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EXISTENCE OF TWO POSITIVE SOLUTIONS FOR A SINGULAR NEUMANN PROBLEM

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Abstract. We obtain two positive solutions for Neumann boundary problems with singularity and subcritical term, by using the Nehari method.

1. Introduction and main result

In this article, we consider the Neumann problem

$$
\begin{gathered}
-\Delta u+u=\lambda P(x)u^p+Q(x)u^{-\gamma}, \quad \text{in } \Omega,\\
u>0, \quad \text{in } \Omega,\\
\frac{\partial u}{\partial \nu}=0, \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.1}
$$

where $\Omega\subset R^N$ $(N\geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$ and λ is a positive parameter. The exponent p of the superlinear satisfies $1<p<2^*-1$, where $2^*=\frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H^1(\Omega)$ into $L^q(\Omega)$ for every $q\in[1,\frac{2N}{N-2}]$. The exponent γ of the singular term satisfies $0<\gamma<1$. The coefficient functions $P\in L^{r_1}(\Omega), Q\in L^{r_2}(\Omega)$ are nonzero and nonnegative, where $r_1>\frac{2^*}{2^*-p-1}$ and $r_2>\frac{2^*}{2^*+\gamma-1}$ are two constants.

A function $u\in H^1(\Omega)$ is called a weak solution of problem (1.1) if $u(x)>0$ in Ω satisfies

$$
\int_\Omega\big((\nabla u,\nabla\phi)+u\phi-\lambda P(x)u^p\phi-Q(x)u^{-\gamma}\phi\big)dx=0, \quad \forall\phi\in H^1(\Omega), \tag{1.2}
$$

where $H^1(\Omega)$ is a Sobolev space equipped with the norm $\|u\|=[\int_\Omega(|\nabla u|^2+u^2)dx]^{1/2}$. This is the space we work on in this paper.

The Dirichlet boundary value problem

$$
\begin{gathered}
-\Delta u=u^p+\lambda u^{-\gamma}, \quad \text{in } \Omega,\\
u>0, \quad \text{in } \Omega,\\
u=0, \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.3}
$$

have been extensively studied in [2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 17, 18, 19, 20]. In particular, in [3] it has been shown that problem (1.3) possesses at least one solution for $\lambda>0$ small enough, and has no solution when λ is large. This result

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has been extended in [4, 8, 11, 12, 13, 14, 15, 17, 18, 19, 20]. When the exponent satisfies $0 < p < 1$, similar results of [3] have been obtained in [7, 10, 18, 19, 20]. Especially, Shi and Yao in [10] studied the case where the coefficient of the singular term changes sign. Using sub-supersolution method, they proved that problem (1.3) has at least one solution for λ large enough and has no solution for λ small enough. When the exponent satisfies $1 < p < 2^* - 1$, the multiplicity of positive solutions has been considered in [14] and [12]. They obtained two positive solutions for problem (1.3) when $\lambda > 0$ is small enough by the Nehari manifold. When the exponent is the critical exponent, the existence and the multiplicity of solutions have been studied in [8, 11, 13, 15, 17].

Recently, Chabrowski in [1] studied the Neumann problems with singular superlinear nonlinearities; that is,

$$
\begin{gathered}
-\Delta u = P(x)u^p + \lambda Q(x)u^{-\gamma}, \quad \text{in } \Omega, \\
u > 0, \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega,
\end{gathered}
$$

where $P \in C(\overline{\Omega})$ changes sign on Ω and satisfies

$$
\int_\Omega P(x)dx < 0,
$$

and $Q \in C(\overline{\Omega})$ with $Q > 0$. When $1 < p < 2^* - 1$ and $0 < \gamma < \min\{p-1, 1\}$, he has obtained two positive solutions for $\lambda > 0$ small enough by approximation and variational methods.

Inspired by [14] and [1], we study problem (1.1) with $1 < p < 2^* - 1$ and $0 < \gamma < 1$, and obtain two positive solutions when $\lambda > 0$ is small by the Nehari method. Moreover, we obtain uniform lower bounds for λ, namely $T_{p,\gamma}$.

We denote by $|\cdot|_q$ the usual L^q-norm. Let S be the best Sobolev constant and $T_{p,\gamma}$ be a constant, respectively

$$
S := \inf\Big\{\frac{\int_\Omega(|\nabla u|^2 + u^2)dx}{(\int_\Omega |u|^{2^*}dx)^{\frac{2}{2^*}}} : u \in H^1(\Omega), u \neq 0\Big\}, \tag{1.4}
$$

$$
T_{p,\gamma} = \frac{1+\gamma}{p-1}\Big(\frac{p-1}{p+\gamma}\Big)^{\frac{p+\gamma}{1+\gamma}} \frac{S^{\frac{p+\gamma}{1+\gamma}}}{|P|_{r_1}|Q|_{r_2}^{\frac{p-1}{1+\gamma}}} |\Omega|^{-\frac{r_1 r_2(p+\gamma)(2^*-2)-2^*[r_1(p-1)+r_2(1-\gamma)]}{2^* r_1 r_2(1+\gamma)}}.
$$

For all $u \in H^1(\Omega)$, we define

$$
I_\lambda(u) = \frac{1}{2}\int_\Omega(|\nabla u|^2 + |u|^2)dx - \frac{\lambda}{p+1}\int_\Omega P(x)|u|^{p+1}dx - \frac{1}{1-\gamma}\int_\Omega Q(x)|u|^{1-\gamma}dx.
$$

It is well known that the singular term leads to the functional $I_\lambda \notin C^1(H^1(\Omega), R)$. However, we may obtain the multiplicity of solutions for problem (1.1) by investigating suitable minimization problems for the functional I_λ. Notice that u is a weak solution of problem (1.1), then $u > 0$ in Ω and satisfies the equation

$$
\int_\Omega(|\nabla u|^2 + u^2)dx - \lambda\int_\Omega P(x)u^{p+1}dx - \int_\Omega Q(x)u^{1-\gamma}dx = 0.
$$

So if such a solution exists then it must lie in Nehari manifold Λ, which is defined by

$$\Lambda = \big\{u \in H^1(\Omega) : \int_\Omega (|\nabla u|^2 + u^2 - \lambda P(x)|u|^{p+1} - Q(x)|u|^{1-\gamma})dx = 0\big\}.$$

To obtain the multiplicity of positive solutions, we split $\Lambda = \Lambda^+ \cup \Lambda^0 \cup \Lambda^-$ where

$$\Lambda^+ = \big\{u \in \Lambda : (1+\gamma)\int_\Omega (|\nabla u|^2 + u^2)dx - \lambda(p+\gamma)\int_\Omega P(x)|u|^{p+1}dx > 0\big\},$$

$$\Lambda^0 = \big\{u \in \Lambda : (1+\gamma)\int_\Omega (|\nabla u|^2 + u^2)dx - \lambda(p+\gamma)\int_\Omega P(x)|u|^{p+1}dx = 0\big\},$$

$$\Lambda^- = \big\{u \in \Lambda : (1+\gamma)\int_\Omega (|\nabla u|^2 + u^2)dx - \lambda(p+\gamma)\int_\Omega P(x)|u|^{p+1}dx < 0\big\}.$$

When $\lambda \in (0, T_{p,\gamma})$, we can prove that $\Lambda^\pm \neq \emptyset$ and $\Lambda^0 = \{0\}$. Then we can find two minimizers of I_λ on Λ^+ and Λ^- respectively, which are local minimizers of I_λ on Λ. Finally, we prove that a local minimizer of I_λ on Λ is indeed a positive solution of (1.1).

The main result can be described as follows.

Theorem 1.1. *Suppose $P \in L^{r_1}(\Omega), Q \in L^{r_2}(\Omega)$ are nonzero and nonnegative, $1 < p < 2^* - 1$ and $0 < \gamma < 1$, then problem (1.1) has at least two positive solutions for all $\lambda \in (0, T_{p,\gamma})$, where $r_1 > \frac{2^*}{2^*-p-1}$ and $r_2 > \frac{2^*}{2^*+\gamma-1}$ are two constants.*

To the best knowledge, up to now there is no study of the exact estimate of λ such that problem (1.1) has at least two positive solutions. For the case $1 < p < 2^* - 1$, Chabrowski obtained two positive solutions restricting the exponent of singular term with $0 < \gamma < \min\{p-1, 1\}$ in [1]. Moreover, we overcome the difficulty of the singular term by Nehari manifold, while [1] used perturbation method to conquer this difficulty.

This article is organized as follow: in Section 2, we give some preliminaries which will be used to prove out main result, and the proof of Theorem 1.1 is given in Section 3.

2. Preliminaries

In this section, we give some lemmas in preparation for the proof of our main result.

Lemma 2.1. *Suppose $\lambda \in (0, T_{p,\gamma})$, then $\Lambda^\pm \neq \emptyset$ and $\Lambda^0 = \{0\}$. Moreover, Λ^- is closed for all $0 < \lambda < T_{p,\gamma}$.*

Proof. According to the assumptions on P and Q, there exists $u \in H^1(\Omega)$ such that $\int_\Omega P(x)|u|^{p+1}dx > 0$ and $\int_\Omega Q(x)|u|^{1-\gamma}dx > 0$. Let $\Phi \in C(R^+, R)$ satisfy

$$\Phi(t) = t^{1-p}\|u\|^2 - t^{-\gamma-p}\int_\Omega Q(x)|u|^{1-\gamma}dx,$$

then

$$\Phi'(t) = (1-p)t^{-p}\|u\|^2 + (p+\gamma)t^{-\gamma-p-1}\int_\Omega Q(x)|u|^{1-\gamma}dx.$$

Let $\Phi'(t) = 0$, we can verify

$$t_{\max} = \Big[\frac{(p+\gamma)\int_\Omega Q(x)|u|^{1-\gamma}dx}{(p-1)\|u\|^2}\Big]^{1/(1+\gamma)}.$$

Easy computations show that $\Phi'(t) > 0$ for all $0 < t < t_{\max}$ and $\Phi'(t) < 0$ for all $t > t_{\max}$. Thus $\Phi(t)$ attains its maximum at $t_{\max}$, that is,

$$\Phi(t_{\max}) = \frac{1+\gamma}{p-1}\Big(\frac{p-1}{p+\gamma}\Big)^{\frac{p+\gamma}{1+\gamma}} \frac{\|u\|^{\frac{2(p+\gamma)}{1+\gamma}}}{\big(\int_\Omega Q(x)|u|^{1-\gamma}dx\big)^{\frac{p-1}{1+\gamma}}}.$$

From (1.4), we have

$$S|u|_{2^*}^2 < \|u\|^2, \tag{2.1}$$

and by Hölder's inequality, one has

$$\int_\Omega P(x)|u|^{p+1}dx \le |P|_{r_1}|u|_{2^*}^{p+1}|\Omega|^{\frac{r_1(2^*-p-1)-2^*}{r_1 2^*}}, \tag{2.2}$$

$$\int_\Omega Q(x)|u|^{1-\gamma}dx \le |Q|_{r_2}|u|_{2^*}^{1-\gamma}|\Omega|^{\frac{r_2(2^*+\gamma-1)-2^*}{r_2 2^*}}. \tag{2.3}$$

Then from (2.1)-(2.3), one gets

$$\begin{aligned}
&\Phi(t_{\max}) - \lambda\int_\Omega P(x)|u|^{p+1}dx \\
&> \frac{1+\gamma}{p-1}\Big(\frac{p-1}{p+\gamma}\Big)^{\frac{p+\gamma}{1+\gamma}} \frac{(S|u|_{2^*}^2)^{\frac{p+\gamma}{1+\gamma}}}{\big(|Q|_{r_2}|u|_{2^*}^{1-\gamma}|\Omega|^{\frac{r_2(2^*+\gamma-1)-2^*}{r_2 2^*}}\big)^{\frac{p-1}{1+\gamma}}} \\
&\quad - \lambda|P|_{r_1}|u|_{2^*}^{p+1}|\Omega|^{\frac{r_1(2^*-p-1)-2^*}{r_1 2^*}} \\
&= \Big[\frac{1+\gamma}{p-1}\Big(\frac{p-1}{p+\gamma}\Big)^{\frac{p+\gamma}{1+\gamma}} \frac{S^{\frac{p+\gamma}{1+\gamma}}}{\big(|Q|_{r_2}|\Omega|^{\frac{r_2(2^*+\gamma-1)-2^*}{r_2 2^*}}\big)^{\frac{p-1}{1+\gamma}}} \\
&\quad - \lambda|P|_{r_1}|\Omega|^{\frac{r_1(2^*-p-1)-2^*}{r_1 2^*}}\Big]|u|_{2^*}^{p+1} \\
&= |P|_{r_1}|\Omega|^{\frac{r_1(2^*-p-1)-2^*}{r_1 2^*}}(T_{p,\gamma} - \lambda)|u|_{2^*}^{p+1} > 0,
\end{aligned} \tag{2.4}$$

for all $\lambda \in (0, T_{p,\gamma})$. Consequently, there exist t_0^+ and t_0^- satisfying $0 < t_0^+ < t_{\max} < t_0^-$ such that

$$\Phi(t_0^+) = \lambda\int_\Omega P(x)|u|^{p+1}dx = \Phi(t_0^-)$$

and

$$\Phi'(t_0^+) < 0 < \Phi'(t_0^-);$$

that is, $t_0^+u \in \Lambda^+$ and $t_0^-u \in \Lambda^-$. Thus $\Lambda^\pm$ are non-empty whenever $\lambda \in (0, T_{p,\gamma})$.

Next, we prove that $\Lambda^0 = \{0\}$ for all $\lambda \in (0, T_{p,\gamma})$. By contradiction, suppose that there exists $u_0 \in \Lambda^0$ and $u_0 \neq 0$. Then it follows that

$$(1+\gamma)\|u_0\|^2 - \lambda(p+\gamma)\int_\Omega P(x)|u_0|^{p+1}dx = 0,$$

and consequently

$$\begin{aligned}
0 &= \|u_0\|^2 - \lambda\int_\Omega P(x)|u_0|^{p+1}dx - \int_\Omega Q(x)|u_0|^{1-\gamma}dx \\
&= \frac{p-1}{p+\gamma}\|u_0\|^2 - \int_\Omega Q(x)|u_0|^{1-\gamma}dx.
\end{aligned}$$

From (2.4), we have

$$
\begin{aligned}
0 &< \Big[\frac{1+\gamma}{p-1}\big(\frac{p-1}{p+\gamma}\big)^{\frac{p+\gamma}{1+\gamma}}\frac{S^{\frac{p+\gamma}{1+\gamma}}}{\big(|Q|_{r_2}|\Omega|^{\frac{r_2(2^*+\gamma-1)-2^*}{r_2 2^*}}\big)^{\frac{p-1}{1+\gamma}}} \\
&\quad - \lambda|P|_{r_1}|\Omega|^{\frac{r_1(2^*-p-1)-2^*}{r_1 2^*}}\Big]|u_0|_{2^*}^{p+1} \\
&< \frac{1+\gamma}{p-1}\big(\frac{p-1}{p+\gamma}\big)^{\frac{p+\gamma}{1+\gamma}}\frac{\|u_0\|^{\frac{2(p+\gamma)}{1+\gamma}}}{\big(\int_\Omega Q(x)|u_0|^{1-\gamma}dx\big)^{\frac{p-1}{1+\gamma}}} - \lambda\int_\Omega P(x)|u_0|^{p+1}dx \\
&= \frac{1+\gamma}{p-1}\big(\frac{p-1}{p+\gamma}\big)^{\frac{p+\gamma}{1+\gamma}}\frac{\|u_0\|^{\frac{2(p+\gamma)}{1+\gamma}}}{\big(\frac{p-1}{p+\gamma}\|u_0\|^2\big)^{\frac{p-1}{1+\gamma}}} - \frac{1+\gamma}{p+\gamma}\|u_0\|^2 = 0,
\end{aligned}
$$

for all $\lambda \in (0, T_{p,\gamma})$, which is impossible. Thus $\Lambda^0 = \{0\}$ for $\lambda \in (0, T_{p,\gamma})$.

Finally, we prove that Λ^- is closed for all $0 < \lambda < T_{p,\gamma}$. That is, suppose $\{u_n\} \subset \Lambda^-$ such that $u_n \to u$ in $H^1(\Omega)$ as $n \to \infty$, then $u \in \Lambda^-$. Since $\{u_n\} \subset \Lambda^-$, from the definition of Λ^-, one has

$$
\begin{gathered}
\|u_n\|^2 - \lambda\int_\Omega P(x)|u_n|^{p+1}dx - \int_\Omega Q(x)|u_n|^{1-\gamma}dx = 0, \\
(1+\gamma)\|u_n\|^2 - \lambda(p+\gamma)\int_\Omega P(x)|u_n|^{p+1}dx < 0,
\end{gathered} \tag{2.5}
$$

and consequently

$$
\begin{gathered}
\|u\|^2 - \lambda\int_\Omega P(x)|u|^{p+1}dx - \int_\Omega Q(x)|u|^{1-\gamma}dx = 0, \\
(1+\gamma)\|u\|^2 - \lambda(p+\gamma)\int_\Omega P(x)|u|^{p+1}dx \le 0,
\end{gathered}
$$

thus $u \in \Lambda^0 \cup \Lambda^-$. If $u \in \Lambda^0$, combining $\Lambda^0 = \{0\}$ it follows that $u = 0$. However, from (2.1), (2.2) and (2.5), one gets

$$
|u_n|_{2^*} \ge \Big[\frac{S(1+\gamma)}{\lambda(p+\gamma)|P|_{r_1}}|\Omega|^{\frac{r_1(2^*-p-1)-2^*}{r_1 2^*}}\Big]^{1/(p-1)}, \quad \forall u_n \in \Lambda^-, \tag{2.6}
$$

which contradicts $u = 0$. Thus $u \in \Lambda^-$ for $\lambda \in (0, T_{p,\gamma})$. Hence the proof is complete. $\square$

Lemma 2.2. *Given $u \in \Lambda^-$ (respectively Λ^+) with $u > 0$, for all $\varphi \in H^1(\Omega)$, $\varphi > 0$, there exist $\varepsilon > 0$ and a continuous function $t = t(s) > 0$, $s \in \mathbb{R}$, $|s| < \varepsilon$ satisfying*

$$
t(0) = 1, \quad t(s)(u + s\varphi) \in \Lambda^- \ \textit{(respectively } \Lambda^+), \quad \forall s \in \mathbb{R},\ |s| < \varepsilon.
$$

Proof. We define $f : \mathbb{R} \times \mathbb{R} \to R$ by:

$$
\begin{aligned}
f(t,s) &= t^{\gamma+1}\int_\Omega \big[|\nabla(u+s\varphi)|^2 + (u+s\varphi)^2\big]dx - \lambda t^{p+\gamma}\int_\Omega P(x)(u+s\varphi)^{p+1}dx \\
&\quad - \int_\Omega Q(x)(u+s\varphi)^{1-\gamma}dx.
\end{aligned}
$$

Then

$$
f_t(t,s) = (\gamma+1)t^\gamma\int_\Omega \big[|\nabla(u+s\varphi)|^2 + (u+s\varphi)^2\big]dx
$$

$$- \lambda(p+\gamma)t^{p+\gamma-1}\int_\Omega P(x)(u+s\varphi)^{p+1}dx,$$

is continuous in $\mathbb{R}\times\mathbb{R}$. Since $u\in\Lambda^-\subset\Lambda$, we have $f(1,0)=0$, and moreover

$$f_t(1,0)=(1+\gamma)\int_\Omega(|\nabla u|^2+u^2)dx-\lambda(p+\gamma)\int_\Omega P(x)u^{p+1}dx<0.$$

Then by applying the implicit function theorem to f at the point $(1,0)$, we obtain $\overline{\varepsilon}>0$ and a continuous function $t=t(s)>0$, $s\in\mathbb{R}$, $|s|<\overline{\varepsilon}$ satisfying that

$$t(0)=1,\quad t(s)(u+s\varphi)\in\Lambda,\quad \forall s\in\mathbb{R},\ |s|<\overline{\varepsilon}.$$

Moreover, taking $\varepsilon>0$ possibly smaller ($\varepsilon<\overline{\varepsilon}$), we obtain

$$t(s)(u+s\varphi)\in\Lambda^-,\quad \forall s\in\mathbb{R},\ |s|<\varepsilon.$$

The case $u\in\Lambda^+$ may be obtained in the same way. Thus the proof is complete. □

3. Proof of main theorem

For all $u\in\Lambda$, we have

$$\begin{aligned} I_\lambda(u)&=\frac12\|u\|^2-\frac{\lambda}{p+1}\int_\Omega P(x)|u|^{p+1}dx-\frac{1}{1-\gamma}\int_\Omega Q(x)|u|^{1-\gamma}dx\\ &=\big(\frac12-\frac{1}{p+1}\big)\|u\|^2-\big(\frac{1}{1-\gamma}-\frac{1}{p+1}\big)\int_\Omega Q(x)|u|^{1-\gamma}dx. \end{aligned}$$

Since $1<p<2^*-1$ and $0<\gamma<1$, from (2.3) and (2.1), we obtain that I_λ is coercive and bounded below on Λ. According to Lemma 2.1 for all $\lambda\in(0,T_{p,\gamma})$

$$m^+=\inf_{u\in\Lambda^+}I_\lambda(u),\quad m^-=\inf_{u\in\Lambda^-}I_\lambda(u)$$

are well defined. Moreover, for all $u\in\Lambda^+$, it follows that

$$(1+\gamma)\|u\|^2-\lambda(p+\gamma)\int_\Omega P(x)|u|^{p+1}dx>0,$$

and consequently, since $2<p+1<2^*$, $0<\gamma<1$ and $u\not\equiv 0$, we have

$$\begin{aligned} I_\lambda(u)&=\frac12\|u\|^2-\frac{\lambda}{p+1}\int_\Omega P(x)|u|^{p+1}dx-\frac{1}{1-\gamma}\int_\Omega Q(x)|u|^{1-\gamma}dx\\ &=\big(\frac12-\frac{1}{1-\gamma}\big)\|u\|^2+\lambda\big(\frac{1}{1-\gamma}-\frac{1}{p+1}\big)\int_\Omega P(x)|u|^{p+1}dx\\ &<-\frac{1+\gamma}{2(1-\gamma)}\|u\|^2+\frac{1+\gamma}{(1-\gamma)(p+1)}\|u\|^2\\ &=-\frac{1+\gamma}{1-\gamma}\big(\frac12-\frac{1}{p+1}\big)\|u\|^2<0. \end{aligned}$$

Thus $m^+=\inf_{u\in\Lambda^+}I_\lambda(u)<0$ for all $\lambda\in(0,T_{p,\gamma})$.

Proof of Theorem 1.1. Let $\lambda\in(0,T_{p,\gamma})$. The following two steps complete the proof of Theorem 1.1.

Step 1. We prove that there exists a positive solution of (1.1) in Λ^+. Applying Ekeland's variational principle to the minimization problem $m^+=\inf_{u\in\Lambda^+}I_\lambda(u)$, there exists a sequence $\{u_n\}\subset\Lambda^+$ with the following properties:

(i) $I_\lambda(u_n)<m^++\frac1n$,
(ii) $I_\lambda(u)\ge I_\lambda(u_n)-\frac1n\|u-u_n\|$, for all $u\in\Lambda^+$

Since $I_\lambda(u) = I_\lambda(|u|)$, we can assume from the beginning that $u_n(x) \geq 0$ for all $x \in \Omega$. Obviously, $\{u_n\}$ is bounded in $H^1(\Omega)$, going if necessary to a subsequence, still denoted by $\{u_n\}$, there exists $u_* \geq 0$ such that

$$\begin{gathered} u_n \rightharpoonup u_*, \quad \text{weakly in } H^1(\Omega), \\ u_n \to u_*, \quad \text{strongly in } L^s(\Omega),\ 1 \leq s < 2^*, \\ u_n(x) \to u_*(x), \quad \text{a.e. in } \Omega, \end{gathered}$$

as $n \to \infty$. Now we will prove that u_* is a positive solution of problem (1.1).

Firstly, we prove that $u_*(x) \not\equiv 0$ in Ω. By Vitali's theorem (see [9, pp. 133]), we claim that

$$\lim_{n\to\infty} \int_\Omega Q(x)|u_n|^{1-\gamma}dx = \int_\Omega Q(x)|u_*|^{1-\gamma}dx. \tag{3.1}$$

Indeed, we only need to prove that $\{\int_\Omega Q(x)|u_n|^{1-\gamma}dx, n \in N\}$ is equi-absolutely-continuous. Note that $\{u_n\}$ is bounded, by the Sobolev embedding theorem, so exists a constant $C > 0$ such that $|u_n|_{2^*} \leq C < \infty$. From (2.3), for every $\varepsilon > 0$, setting

$$\delta = \Big(\frac{\varepsilon}{|Q|_{r_2} C^{1-\gamma}}\Big)^{\frac{r_2 2^*}{r_2(2^*+\gamma-1)-2^*}},$$

when $E \subset \Omega$ with $mesE < \delta$, we have

$$\begin{aligned} \int_E Q(x)|u_n|^{1-\gamma}dx &\leq |Q|_{r_2}|u|_{2^*}^{1-\gamma}\big(\operatorname{meas} E\big)^{\frac{r_2(2^*+\gamma-1)-2^*}{r_2 2^*}} \\ &\leq |Q|_{r_2} C^{1-\gamma} \delta^{\frac{r_2(2^*+\gamma-1)-2^*}{r_2 2^*}} < \varepsilon. \end{aligned}$$

Thus, our claim is true. Similarly,

$$\lim_{n\to\infty} \int_\Omega P(x)|u_n|^{p+1}dx = \int_\Omega P(x)|u_*|^{p+1}dx. \tag{3.2}$$

By the weakly lower semicontinuity of the norm, combining (3.1) and (3.2), we have

$$\begin{aligned} I_\lambda(u_*) &= \frac{1}{2}\|u_*\|^2 - \frac{\lambda}{p+1}\int_\Omega P(x)|u_*|^{p+1}dx - \frac{1}{1-\gamma}\int_\Omega Q(x)|u_*|^{1-\gamma}dx \\ &\leq \liminf_{n\to\infty} \Big[\frac{1}{2}\|u_n\|^2 - \frac{\lambda}{p+1}\int_\Omega P(x)|u_n|^{p+1}dx \\ &\quad - \frac{1}{1-\gamma}\int_\Omega Q(x)|u_n|^{1-\gamma}dx\Big] \\ &= \liminf_{n\to\infty} I_\lambda(u_n) = m^+ < 0, \end{aligned}$$

which implies that $u_*(x) \not\equiv 0$ in Ω.

Secondly, we prove that $u_*(x) > 0$ a.e. in Ω. From $u_n \in \Lambda^+$, we can claim that there exists a constant $C_1 > 0$ such that

$$(1+\gamma)\|u_n\|^2 - \lambda(p+\gamma)\int_\Omega P(x)|u_n|^{p+1}dx \geq C_1. \tag{3.3}$$

In fact, (3.3) is equivalent to

$$(1+\gamma)\int_\Omega Q(x)|u_n|^{1-\gamma}dx - \lambda(p-1)\int_\Omega P(x)|u_n|^{p+1}dx \geq C_1. \tag{3.4}$$

Since $u_n \in \Lambda^+$, one has

$$(1+\gamma)\int_\Omega Q(x)|u_n|^{1-\gamma}dx - \lambda(p-1)\int_\Omega P(x)|u_n|^{p+1}dx > 0,$$

and consequently, from (3.1) and (3.2) it follows that

$$\begin{aligned}&\lim_{n\to\infty}\Big[(1+\gamma)\int_\Omega Q(x)|u_n|^{1-\gamma}dx - \lambda(p-1)\int_\Omega P(x)|u_n|^{p+1}dx\Big]\\ &= (1+\gamma)\int_\Omega Q(x)|u_*|^{1-\gamma}dx - \lambda(p-1)\int_\Omega P(x)|u_*|^{p+1}dx \geq 0.\end{aligned}$$

Thus we only need to prove that

$$(1+\gamma)\int_\Omega Q(x)|u_*|^{1-\gamma}dx - \lambda(p-1)\int_\Omega P(x)|u_*|^{p+1}dx > 0. \tag{3.5}$$

By contradiction, we assume that

$$(1+\gamma)\int_\Omega Q(x)|u_*|^{1-\gamma}dx - \lambda(p-1)\int_\Omega P(x)|u_*|^{p+1}dx = 0. \tag{3.6}$$

Since

$$\|u_n\|^2 - \lambda\int_\Omega P(x)|u_n|^{p+1}dx - \int_\Omega Q(x)|u_n|^{1-\gamma}dx = 0, \tag{3.7}$$

by the weakly lower semicontinuity of the norm, and combining (3.1)-(3.2) and (3.6), we have

$$\begin{aligned}0 &\geq \|u_*\|^2 - \lambda\int_\Omega P(x)|u_*|^{p+1}dx - \int_\Omega Q(x)|u_*|^{1-\gamma}dx\\ &= \|u_*\|^2 - \frac{p+\gamma}{p-1}\int_\Omega Q(x)|u_*|^{1-\gamma}dx\\ &= \|u_*\|^2 - \frac{\lambda(p+\gamma)}{1+\gamma}\int_\Omega P(x)|u_*|^{p+1}dx,\end{aligned} \tag{3.8}$$

and consequently, from (2.4) one has

$$\begin{aligned}0 &< \Big[\frac{1+\gamma}{p-1}\big(\frac{p-1}{p+\gamma}\big)^{\frac{p+\gamma}{1+\gamma}}\frac{S^{\frac{p+\gamma}{1+\gamma}}}{(|Q|_{r_2}|\Omega|^{\frac{r_2(2^*+\gamma-1)-2^*}{r_2 2^*}})^{\frac{p-1}{1+\gamma}}}\\ &\quad - \lambda|P|_{r_1}|\Omega|^{\frac{r_1(2^*-p-1)-2^*}{r_1 2^*}}\Big]|u_*|_{2^*}^{p+1}\\ &< \frac{1+\gamma}{p-1}\big(\frac{p-1}{p+\gamma}\big)^{\frac{p+\gamma}{1+\gamma}}\frac{\|u_*\|^{\frac{2(p+\gamma)}{1+\gamma}}}{\big(\int_\Omega Q(x)|u_*|^{1-\gamma}dx\big)^{\frac{p-1}{1+\gamma}}} - \lambda\int_\Omega P(x)|u_*|^{p+1}dx\\ &= \frac{1+\gamma}{p-1}\big(\frac{p-1}{p+\gamma}\big)^{\frac{p+\gamma}{1+\gamma}}\frac{\|u_*\|^{\frac{2(p+\gamma)}{1+\gamma}}}{\big(\frac{p-1}{p+\gamma}\|u_*\|^2\big)^{\frac{p-1}{1+\gamma}}} - \frac{1+\gamma}{p+\gamma}\|u_*\|^2 = 0\end{aligned}$$

for all $\lambda \in (0, T_{p,\gamma})$, which is impossible. So (3.5) is obtained and our claim is true. Applying Lemma 2.2 with $u = u_n$, and $\varphi \in H^1(\Omega)$, $\varphi \geq 0$, $t > 0$ small enough, we find a sequence of continuous functions $t_n = t_n(s)$ such that $t_n(0) = 1$ and $t_n(s)(u_n + s\varphi) \in \Lambda^+$. Noting that $t_n(s)(u_n + s\varphi) \in \Lambda^+$ and $u_n \in \Lambda^+$, one has

$$t_n^2(s)\|u_n + s\varphi\|^2 - \lambda t_n^{p+1}(s)\int_\Omega P(x)|u_n + s\varphi|^{p+1}dx$$

$$- t_n^{1-\gamma}(s) \int_\Omega Q(x)(u_n + s\varphi)^{1-\gamma} dx = 0,$$

consequently, from (3.7) it follows that

$$\begin{aligned}
0 &= [t_n^2(s) - 1]\|u_n + s\varphi\|^2 + (\|u_n + s\varphi\|^2 - \|u_n\|^2) \\
&\quad - \lambda[t_n^{p+1}(s) - 1] \int_\Omega P(x)|u_n + s\varphi|^{p+1} dx \\
&\quad - \lambda \int_\Omega P(x)(|u_n + s\varphi|^{p+1} - |u_n|^{p+1}) dx \\
&\quad - [t_n^{1-\gamma}(s) - 1] \int_\Omega Q(x)(u_n + s\varphi)^{1-\gamma} dx \\
&\quad - \int_\Omega Q(x)[(u_n + s\varphi)^{1-\gamma} - |u_n|^{1-\gamma}] dx \\
&\le [t_n^2(s) - 1]\|u_n + s\varphi\|^2 + (\|u_n + s\varphi\|^2 - \|u_n\|^2) \\
&\quad - \lambda[t_n^{p+1}(s) - 1] \int_\Omega P(x)|u_n + s\varphi|^{p+1} dx \\
&\quad - \lambda \int_\Omega P(x)(|u_n + s\varphi|^{p+1} - |u_n|^{p+1}) dx \\
&\quad - [t_n^{1-\gamma}(s) - 1] \int_\Omega Q(x)(u_n + s\varphi)^{1-\gamma} dx,
\end{aligned}$$

then dividing by $s > 0$, we have

$$\begin{aligned}
0 \le \Big[&(t_n(s) + 1)\|u_n + s\varphi\|^2 - \lambda \frac{t_n^{p+1}(s) - 1}{t_n(s) - 1} \int_\Omega P(x)|u_n + s\varphi|^{p+1} dx \\
&- \frac{t_n^{1-\gamma}(s) - 1}{t_n(s) - 1} \int_\Omega Q(x)(u_n + s\varphi)^{1-\gamma} dx\Big] \frac{t_n(s) - 1}{s} + s\|\varphi\|^2 \\
&+ 2 \int_\Omega ((\nabla u_n, \nabla\varphi) + u_n\varphi) dx - \lambda \int_\Omega P(x) \frac{|u_n + s\varphi|^{p+1} - |u_n|^{p+1}}{s} dx.
\end{aligned} \tag{3.9}$$

Let

$$A_n(s) = \frac{t_n(s) - 1}{s}, \tag{3.10}$$

$$\begin{aligned}
K_{1,n}(s) = (t_n(s) + 1)\|u_n + s\varphi\|^2 &- \lambda \frac{t_n^{p+1}(s) - 1}{t_n(s) - 1} \int_\Omega P(x)|u_n + s\varphi|^{p+1} dx \\
&- \frac{t_n^{1-\gamma}(s) - 1}{t_n(s) - 1} \int_\Omega Q(x)(u_n + s\varphi)^{1-\gamma} dx,
\end{aligned}$$

and

$$\begin{aligned}
K_{2,n}(s) = s\|\varphi\|^2 &+ 2 \int_\Omega ((\nabla u_n, \nabla\varphi) + u_n\varphi) dx \\
&- \lambda \int_\Omega P(x) \frac{|u_n + s\varphi|^{p+1} - |u_n|^{p+1}}{s} dx.
\end{aligned}$$

Then, according to (3.7) and (3.3) we have

$$\lim_{s\to 0^+} K_{1,n}(s) = 2\|u_n\|^2 - \lambda(p+1) \int_\Omega P(x)u_n^{p+1} dx - (1-\gamma) \int_\Omega Q(x)u_n^{1-\gamma} dx$$

$$
\begin{aligned}
&= (1+\gamma)\|u_n\|^2 - \lambda(p+\gamma)\int_\Omega P(x)u_n^{p+1}dx \\
&=: K_{1,n} \geq C_1 > 0,
\end{aligned}
$$

and

$$
\lim_{s\to 0^+} K_{2,n}(s) = 2\int_\Omega ((\nabla u_n, \nabla\varphi) + u_n\varphi)dx - \lambda(p+1)\int_\Omega P(x)u_n^p\varphi dx =: K_{2,n}.
$$

Thus, from (3.9) and the continuity of $K_{1,n}(s)$, one obtains

$$
A_n(s) \geq \frac{-K_{2,n}(s)}{K_{1,n}(s)},
$$

for $s > 0$ small. Since $\{u_n\}$ is bounded in $H^1(\Omega)$ there exists a positive constant C_2 such that $|K_{2,n}| < C_2$ for all $n \in N^+$. Therefore,

$$
\liminf_{s\to 0^+} A_n(s) \geq \frac{-K_{2,n}}{K_{1,n}} \geq \frac{-|K_{2,n}|}{K_{1,n}} \geq -\frac{C_2}{C_1} \tag{3.11}
$$

By the subadditivity of norm we have

$$
\|t_n(s)(u_n + s\varphi) - u_n\| \leq |t_n(s) - 1| \cdot \|u_n\| + st_n(s)\|\varphi\|.
$$

Thus from condition (ii) it follows that

$$
\begin{aligned}
&|t_n(s) - 1|\frac{\|u_n\|}{n} + st_n(s)\frac{\|\varphi\|}{n} \\
&\geq I_\lambda(u_n) - I_\lambda[t_n(s)(u_n + s\varphi)] \\
&= -\frac{1+\gamma}{2(1-\gamma)}\|u_n\|^2 + \lambda\frac{p+\gamma}{(p+1)(1-\gamma)}\int_\Omega P(x)u_n^{p+1}dx \\
&\quad + \frac{1+\gamma}{2(1-\gamma)}t_n^2(s)\|u_n + s\varphi\|^2 - \lambda\frac{p+\gamma}{(p+1)(1-\gamma)}t_n^{p+1}(s)\int_\Omega P(x)|u_n + s\varphi|^{p+1}dx \\
&= \frac{1+\gamma}{2(1-\gamma)}(\|u_n + s\varphi\|^2 - \|u_n\|^2) + \frac{1+\gamma}{2(1-\gamma)}[t_n(s) - 1]\|u_n + s\varphi\|^2 \\
&\quad - \lambda\frac{p+\gamma}{(p+1)(1-\gamma)}t_n^{p+1}(s)\int_\Omega P(x)(|u_n + s\varphi|^{p+1} - |u_n|^{p+1})dx \\
&\quad - \lambda\frac{p+\gamma}{(p+1)(1-\gamma)}[t_n^{p+1}(s) - 1]\int_\Omega P(x)u_n^{p+1}dx.
\end{aligned}
$$

Then dividing by $s > 0$, it follows that

$$
\begin{aligned}
&\frac{|t_n(s) - 1|}{s}\frac{\|u_n\|}{n} + t_n(s)\frac{\|\varphi\|}{n} \\
&\geq \frac{1}{1-\gamma}\Big[\frac{1+\gamma}{2}\|u_n + s\varphi\|^2 \\
&\quad - \lambda\frac{p+\gamma}{p+1}\frac{t_n^{p+1}(s) - 1}{t_n(s) - 1}\int_\Omega P(x)u_n^{p+1}dx\Big]\frac{t_n(s) - 1}{s} \\
&\quad + \frac{1+\gamma}{2(1-\gamma)}\frac{\|u_n + s\varphi\|^2 - \|u_n\|^2}{s} \\
&\quad - \lambda\frac{p+\gamma}{(p+1)(1-\gamma)}t_n^{p+1}(s)\int_\Omega P(x)\frac{|u_n + s\varphi|^{p+1} - |u_n|^{p+1}}{s}dx.
\end{aligned} \tag{3.12}
$$

Let

$$K_{3,n}(s)=\frac{1+\gamma}{2}\|u_n+s\varphi\|^2-\lambda\frac{p+\gamma}{p+1}\frac{t_n^{p+1}(s)-1}{t_n(s)-1}\int_\Omega P(x)u_n^{p+1}dx,$$

and

$$\begin{aligned}K_{4,n}(s)=&\frac{1+\gamma}{2(1-\gamma)}\frac{\|u_n+s\varphi\|^2-\|u_n\|^2}{s}\\&-\lambda\frac{p+\gamma}{(p+1)(1-\gamma)}t_n^{p+1}(s)\int_\Omega P(x)\frac{|u_n+s\varphi|^{p+1}-|u_n|^{p+1}}{s}dx.\end{aligned}$$

Then from (3.7) and (3.3), one has

$$\lim_{s\to0^+}K_{3,n}(s)=(1+\gamma)\|u_n\|^2-\lambda(p+\gamma)\int_\Omega P(x)u_n^{p+1}dx=K_{1,n}\ge C_1>0,$$

and

$$\lim_{s\to0^+}K_{4,n}(s)=\frac{1+\gamma}{1-\gamma}\int_\Omega((\nabla u_n,\nabla\varphi)+u_n\varphi)dx-\lambda\frac{p+\gamma}{1-\gamma}\int_\Omega P(x)u_n^p\varphi dx=:K_{4,n}.$$

From (3.12) we have

$$|A_n(s)|\frac{\|u_n\|}{n}+t_n(s)\frac{\|\varphi\|}{n}\ge K_{3,n}(s)A_n(s)+K_{4,n}(s).$$

If $A_n(s)\ge0$, then

$$A_n(s)\le\frac{t_n(s)\frac{\|\varphi\|}{n}-K_{4,n}(s)}{K_{3,n}(s)-\frac{\|u_n\|}{n}}\le\frac{t_n(s)\frac{\|\varphi\|}{n}+|K_{4,n}(s)|}{K_{3,n}(s)-\frac{\|u_n\|}{n}}.$$

If $A_n(s)<0$, then

$$A_n(s)\le\frac{t_n(s)\frac{\|\varphi\|}{n}-K_{4,n}(s)}{K_{3,n}(s)+\frac{\|u_n\|}{n}}\le\frac{t_n(s)\frac{\|\varphi\|}{n}+|K_{4,n}(s)|}{K_{3,n}(s)+\frac{\|u_n\|}{n}}.$$

Hence

$$A_n(s)\le\frac{t_n(s)\frac{\|\varphi\|}{n}+|K_{4,n}(s)|}{K_{3,n}(s)-\frac{\|u_n\|}{n}},$$

and consequently, for n large enough we have

$$\limsup_{s\to0^+}A_n(s)\le\frac{\frac{\|\varphi\|}{n}+|K_{4,n}|}{K_{1,n}-\frac{\|u_n\|}{n}}\le2\frac{1+|K_{4,n}|}{K_{1,n}}\le2\frac{1+C_3}{C_1},\tag{3.13}$$

where $C_3>0$ is a constant such that $|K_{4,n}|<C_3$ by the boundedness of $\{u_n\}$. Thus, according to (3.11) and (3.13), there exists a positive constant C_4 such that

$$\limsup_{s\to0^+}|A_n(s)|\le C_4\tag{3.14}$$

for n large enough.

By the subadditivity of norm, from (ii), we obtain

$$\begin{aligned}&\frac{1}{n}[|t_n(s)-1|\cdot\|u_n\|+st_n(s)\|\varphi\|]\\&\ge\frac{1}{n}\|t_n(s)(u_n+s\varphi)-u_n\|\\&\ge I_\lambda(u_n)-I_\lambda[t_n(s)(u_n+s\varphi)]\end{aligned}$$

$$
\begin{aligned}
&= -\frac{t_n^2(s)-1}{2}\|u_n\|^2 + \lambda\frac{t_n^{p+1}(s)-1}{p+1}\int_\Omega P(x)(u_n+s\varphi)^{p+1}dx \\
&\quad + \frac{t_n^{1-\gamma}(s)-1}{1-\gamma}\int_\Omega Q(x)(u_n+s\varphi)^{1-\gamma}dx + \frac{t_n^2(s)}{2}\big(\|u_n\|^2-\|u_n+s\varphi\|^2\big) \\
&\quad + \frac{\lambda}{p+1}\int_\Omega P(x)\big[(u_n+s\varphi)^{p+1}-u_n^{p+1}\big]dx \\
&\quad + \frac{1}{1-\gamma}\int_\Omega Q(x)\big[(u_n+s\varphi)^{1-\gamma}-u_n^{1-\gamma}\big]dx,
\end{aligned}
$$

and dividing by $s>0$, we have

$$
\begin{aligned}
&\frac{1}{n}(|A_n(s)|\cdot\|u_n\|+\|\varphi\|) \\
&\geq -\Big[\frac{t_n(s)+1}{2}\|u_n\|^2 - \lambda\frac{t_n^{p+1}(s)-1}{(p+1)(t_n(s)-1)}\int_\Omega P(x)(u_n+s\varphi)^{p+1}dx \\
&\quad - \frac{t_n^{1-\gamma}(s)-1}{(1-\gamma)(t_n(s)-1)}\int_\Omega Q(x)(u_n+s\varphi)^{1-\gamma}dx\Big]A_n(s) \\
&\quad + \frac{t_n^2(s)}{2}\frac{\|u_n\|^2-\|u_n+s\varphi\|^2}{s} \\
&\quad + \frac{\lambda}{p+1}\int_\Omega P(x)\frac{(u_n+s\varphi)^{p+1}-u_n^{p+1}}{s}dx \\
&\quad + \frac{1}{1-\gamma}\int_\Omega Q(x)\frac{(u_n+s\varphi)^{1-\gamma}-u_n^{1-\gamma}}{s}dx.
\end{aligned} \tag{3.15}
$$

Let

$$
\begin{aligned}
K_{5,n}(s) = \frac{t_n(s)+1}{2}\|u_n\|^2 - \lambda\frac{t_n^{p+1}(s)-1}{(p+1)(t_n(s)-1)}\int_\Omega P(x)(u_n+s\varphi)^{p+1}dx \\
- \frac{t_n^{1-\gamma}(s)-1}{(1-\gamma)(t_n(s)-1)}\int_\Omega Q(x)(u_n+s\varphi)^{1-\gamma}dx,
\end{aligned}
$$

and

$$
K_{6,n}(s) = \frac{t_n^2(s)}{2}\frac{\|u_n\|^2-\|u_n+s\varphi\|^2}{s} + \frac{\lambda}{p+1}\int_\Omega P(x)\frac{(u_n+s\varphi)^{p+1}-u_n^{p+1}}{s}dx.
$$

Then from (3.7), we have

$$
\lim_{s\to0^+}K_{5,n}(s) = \|u_n\|^2 - \lambda\int_\Omega P(x)u_n^{p+1}dx - \int_\Omega Q(x)u_n^{1-\gamma}dx = 0.
$$

and

$$
\lim_{s\to0^+}K_{6,n}(s) = -\int_\Omega((\nabla u_n,\nabla\varphi)+u_n\varphi)dx + \lambda\int_\Omega P(x)u_n^p\varphi dx.
$$

Thus from (3.15) we deduce

$$
\begin{aligned}
&\frac{1}{1-\gamma}\int_\Omega Q(x)\frac{(u_n+s\varphi)^{1-\gamma}-u_n^{1-\gamma}}{s}dx \\
&\leq |K_{5,n}(s)|\cdot|A_n(s)| - K_{6,n}(s) + \frac{|A_n(s)|\cdot\|u_n\|+\|\varphi\|}{n}.
\end{aligned} \tag{3.16}
$$

Since

$$
Q(x)[(u_n+s\varphi)^{1-\gamma}-u_n^{1-\gamma}] \geq 0, \quad \forall x\in\Omega,\ \forall s>0,
$$

then by Fatou's Lemma we have

$$\int_\Omega Q(x)u_n^{-\gamma}\varphi dx \le \liminf_{s\to 0^+} \frac{1}{1-\gamma}\int_\Omega Q(x)\frac{(u_n+s\varphi)^{1-\gamma}-u_n^{1-\gamma}}{s}dx.$$

Consequently, combining with (3.16) and (3.14), it follows that

$$\begin{aligned}\int_\Omega Q(x)u_n^{-\gamma}\varphi dx &\le \int_\Omega ((\nabla u_n,\nabla\varphi)+u_n\varphi)dx - \lambda\int_\Omega P(x)u_n^p\varphi dx\\ &\quad + \frac{C_4\|u_n\|+\|\varphi\|}{n}\end{aligned}$$

for n large enough which implies that

$$\liminf_{n\to\infty}\int_\Omega Q(x)u_n^{-\gamma}\varphi dx \le \int_\Omega ((\nabla u_*,\nabla\varphi)+u_*\varphi)dx - \lambda\int_\Omega P(x)u_*^p\varphi dx.$$

Then applying Fatou's Lemma again, one obtains

$$\int_\Omega Q(x)u_*^{-\gamma}\varphi dx \le \int_\Omega ((\nabla u_*,\nabla\varphi)+u_*\varphi)dx - \lambda\int_\Omega P(x)u_*^p\varphi dx;$$

that is,

$$\int_\Omega ((\nabla u_*,\nabla\varphi)+u_*\varphi-\lambda P(x)u_*^p\varphi - Q(x)u_*^{-\gamma}\varphi)dx \ge 0, \tag{3.17}$$

for all $\varphi\in H^1(\Omega)$, $\varphi\ge 0$. This means u_* satisfies in the weak sense that

$$-\Delta u_* + u_* \ge 0, \forall x\in\Omega.$$

Since $u_*\ge 0$ and $u_*\not\equiv 0$ in Ω, by the strong maximum principle we have

$$u_*(x) > 0, \quad \text{a.e. } x\in\Omega. \tag{3.18}$$

Thirdly, we prove that $u_*\in\Lambda^+$. On one hand, from (3.18), choosing $\varphi = u_*$ in (3.17), one has

$$\|u_*\|^2 \ge \lambda\int_\Omega P(x)u_*^{p+1}dx + \int_\Omega Q(x)u_*^{1-\gamma}dx.$$

On the other hand, it follows from (3.8) that

$$\|u_*\|^2 \le \lambda\int_\Omega P(x)u_*^{p+1}dx + \int_\Omega Q(x)u_*^{1-\gamma}dx.$$

Thus

$$\|u_*\|^2 = \lambda\int_\Omega P(x)u_*^{p+1}dx + \int_\Omega Q(x)u_*^{1-\gamma}dx, \tag{3.19}$$

and this implies $u_*\in\Lambda$. Moreover from (3.7), one gets

$$\lim_{n\to\infty}\|u_n\| = \lambda\int_\Omega P(x)u_*^{p+1}dx + \int_\Omega Q(x)u_*^{1-\gamma}dx.$$

Hence according to (3.19), we have $u_n\to u_*$ in $H^1(\Omega)$ as $n\to\infty$. In particular, combining (3.19) with (3.5), we obtain

$$(1+\gamma)\|u_*\|^2 - \lambda(p+\gamma)\int_\Omega P(x)|u_*|^{p+1}dx > 0,$$

and therefore $u_*\in\Lambda^+$.

Finally, we prove that u_* is a solution of problem (1.1); that is, u_* satisfies (1.2). In fact, we only need prove that (3.17) is true for all $\varphi\in H^1(\Omega)$. Our proof is inspired by [14]. For the convenience of the reader, we sketch the main steps here. Suppose $\phi \in H^1(\Omega)$ and $t > 0$. We define $\Psi \in H^1(\Omega)$ by

$$\Psi \equiv (u_* + t\phi)^+$$

where $(u_* + t\phi)^+ = \max\{u_* + t\phi, 0\}$. Obviously, $\Psi \geq 0$, so we can replace φ with Ψ in (3.17). Combining with (3.19) we deduce that

$$\begin{aligned}
0 &\leq \int_\Omega \big((\nabla u_*, \nabla\Psi) + u_*\Psi - \lambda P(x)u_*^p\Psi - Q(x)u_*^{-\gamma}\Psi\big)dx \\
&= \int_{\{x|u_*+t\phi\geq 0\}} \Big[(\nabla u_*, \nabla(u_* + t\phi)) + u_*(u_* + t\phi) - \lambda P(x)u_*^p(u_* + t\phi)\Big]dx \\
&\quad - \int_{\{x|u_*+t\phi\geq 0\}} Q(x)u_*^{-\gamma}(u_* + t\phi)dx \\
&= \Big(\|u_*\|^2 - \lambda P(x)u_*^{p+1} - \int_\Omega Q(x)|u_*|^{1-\gamma}dx\Big) \\
&\quad + t\int_\Omega \big((\nabla u_*, \nabla\phi) + u_*\phi - \lambda P(x)u_*^p\phi - Q(x)u_*^{-\gamma}\phi\big)dx \\
&\quad - \int_{\{x|u_*+t\phi<0\}} \big[(\nabla u_*, \nabla(u_* + t\phi)) - \lambda P(x)u_*^p(u_* + t\phi)\big]dx \\
&\quad + \int_{\{x|u_*+t\phi<0\}} Q(x)u_*^{-\gamma}(u_* + t\phi)dx \\
&= t\int_\Omega \big((\nabla u_*, \nabla\phi) + u_*\phi - \lambda P(x)u_*^p\phi - Q(x)u_*^{-\gamma}\phi\big)dx \\
&\quad - \int_{\{x|u_*+t\phi<0\}} \big[(\nabla u_*, \nabla(u_* + t\phi)) - \lambda P(x)u_*^p(u_* + t\phi)\big]dx \\
&\quad + \int_{\{x|u_*+t\phi<0\}} Q(x)u_*^{-\gamma}(u_* + t\phi)dx \\
&\leq t\int_\Omega \big((\nabla u_*, \nabla\phi) + u_*\phi - \lambda P(x)u_*^p\phi - Q(x)u_*^{-\gamma}\phi\big)dx \\
&\quad - t\int_{\{x|u_*+t\phi<0\}} (\nabla u_*, \nabla\phi)dx.
\end{aligned}$$

Since the measure of the domain of integration $\{x : u_* + t\phi < 0\}$ tends to zero as $t \to 0^+$, it follows that $\int_{\{x|u_*+t\phi<0\}}(\nabla u_*, \nabla\phi)dx \to 0$ as $t \to 0^+$. Dividing by t and letting $t \to 0^+$, we deduce that

$$\int_\Omega \big((\nabla u_*, \nabla\phi) + u_*\phi - \lambda P(x)u_*^p\phi - u_*^{-\gamma}\phi\big)dx \geq 0.$$

We note that $\phi \in H^1(\Omega)$ is arbitrary, which implies that u_* is a positive solution of problem (1.1).

Step 2. We prove that there exists a positive solution of problem (1.1) in Λ^-. Similarly to Step 1, applying Ekeland's variational principle to the minimization problem $m^- = \inf_{u\in\Lambda^-} I_\lambda(u)$, there exists a sequence $\{w_n\} \subset \Lambda^-$ with the following properties:

(i) $I_\lambda(w_n) < m^- + \frac{1}{n}$,

(ii) $I_\lambda(w) \geq I_\lambda(w_n) - \frac{1}{n}\|w - w_n\|$, for all $w \in \Lambda^-$.

Since $I_\lambda(u) = I_\lambda(|u|)$, we may assume that $w_n(x) \geq 0$ for all $x \in \Omega$. Obviously, $\{w_n\}$ is bounded in $H^1(\Omega)$, going if necessary to a subsequence, still denoted by $\{w_n\}$, there exists $u_{**} \geq 0$ such that

$$\begin{gathered} w_n \rightharpoonup u_{**}, \quad \text{weakly in } H^1(\Omega), \\ w_n \to u_{**}, \quad \text{strongly in } L^s(\Omega),\ 1 \leq s < 2^*, \\ w_n(x) \to u_{**}(x), \quad \text{a. e. in } \Omega, \end{gathered}$$

as $n \to \infty$. Now we will prove that u_{**} is a positive solution of problem (1.1).

First, we prove that $u_{**}(x) \not\equiv 0$ in Ω. From (2.6), one gets

$$|w_n|_{2^*} \geq \Big[\frac{S(1+\gamma)}{\lambda(p+\gamma)|P|_{r_1}}|\Omega|^{\frac{r_1(2^*-p-1)-2^*}{r_1 2^*}}\Big]^{1/(p-1)},$$

and we obtain $u_{**} \geq 0$ and $u_{**} \not\equiv 0$ in Ω.

Second, we prove that $u_{**}(x) > 0$ a.e. in Ω. Similarly to the arguments in Step 1, we claim that

$$(1+\gamma)\|w_n\|^2 - \lambda(p+\gamma)\int_\Omega P(x)|w_n|^{p+1}dx \leq -C_5, n = 1, 2, \cdots, \tag{3.20}$$

where $C_5 > 0$ is a constant. Since $w_n \in \Lambda$, thus (3.20) is to

$$(1+\gamma)\int_\Omega Q(x)|w_n|^{1-\gamma}dx - \lambda(p-1)\int_\Omega P(x)|w_n|^{p+1}dx \leq -C_5. \tag{3.21}$$

From $w_n \in \Lambda^-$, we have

$$(1+\gamma)\int_\Omega Q(x)|w_n|^{1-\gamma}dx - \lambda(p-1)\int_\Omega P(x)|w_n|^{p+1}dx < 0,$$

and combining with (3.1) and (3.2), it follows that

$$\begin{aligned} &\lim_{n\to\infty}\Big[(1+\gamma)\int_\Omega Q(x)|w_n|^{1-\gamma}dx - \lambda(p-1)\int_\Omega P(x)|w_n|^{p+1}dx\Big] \\ &= (1+\gamma)\int_\Omega Q(x)|u_{**}|^{1-\gamma}dx - \lambda(p-1)\int_\Omega P(x)|u_{**}|^{p+1}dx \leq 0. \end{aligned}$$

Thus we only need prove that

$$(1+\gamma)\int_\Omega Q(x)|u_{**}|^{1-\gamma}dx - \lambda(p-1)\int_\Omega P(x)|u_{**}|^{p+1}dx < 0.$$

By repeating the proof of (3.5) in Step 1.

From Lemma 2.2, choosing $u = w_n$, and $\varphi \in H^1(\Omega)$, $\varphi \geq 0$, $t > 0$ small enough, we find a sequence of continuous functions $t_n = t_n(s)$ such that $t_n(0) = 1$ and $t_n(s)(w_n + s\varphi) \in \Lambda^-$. Similarly to the arguments in Step 1, we also obtain that there exists a constant $C_6 > 0$, such that

$$\limsup_{s\to 0^+}|A_n(s)| \leq C_6 \tag{3.22}$$

for n large enough. Here $A_n(s)$ is also defined by (3.10). In the same manner in Step 1, applying (ii) and (3.22), we have

$$\int_\Omega(\nabla u_{**}\nabla\varphi + u_{**}\varphi - \lambda P(x)u_{**}^p\varphi - Q(x)u_{**}^{-\gamma}\varphi)dx \geq 0, \tag{3.23}$$

for all $\varphi \in H^1(\Omega), \varphi \geq 0$, which means u_{**} satisfies in the weak sense that

$$-\Delta u_{**} + u_{**} \geq 0, \quad \forall x \in \Omega.$$

Since $u_{**} \geq 0$ and $u_{**} \not\equiv 0$ in Ω, by the strong maximum principle, one has

$$u_{**}(x) > 0, \quad \text{a.e.} x \in \Omega. \tag{3.24}$$

Finally, according to (3.23) and (3.24), we can repeat the arguments of Step 1, and obtain that $u_{**} \in \Lambda^-$ is a positive solution of problem (1.1). This complete the proof of Theorem 1.1. □

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References

[1] J. Chabrowski; *On the Neumann problem with singular and superlinear nonlinearities*. Commun. Appl. Anal. 13 (2009) 327–339.
[2] M. M. Coclite; *On a singular nonlinear Dirichlet problem II*, Boll. Unione Mat. Ital. Sez. B (7) 5 (1991) 955–975.
[3] M. M. Coclite, G. Palmieri; *On a singular nonlinear Dirichlet problem*, Comm. Partial Differential Equations 14 (1989) 1315–1327.
[4] M. A. del Pino; *A global estimate for the gradient in a singular elliptic boundary value problem*, Proc. Roy. Soc. Edinburgh Sect. A 122 (1992) 341–352.
[5] A. C. Lazer, P. J. Mckenna; *On a singular nonlinear elliptic boundary value problem*, Proc. Amer. Math. Soc. 111 (1991) 721–730.
[6] A. V. Lair, A. W. Shaker; *Classical and weak solutions of a singular semilinear elliptic problem*, J. Math. Anal. Appl. 211 (1997) 371–385.
[7] J. Liao, P. Zhang; *Existence and nonexistence of solutions for some singular semilinear elliptic problem*. J. Math. (Wuhan) 31 (2011) 777–784.
[8] H. Norimichi, S. Claudio, S. Naoki; *Existence of multiple positive solutions for singular elliptic problems with concave and convex nonlinearities*, Adv. Differential Equations 9 (2004) 197–220.
[9] W. Rudin; *Real and complex analysis*, McGraw-Hill, New York, London etc. 1966.
[10] J. Shi, M. Yao; *On a singular nonlinear semilinear elliptic problem*, Proc. Roy. Soc. Edinburgh Sect. 128 (1998) 1389–1401.
[11] Y. Sun, S. Li; *Structure of ground state solutions of singular semilinear elliptic equations*, Nonlinear Anal. 55 (2003) 399–417.
[12] Y. Sun; S. Li; *Some remarks on a superlinear-singular problem: estimates of λ^**, Nonlinear Anal. 69 (2008) 2636–2650.
[13] Y. Sun, S. Wu; *An exact estimate result for a class of singular equations with critical exponents*, J. Funct. Anal. 260 (2011) 1257–1284.
[14] Y. Sun, S. Wu, Y. M. Long; *Combined effects of singular and superlinear nonlinearities in some singular boundary value problems*, J. Differential Equations 176 (2001) 511–531.
[15] X. Wang, L. Zhao, P. Zhao; *Combined effects of singular and critical nonlinearities in elliptic problems*, Nonlinear Anal. 87 (2013) 1–10.
[16] M. Willem; *Minimax Theorems*, Birkhauser, Boston, 1996.
[17] H. Yang; *Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem*, J. Differential Equations 189 (2003) 487–512.
[18] Z. Zhang; *On a Dirichlet problem with a singular nonlinearity*, J. Math. Anal. Appl. 194 (1995) 103–113.

[19] P. Zhang, J. Liao; *Existence and nonexistence results for classes of singular elliptic problem*, Abstr. Appl. Anal. (2010) Art. ID 435083, 10 pp.

[20] Z. Zhang, J. Yu; *On a singular nonlinear Dirichlet problem with a convection term*, SIAM J. Math. Anal. 32 (2000) 916–927.

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