, Q_1) and (P_2, Q_2) be two C^1 vector fields defined in an open subset $U \subset \mathbb{R}^2$, which have the same inverse integrating factor $V(x, y)$; i.e.,*

$$\begin{aligned} P_1 \frac{\partial V}{\partial x} + Q_1 \frac{\partial V}{\partial y} - \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} \right) V &= 0, \\ P_2 \frac{\partial V}{\partial x} + Q_2 \frac{\partial V}{\partial y} - \left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} \right) V &= 0, \end{aligned}$$

then the vector field $(P_1 + \lambda P_2, Q_1 + \lambda Q_2)$ has also the function $V(x, y)$ as inverse integrating factor, for arbitrary values of the parameter λ .

Using this proposition we can establish the following result.

Proposition 4.3. *Consider the differential system*

$$\dot{x} = x + 8xy - 5y^2, \quad \dot{y} = 6x - 2y - 2y^2. \quad (4.4)$$

System (4.4) has no limit cycles in all \mathbb{R}^2 .

Proof. To obtain the result we decompose the system into two homogeneous systems.

$$P_1 = x, \quad Q_1 = 6x - 2y, \quad P_2 = 8xy - 5y^2, \quad Q_2 = -2y^2.$$

Now we try to compute the inverse integrating factor for each homogeneous system. It is well-known that the inverse integrating factor of any homogeneous system is given by $V = yP_n - xQ_n$, see [2]. Therefore the vector field (P_1, Q_1) has the inverse integrating factor $V_1 = x(2x - y)$ and the vector field (P_2, Q_2) has the inverse integrating factor $V_2 = y^2(2x - y)$. Moreover the respective first integrals

associated to each integrating factor are $H_1 = x^2(2x - y)$ and $H_2 = y^4(2x - y)$ respectively.

We cannot directly apply Proposition 4.2 because the two vector fields have a different inverse integrating factor. Now we apply another well-known result which says that if V is an inverse integrating factor and if H is a first integral then VH is also an inverse integrating factor. Hence we look for an inverse integrating factor of system (4.4) that satisfies $V = V_1\mathcal{F}_1(H_1) = V_2\mathcal{F}_2(H_2)$, where $\mathcal{F}_1(H_1)$ and $\mathcal{F}_2(H_2)$ are arbitrary functions of H_1 and H_2 respectively. In our case we have

$$V = x(2x - y)\mathcal{F}_1(x^2(2x - y)) = y^2(2x - y)\mathcal{F}_2(y^4(2x - y)), \quad (4.5)$$

which simplifying implies

$$\mathcal{F}_1(x^2(2x - y)) = \frac{y^2}{x}\mathcal{F}_2(y^4(2x - y)). \quad (4.6)$$

Now we take $u = y^2/x$ and equation (4.6) takes the form

$$\mathcal{F}_1(\beta) = u\mathcal{F}_2(u^2\beta), \quad (4.7)$$

where $\beta = x^2(2x - y)$. For equation (4.7) be satisfied we need to take

$$\mathcal{F}_2(u^2\beta) = \frac{1}{\sqrt{u^2\beta}} = \frac{1}{u\sqrt{\beta}}, \quad \text{with } u > 0,$$

which implies $\mathcal{F}_1(\beta) = 1/\sqrt{\beta}$ and substituting in (4.5) we have

$$V = x(2x - y)\frac{1}{x\sqrt{2x - y}} = \sqrt{2x - y}, \quad \text{with } 2x - y > 0, \quad (4.8)$$

Now we can apply Proposition 4.2 because the two vector fields (P_1, Q_1) and (P_2, Q_2) have the same inverse integrating factor and consequently (4.8) is also the inverse integrating factor of system (4.4). Moreover, for the case $y - 2x > 0$, it is easy to see that $V = \sqrt{y - 2x}$ is also an inverse integrating factor of system (4.4). Now it is clear that we could have applied Theorem 2.1 with $z = 2x + y$ in order to find the inverse integrating factor. Finally we can apply Theorem 1.1 and we obtain that in the domain $2x - y \geq 0$ there are not limit cycles and in the domain $y - 2x \geq 0$. Moreover the line $2x - y = 0$ is an invariant algebraic curve of system (4.8), and consequently no limit cycle can cross this line. Therefore system (4.8) has no limit cycles in all \mathbb{R}^2 . \square

The method developed in the previous example can also applied to the example of Proposition 4.1. In that case the common inverse integrating factor is $V = \sqrt{x + y}$ which ensures the nonexistence of limit cycles in the domain $x + y \geq 0$.

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