Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 80, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

LAPLACE TRANSFORM AND GENERALIZED HYERS-ULAM STABILITY OF LINEAR DIFFERENTIAL EQUATIONS

QUSUAY H. ALQIFIARY, SOON-MO JUNG

Abstract. By applying the Laplace transform method, we prove that the linear differential equation

$$y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) = f(t)$$

has the generalized Hyers-Ulam stability, where α_k is a scalar, y and f are n times continuously differentiable and of exponential order.

1. Introduction

In 1940, Ulam [24] posed a problem concerning the stability of functional equations: "Give conditions in order for a linear function near an approximately linear function to exist." A year later, Hyers [5] gave an answer to the problem of Ulam for additive functions defined on Banach spaces: Let X_1 and X_2 be real Banach spaces and $\varepsilon > 0$. Then for every function $f : X_1 \to X_2$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \quad (x, y \in X_1),$$

there exists a unique additive function $A : X_1 \to X_2$ with the property

$$\|f(x) - A(x)\| \le \varepsilon \quad (x \in X_1).$$

After Hyers's result, many mathematicians have extended Ulam's problem to other functional equations and generalized Hyers's result in various directions (see [3, 6, 10, 18]). A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation $\varphi(f, y, y', \dots, y^{(n)}) = 0$ has Hyers-Ulam stability if for a given $\varepsilon > 0$ and a function y such that $|\varphi(f, y, y', \dots, y^{(n)})| \le \varepsilon$, there exists a solution y_a of the differential equation such that $|y(t) - y_a(t)| \le K(\varepsilon)$ and $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$. If the preceding statement is also true when we replace ε and $K(\varepsilon)$ by $\varphi(t)$ and $\Phi(t)$, where φ, Φ are appropriate functions not depending on y and y_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability (or Hyers-Ulam-Rassias stability).

2000 *Mathematics Subject Classification.* 44A10, 39B82, 34A40, 26D10.
Key words and phrases. Laplace transform method; differential equations; generalized Hyers-Ulam stability.

Submitted March 5, 2014. Published March 21, 2014.

Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [14, 15]). Thereafter, Alsina and Ger published their paper [1], which handles the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$: If a differentiable function $y(t)$ is a solution of the inequality $|y'(t) - y(t)| \leq \varepsilon$ for any $t \in (a, \infty)$, then there exists a constant c such that $|y(t) - ce^t| \leq 3\varepsilon$ for all $t \in (a, \infty)$.

Those previous results were extended to the Hyers-Ulam stability of linear differential equations of first order and higher order with constant coefficients in [12, 22, 23] and in [13], respectively. Furthermore, Jung has also proved the Hyers-Ulam stability of linear differential equations (see [7, 8, 9]). Rus investigated the Hyers-Ulam stability of differential and integral equations using the Gronwall lemma and the technique of weakly Picard operators (see [20, 21]). Recently, the Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied by using the method of integral factors (see [11, 25]). The results given in [8, 11, 12] have been generalized by Cimpean and Popa [2] and by Popa and Raşa [16, 17] for the linear differential equations of nth order with constant coefficients.

Recently, Rezaei, Jung and Rassias have proved the Hyers-Ulam stability of linear differential equations by using the Laplace transform method (see [19]).

In this paper, by using the Laplace transform method, we prove that the linear differential equation of the nth order

$$y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) = f(t)$$

has the generalized Hyers-Ulam stability, where α_k is a scalar, y and f are n times continuously differentiable and of exponential order, respectively.

2. Preliminaries

Throughout this paper, $\mathbb{F}$ will denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A function $f : (0, \infty) \to \mathbb{F}$ is said to be of exponential order if there are constants $A, B \in \mathbb{R}$ such that

$$|f(t)| \leq Ae^{tB}$$

for all $t > 0$. For each function $f : (0, \infty) \to \mathbb{F}$ of exponential order, we define the Laplace transform of f by

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

There exists a unique number $-\infty \leq \sigma < \infty$ such that this integral converges if $\Re(s) > \sigma$ and diverges if $\Re(s) < \sigma$, where $\Re(s)$ denotes the real part of the (complex) number s. The number σ is called the abscissa of convergence and denoted by σ_f. It is well known that $|F(s)| \to 0$ as $\Re(s) \to \infty$. Furthermore, f is analytic on the open right half plane $\{s \in \mathbb{C} : \Re(s) > \sigma\}$ and we have

$$\frac{d}{ds}F(s) = -\int_0^\infty te^{-st}f(t)dt \quad (\Re(s) > \sigma).$$

The Laplace transform of f is sometimes denoted by $\mathcal{L}(f)$. It is well known that $\mathcal{L}$ is linear and one-to-one.

Conversely, let $f(t)$ be a continuous function whose Laplace transform $F(s)$ has the abscissa of convergence σ_f, then the formula for the inverse Laplace transforms yields

$$f(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\alpha-iT}^{\alpha+iT} F(s)e^{st}ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\alpha+iy)t}F(\alpha+iy)dy$$

for any real constant $\alpha > \sigma_f$, where the first integral is taken along the vertical line $\Re(s) = \alpha$ and converges as an improper Riemann integral and the second integral is used as an alternative notation for the first integral (see [4]). Hence, we have

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}dt \quad (\Re(s) > \sigma_f)$$
$$\mathcal{L}^{-1}(F)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\alpha+iy)t}F(\alpha+iy)dy \quad (\alpha > \sigma_f).$$

The convolution of two integrable functions $f, g : (0,\infty) \to \mathbb{F}$ is defined by

$$(f * g)(t) := \int_0^t f(t-x)g(x)dx.$$

Then $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$.

Lemma 2.1 ([19]). *Let $P(s) = \sum_{k=0}^n \alpha_k s^k$ and $Q(s) = \sum_{k=0}^m \beta_k s^k$, where m, n are nonnegative integers with $m < n$ and α_k, β_k are scalars. Then there exists an infinitely differentiable function $g : (0,\infty) \to \mathbb{F}$ such that*

$$\mathcal{L}(g) = \frac{Q(s)}{P(s)} \quad (\Re(s) > \sigma_P)$$

and

$$g^{(i)}(0) = \begin{cases} 0 & \text{for } i \in \{0, 1, \dots, n-m-2\}, \\ \beta_m/\alpha_n & \text{for } i = n-m-1 \end{cases}$$

where $\sigma_P = \max\{\Re(s) : P(s) = 0\}$.

Lemma 2.2 ([19]). *Given an integer $n > 1$, let $f : (0,\infty) \to \mathbb{F}$ be a continuous function and let $P(s)$ be a complex polynomial of degree n. Then there exists an n times continuously differentiable function $h : (0,\infty) \to \mathbb{F}$ such that*

$$\mathcal{L}(h) = \frac{\mathcal{L}(f)}{P(s)} \quad (\Re(s) > \max\{\sigma_P, \sigma_f\}),$$

where $\sigma_P = \max\{\Re(s) : P(s) = 0\}$ and σ_f is the abscissa of convergence for f. In particular, it holds that $h^{(i)}(0) = 0$ for every $i \in \{0, 1, \dots, n-1\}$.

3. Main Results

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. In the following theorem, using the Laplace transform method, we investigate the generalized Hyers-Ulam stability of the linear differential equation of first order

$$y'(t) + \alpha y(t) = f(t). \tag{3.1}$$

Theorem 3.1. *Let α be a constant in $\mathbb{F}$ and let $\varphi : (0,\infty) \to (0,\infty)$ be an integrable function. If a continuously differentiable function $y : (0,\infty) \to \mathbb{F}$ satisfies the inequality*

$$|y'(t) + \alpha y(t) - f(t)| \le \varphi(t) \tag{3.2}$$

for all $t > 0$, then there exists a solution $y_\alpha : (0, \infty) \to \mathbb{F}$ of the differential equation (3.1) *such that*

$$|y(t) - y_\alpha(t)| \le e^{-\Re(\alpha)t} \int_0^t e^{\Re(\alpha)x} \varphi(x) dx$$

for any $t > 0$.

Proof. If we define a function $z : (0, \infty) \to \mathbb{F}$ by $z(t) := y'(t) + \alpha y(t) - f(t)$ for each $t > 0$, then

$$\mathcal{L}(y) - \frac{y(0) + \mathcal{L}(f)}{s + \alpha} = \frac{\mathcal{L}(z)}{s + \alpha}. \tag{3.3}$$

If we set $y_\alpha(t) := y(0)e^{-\alpha t} + (E_{-\alpha} * f)(t)$, where $E_{-\alpha}(t) = e^{-\alpha t}$, then $y_\alpha(0) = y(0)$ and

$$\mathcal{L}(y_\alpha) = \frac{y(0) + \mathcal{L}(f)}{s + \alpha} = \frac{y_\alpha(0) + \mathcal{L}(f)}{s + \alpha}. \tag{3.4}$$

Hence, we get

$$\mathcal{L}\big(y'_\alpha(t) + \alpha y_\alpha(t)\big) = s\mathcal{L}(y_\alpha) - y_\alpha(0) + \alpha \mathcal{L}(y_\alpha) = \mathcal{L}(f).$$

Since $\mathcal{L}$ is a one-to-one operator, it holds that

$$y'_\alpha(t) + \alpha y_\alpha(t) = f(t).$$

Thus, y_α is a solution of (3.1).

Moreover, by (3.3) and (3.4), we obtain $\mathcal{L}(y) - \mathcal{L}(y_\alpha) = \mathcal{L}(E_{-\alpha} * z)$. Therefore, we have

$$y(t) - y_\alpha(t) = (E_{-\alpha} * z)(t). \tag{3.5}$$

In view of (3.2), it holds that

$$|z(t)| \le \varphi(t) \tag{3.6}$$

for all $t > 0$, and it follows from the definition of convolution, (3.5), and (3.6) that

$$\begin{aligned} |y(t) - y_\alpha(t)| &= |(E_{-\alpha} * z)(t)| \\ &= \Big| \int_0^t E_{-\alpha}(t - x) z(x) dx \Big| \\ &\le \int_0^t \big| e^{-\alpha(t-x)} \big| \varphi(x) dx \\ &\le e^{-\Re(\alpha)t} \int_0^t e^{\Re(\alpha)x} \varphi(x) dx \end{aligned}$$

for all $t > 0$. (We remark that $\int_0^t e^{\Re(\alpha)x} \varphi(x) dx$ exists for each $t > 0$ provided φ is an integrable function.) □

Corollary 3.2. *Let α be a constant in $\mathbb{F}$ and let $\varphi : (0, \infty) \to (0, \infty)$ be an integrable function such that*

$$\int_0^t e^{\Re(\alpha)(x-t)} \varphi(x) dx \le K\varphi(t) \tag{3.7}$$

for all $t > 0$ and for some positive real constant K. If a continuously differentiable function $y : (0, \infty) \to \mathbb{F}$ satisfies the inequality (3.2) *for all $t > 0$, then there exists a solution $y_\alpha : (0, \infty) \to \mathbb{F}$ of the differential equation* (3.1) *such that*

$$|y(t) - y_\alpha(t)| \le K\varphi(t)$$

for any $t > 0$.

In the following remark, we show that there exists an integrable function $\varphi : (0,\infty) \to (0,\infty)$ satisfying the condition (3.7).

Remark 3.3. Let α be a constant in $\mathbb{F}$ with $\Re(\alpha) > -1$. If we define $\varphi(t) = Ae^t$ for all $t > 0$ and for some $A > 0$, then we have

$$\begin{aligned} \int_0^t e^{\Re(\alpha)(x-t)}\varphi(x)dx &= \int_0^t e^{\Re(\alpha)(x-t)}Ae^x dx \\ &= \frac{1}{1+\Re(\alpha)}\Big(Ae^t - Ae^{-\Re(\alpha)t}\Big) \\ &\le \frac{1}{1+\Re(\alpha)}\varphi(t) \end{aligned}$$

for each $t > 0$.

Now, we apply the Laplace transform method to the proof of the generalized Hyers-Ulam stability of the linear differential equation of second order

$$y''(t) + \beta y'(t) + \alpha y(t) = f(t). \tag{3.8}$$

Theorem 3.4. *Let α and β be constants in $\mathbb{F}$ such that there exist $a, b \in \mathbb{F}$ with $a + b = -\beta$, $ab = \alpha$, and $a \neq b$. Assume that $\varphi : (0,\infty) \to (0,\infty)$ is an integrable function. If a twice continuously differentiable function $y : (0,\infty) \to \mathbb{F}$ satisfies the inequality*

$$|y''(t) + \beta y'(t) + \alpha y(t) - f(t)| \le \varphi(t) \tag{3.9}$$

for all $t > 0$, then there exists a solution $y_c : (0,\infty) \to \mathbb{F}$ of the differential equation (3.8) *such that*

$$|y(t) - y_c(t)| \le \frac{e^{\Re(a)t}}{|a-b|}\int_0^t e^{-\Re(a)x}\varphi(x)dx + \frac{e^{\Re(b)t}}{|a-b|}\int_0^t e^{-\Re(b)x}\varphi(x)dx$$

for all $t > 0$.

Proof. If we define a function $z : (0,\infty) \to \mathbb{F}$ by $z(t) := y''(t) + \beta y'(t) + \alpha y(t) - f(t)$ for each $t > 0$, then we have

$$\mathcal{L}(z) = \big(s^2 + \beta s + \alpha\big)\mathcal{L}(y) - [sy(0) + \beta y(0) + y'(0)] - \mathcal{L}(f). \tag{3.10}$$

In view of (3.10), a function $y_0 : (0,\infty) \to \mathbb{F}$ is a solution of (3.8) if and only if

$$\big(s^2 + \beta s + \alpha\big)\mathcal{L}(y_0) - sy_0(0) - [\beta y_0(0) + y_0'(0)] = \mathcal{L}(f). \tag{3.11}$$

Now, since $s^2 + \beta s + \alpha = (s-a)(s-b)$, (3.10) implies that

$$\mathcal{L}(y) - \frac{sy(0) + [\beta y(0) + y'(0)] + \mathcal{L}(f)}{(s-a)(s-b)} = \frac{\mathcal{L}(z)}{(s-a)(s-b)}. \tag{3.12}$$

If we set

$$y_c(t) := y(0)\frac{ae^{at} - be^{bt}}{a-b} + [\beta y(0) + y'(0)]E_{a,b}(t) + (E_{a,b} * f)(t), \tag{3.13}$$

where $E_{a,b}(t) := \frac{e^{at}-e^{bt}}{a-b}$, then $y_c(0) = y(0)$. Moreover, since

$$\begin{aligned} y_c'(t) &= y(0)\frac{a^2e^{at} - b^2e^{bt}}{a-b} + [\beta y(0) + y'(0)]\frac{ae^{at} - be^{bt}}{a-b} + \frac{d}{dt}(E_{a,b} * f)(t), \\ (E_{a,b} * f)(t) &= \frac{e^{at}}{a-b}\int_0^t e^{-ax}f(x)dx - \frac{e^{bt}}{a-b}\int_0^t e^{-bx}f(x)dx, \end{aligned}$$

we have

$$\begin{aligned}
y_c'(0) &= y(0)\frac{a^2-b^2}{a-b} + [\beta y(0) + y'(0)]\frac{a-b}{a-b} \\
&= (a+b)y(0) + \beta y(0) + y'(0) \\
&= y'(0).
\end{aligned}$$

It follows from (3.13) that

$$\mathcal{L}(y_c) = \frac{sy_c(0) + [\beta y_c(0) + y_c'(0)] + \mathcal{L}(f)}{(s-a)(s-b)}. \tag{3.14}$$

Now, (3.11) and (3.14) imply that y_c is a solution of (3.8). Applying (3.12) and (3.14) and considering the facts that $y_c(0) = y(0)$, $y_c'(0) = y'(0)$, and $\mathcal{L}(E_{a,b} * z) = \frac{\mathcal{L}(z)}{(s-a)(s-b)}$, we obtain $\mathcal{L}(y) - \mathcal{L}(y_c) = \mathcal{L}(E_{a,b} * z)$ or equivalently, $y(t) - y_c(t) = (E_{a,b} * z)(t)$.

In view of (3.9), it holds that $|z(t)| \le \varphi(t)$, and it follows from the definition of the convolution that

$$\begin{aligned}
|y(t) - y_c(t)| &= |(E_{a,b} * z)(t)| \\
&\le \frac{e^{\Re(a)t}}{|a-b|}\int_0^t e^{-\Re(a)x}\varphi(x)dx + \frac{e^{\Re(b)t}}{|a-b|}\int_0^t e^{-\Re(b)x}\varphi(x)dx
\end{aligned}$$

for any $t > 0$. We remark that $\int_0^t e^{-\Re(a)x}\varphi(x)dx$ and $\int_0^t e^{-\Re(b)x}\varphi(x)dx$ exist for any $t > 0$ provided φ is an integrable function. □

Corollary 3.5. *Let α and β be constants in $\mathbb{F}$ such that there exist $a, b \in \mathbb{F}$ with $a + b = -\beta$, $ab = \alpha$, and $a \ne b$. Assume that $\varphi : (0,\infty) \to (0,\infty)$ is an integrable function for which there exists a positive real constant K with*

$$\int_0^t \left(e^{\Re(a)(t-x)} + e^{\Re(b)(t-x)}\right)\varphi(x)dx \le K\varphi(t) \tag{3.15}$$

for all $t > 0$. If a twice continuously differentiable function $y : (0,\infty) \to \mathbb{F}$ satisfies the inequality (3.9) for all $t > 0$, then there exists a solution $y_c : (0,\infty) \to \mathbb{F}$ of the differential equation (3.8) such that

$$|y(t) - y_c(t)| \le \frac{K}{|a-b|}\varphi(t)$$

for all $t > 0$.

We now show that there exists an integrable function $\varphi : (0,\infty) \to (0,\infty)$ which satisfies the condition (3.15).

Remark 3.6. Let α and β be constants in $\mathbb{F}$ such that there exist $a, b \in \mathbb{F}$ with $a + b = -\beta$, $ab = \alpha$, $\Re(a) < 1$, $\Re(b) < 1$, and $a \ne b$. If we define $\varphi(t) = Ae^t$ for all $t > 0$ and for some $A > 0$, then we get

$$\begin{aligned}
&\int_0^t \left(e^{\Re(a)(t-x)} + e^{\Re(b)(t-x)}\right)\varphi(x)dx \\
&= \int_0^t \left(e^{\Re(a)(t-x)} + e^{\Re(b)(t-x)}\right)Ae^x dx \\
&= \frac{A}{1-\Re(a)}\left(e^t - e^{\Re(a)t}\right) + \frac{A}{1-\Re(b)}\left(e^t - e^{\Re(b)t}\right)
\end{aligned}$$

$$\leq \Big(\frac{1}{1-\Re(a)} + \frac{1}{1-\Re(b)}\Big)\varphi(t)$$

for all $t > 0$.

Similarly, we apply the Laplace transform method to investigate the generalized Hyers-Ulam stability of the linear differential equation of nth order

$$y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) = f(t) \tag{3.16}$$

Theorem 3.7. *Let $\alpha_0, \alpha_1, \dots, \alpha_n$ be scalars in $\mathbb{F}$ with $\alpha_n = 1$, where n is an integer larger than 1. Assume that $\varphi : (0,\infty) \to (0,\infty)$ is an integrable function of exponential order. If an n times continuously differentiable function $y : (0,\infty) \to \mathbb{F}$ satisfies the inequality*

$$\big|y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) - f(t)\big| \leq \varphi(t) \tag{3.17}$$

for all $t > 0$, then there exist real constants $M > 0$ and σ_g and a solution $y_c : (0,\infty) \to \mathbb{F}$ of the differential equation (3.16) *such that*

$$|y(t) - y_c(t)| \leq M \int_0^t e^{\alpha(t-x)}\varphi(x)dx$$

for all $t > 0$ and $\alpha > \sigma_g$.

Proof. Applying integration by parts repeatedly, we derive

$$\mathcal{L}\big(y^{(k)}\big) = s^k \mathcal{L}(y) - \sum_{j=1}^{k} s^{k-j} y^{(j-1)}(0)$$

for any integer $k > 0$. Using this formula, we can prove that a function $y_0 : (0,\infty) \to \mathbb{F}$ is a solution of (3.16) if and only if

$$\begin{aligned}
\mathcal{L}(f) &= \sum_{k=0}^{n} \alpha_k s^k \mathcal{L}(y_0) - \sum_{k=1}^{n} \alpha_k \sum_{j=1}^{k} s^{k-j} y_0^{(j-1)}(0) \\
&= \sum_{k=0}^{n} \alpha_k s^k \mathcal{L}(y_0) - \sum_{j=1}^{n}\sum_{k=j}^{n} \alpha_k s^{k-j} y_0^{(j-1)}(0) \\
&= P_{n,0}(s)\mathcal{L}(y_0) - \sum_{j=1}^{n} P_{n,j}(s) y_0^{(j-1)}(0),
\end{aligned} \tag{3.18}$$

where $P_{n,j}(s) := \sum_{k=j}^{n} \alpha_k s^{k-j}$ for $j \in \{0, 1, \dots, n\}$.

Let us define a function $z : (0,\infty) \to \mathbb{F}$ by

$$z(t) := y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) - f(t) \tag{3.19}$$

for all $t > 0$. Then, similarly as in (3.18), we obtain

$$\mathcal{L}(z) = P_{n,0}(s)\mathcal{L}(y) - \sum_{j=1}^{n} P_{n,j}(s) y^{(j-1)}(0) - \mathcal{L}(f).$$

Hence, we get

$$\mathcal{L}(y) - \frac{1}{P_{n,0}(s)}\Big(\sum_{j=1}^{n} P_{n,j}(s)y^{(j-1)}(0) + \mathcal{L}(f)\Big) = \frac{\mathcal{L}(z)}{P_{n,0}(s)}. \tag{3.20}$$

Let σ_f be the abscissa of convergence for f, let $s_1, s_2, \dots, s_n$ be the roots of the polynomial $P_{n,0}(s)$, and let $\sigma_{P} = \max\{\Re(s_k) : k \in \{1, 2, \dots, n\}\}$. For any s with $\Re(s) > \max\{\sigma_f, \sigma_{P}\}$, we set

$$G(s) := \frac{1}{P_{n,0}(s)}\Big(\sum_{j=1}^{n} P_{n,j}(s)y^{(j-1)}(0) + \mathcal{L}(f)\Big). \tag{3.21}$$

By Lemma 2.2, there exists an n times continuously differentiable function f_0 such that

$$\mathcal{L}(f_0) = \frac{\mathcal{L}(f)}{P_{n,0}(s)} \tag{3.22}$$

for all s with $\Re(s) > \max\{\sigma_f, \sigma_{P}\}$ and

$$f_0^{(i)}(0) = 0 \tag{3.23}$$

for any $i \in \{0, 1, \dots, n-1\}$.

For $j \in \{1, 2, \dots, n\}$, we note that

$$\frac{P_{n,j}(s)}{P_{n,0}(s)} = \frac{1}{s^j} - \frac{\sum_{k=0}^{j-1} \alpha_k s^k}{s^j P_{n,0}(s)} \tag{3.24}$$

for every s with $\Re(s) > \max\{0, \sigma_{P}\}$. Applying Lemma 2.1 for the case of $Q(s) = \sum_{k=0}^{j-1} \alpha_k s^k$ and $P(s) = s^j P_{n,0}(s)$, we can find an infinitely differentiable function g_j such that

$$\mathcal{L}(g_j) = \frac{\sum_{k=0}^{j-1} \alpha_k s^k}{s^j P_{n,0}(s)} \tag{3.25}$$

and $g_j^{(k)}(0) = 0$ for $k \in \{0, 1, \dots, n-1\}$.

Let

$$f_j(t) := \frac{t^{j-1}}{(j-1)!} - g_j(t) \tag{3.26}$$

for $j \in \{1, 2, \dots, n\}$. Then we have

$$f_j^{(i)}(0) = \begin{cases} 0 & \text{for } i \in \{0, 1, \dots, j-2, j, j+1, \dots, n-1\}, \\ 1 & \text{for } i = j-1. \end{cases} \tag{3.27}$$

If we define

$$y_c(t) := \sum_{j=1}^{n} y^{(j-1)}(0) f_j(t) + f_0(t),$$

then the conditions (3.23) and (3.27) imply that

$$y_c^{(i)}(0) = y^{(i)}(0) \tag{3.28}$$

for every $i \in \{0, 1, \dots, n-1\}$. Moreover, it follows from (3.21)–(3.28) that

$$\begin{aligned}
\mathcal{L}(y_c) &= \sum_{j=1}^{n} y^{(j-1)}(0)\mathcal{L}(f_j) + \mathcal{L}(f_0) \\
&= \sum_{j=1}^{n} y^{(j-1)}(0)\Big(\frac{1}{s^j} - \mathcal{L}(g_j)\Big) + \frac{\mathcal{L}(f)}{P_{n,0}(s)} \\
&= \frac{1}{P_{n,0}(s)}\Big(\sum_{j=1}^{n} P_{n,j}(s) y^{(j-1)}(0) + \mathcal{L}(f)\Big)
\end{aligned} \tag{3.29}$$

for each s with $\Re(s) > \max\{0, \sigma_f, \sigma_P\}$.

Now, (3.18) implies that y_c is a solution of (3.16). Moreover, by (3.20) and (3.29), we have

$$\mathcal{L}(y) - \mathcal{L}(y_c) = \frac{\mathcal{L}(z)}{P_{n,0}(s)}. \tag{3.30}$$

Applying Lemma 2.1 for the case of $Q(s) = 1$ and $P(s) = P_{n,0}(s)$, we find an infinitely differentiable function $g : (0, \infty) \to \mathbb{F}$ such that

$$\mathcal{L}(g) = \frac{1}{P_{n,0}(s)} \tag{3.31}$$

which implies that

$$g(t) = \mathcal{L}^{-1}\Big(\frac{1}{P_{n,0}(s)}\Big) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+iy)t}\frac{1}{P_{n,0}(\alpha+iy)}dy$$

for any real constant $\alpha > \sigma_g$. Moreover, it holds that

$$\begin{aligned}
|g(t-x)| &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} \big|e^{(\alpha+iy)(t-x)}\big|\frac{1}{|P_{n,0}(\alpha+iy)|}dy \\
&\le \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\alpha(t-x)}\frac{1}{|P_{n,0}(\alpha+iy)|}dy \\
&\le \frac{1}{2\pi}e^{\alpha(t-x)}\int_{-\infty}^{\infty} \frac{1}{|P_{n,0}(\alpha+iy)|}dy \\
&\le Me^{\alpha(t-x)}
\end{aligned} \tag{3.32}$$

for all $\alpha > \sigma_g$, where

$$M = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{|P_{n,0}(\alpha+iy)|}dy < \infty,$$

because n is an integer larger than 1. By (3.17) and (3.19), it also holds that $|z(t)| \le \varphi(t)$ for all $t > 0$.

In view of (3.30), (3.31), and (3.32), we obtain

$$\mathcal{L}(y) - \mathcal{L}(y_c) = \mathcal{L}(g)\mathcal{L}(z) = \mathcal{L}(g * z).$$

Consequently, we have $y(t) - y_c(t) = (g * z)(t)$ for any $t > 0$. Hence, it follows from (3.17), (3.19), and (3.32) that

$$|y(t) - y_c(t)| = |(g * z)(t)| \le \int_0^t |g(t-x)||z(x)|dx \le M\int_0^t e^{\alpha(t-x)}\varphi(x)dx$$

for all $t > 0$ and for any real constant $\alpha > \sigma_g$, which completes the proof. $\square$

Corollary 3.8. *Let $\alpha_0, \alpha_1, \dots, \alpha_n$ be scalars in $\mathbb{F}$ with $\alpha_n = 1$, where n is an integer larger than 1. Assume that there exist real constants α and $K > 0$ such that a function $\varphi : (0, \infty) \to (0, \infty)$ satisfies*

$$\int_0^t e^{\alpha(t-x)}\varphi(x)dx \le K\varphi(t)$$

for all $t > 0$. Moreover, assume that the constant σ_g given in Theorem 3.7 is less than α. If an n times continuously differentiable function $y : (0, \infty) \to \mathbb{F}$ satisfies the inequality (3.17) for all $t > 0$, then there exist a real constants $M > 0$ and a solution $y_c : (0, \infty) \to \mathbb{F}$ of the differential equation (3.16) such that

$$|y(t) - y_c(t)| \le KM\varphi(t)$$

for all $t > 0$.

Remark 3.9. Assume that $\alpha < 1$. If we define $\varphi(t) = Ae^t$ for all $t > 0$ and for some $A > 0$, then we get

$$\int_0^t e^{\alpha(t-x)}\varphi(x)dx = \int_0^t e^{\alpha(t-x)}Ae^x dx = \frac{A}{1-\alpha}\big(e^t - e^{\alpha t}\big) \le \frac{1}{1-\alpha}\varphi(t)$$

for all $t > 0$.

Acknowledgements. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2013R1A1A2005557).

References

[1] C. Alsina, R. Ger; *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. **2**, pp. 373–380, 1998.

[2] D. S. Cimpean, D. Popa; *On the stability of the linear differential equation of higher order with constant coefficients*, Appl. Math. Comput. **217**, pp. 4141–4146, 2010.

[3] S. Czerwik; *Functional Equations and Inequalities in Several Variables*, World Scientific, Singapore, 2002.

[4] B. Davies; *Integral Transforms and Their Apllications*, Springer, New York, 2001.

[5] D. H. Hyers; *On the stability of the linear functional equation*, Proc. Natl. Soc. USA **27**, pp. 222–224, 1941.

[6] D. H. Hyers, G. Isac, Th. M. Rassias; *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 1998.

[7] S.-M. Jung; *Hyers-Ulam stability of linear differential equations of first order*, Appl. Math. Lett. **17**, pp. 1135–1140, 2004.

[8] S.-M. Jung; *Hyers-Ulam stability of linear differential equations of first order, III*, J. Math. Anal. Appl. **311**, pp. 139–146, 2005.

[9] S.-M. Jung; *Hyers-Ulam stability of linear differential equations of first order, II*, Appl. Math. Lett. **19**, pp. 854–858, 2006.

[10] S.-M. Jung; *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.

[11] Y. Li, Y. Shen; *Hyers-Ulam stability of linear differential equations of second order*, Appl. Math. Lett. **23**, pp. 306–309, 2010.

[12] T. Miura, S. Miyajima, S.-E. Takahasi; *A characterization of Hyers-Ulam stability of first order linear differential operators*, J. Math. Anal. Appl. **286**, pp. 136–146, 2003.

[13] T. Miura, S. Miyajima, S.-E. Takahasi; *Hyers-Ulam stability of linear differential operator with constant coefficients*, Math. Nachr. **258**, pp. 90–96, 2003.

[14] M. Obłoza; *Hyers stability of the linear differential equation*, Rocznik Nauk.-Dydakt. Prace Mat. **13**, pp. 259–270, 1993.

[15] M. Obłoza; *Connections between Hyers and Lyapunov stability of the ordinary differential equations*, Rocznik Nauk.-Dydakt. Prace Mat. **14**, pp. 141–146, 1997.

[16] D. Popa, I. Raşa; *On the Hyers-Ulam stability of the linear differential equation*, J. Math. Anal. Appl. **381**, pp. 530–537, 2011.
[17] D. Popa, I. Raşa; *Hyers-Ulam stability of the linear differential operator with non-constant coefficients*, Appl. Math. Comput. **219**, pp. 1562–1568, 2012.
[18] Th. M. Rassias; *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72**, pp. 297–300, 1978.
[19] H. Rezaei, S.-M. Jung, Th. M. Rassias; *Laplace transform and Hyers-Ulam stability of linear differential equations*, J. Math. Anal. Appl. **403**, pp. 244–251, 2013.
[20] I. A. Rus; *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory **10**, pp. 305–320, 2009.
[21] I. A. Rus; *Ulam stability of ordinary differential equations*, Stud. Univ. Babes-Bolyai Math. **54**, pp. 125–134, 2009.
[22] S.-E. Takahasi, T. Miura, S. Miyajima; *On the Hyers-Ulam stability of the Banach space-valued differential equation* $y' = \lambda y$, Bull. Korean Math. Soc. **39**, pp. 309–315, 2002.
[23] S.-E. Takahasi, H. Takagi, T. Miura, S. Miyajima; *The Hyers-Ulam stability constants of first order linear differential operators*, J. Math. Anal. Appl. **296**, pp. 403–409, 2004.
[24] S. M. Ulam; *Problems in Modern Mathematics, Chapter VI*, Scince Editors, Wiley, New York, 1960.
[25] G. Wang, M. Zhou, L. Sun; *Hyers-Ulam stability of linear differential equations of first order*, Appl. Math. Lett. **21**, pp. 1024–1028, 2008.

Qusuay H. Alqifiary
Department of Mathematics, University of Belgrade, Belgrade, Serbia.
University of Al-Qadisiyah, Al-Diwaniya, Iraq
E-mail address: qhaq2010@gmail.com

Soon-Mo Jung
Mathematics Section, College of Science and Technology, Hongik University, 339–701 Sejong, Korea
E-mail address: smjung@hongik.ac.kr